

## A PHASE TRANSITION IN A MODEL FOR THE SPREAD OF AN INFECTION

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**ABSTRACT.** We show that a certain model for the spread of an infection has a phase transition in the recuperation rate. The model is as follows: There are particles or individuals of type  $A$  and type  $B$ , interpreted as healthy and infected, respectively. All particles perform independent, continuous time, simple random walks on  $\mathbb{Z}^d$  with the same jump rate  $D$ . The only interaction between the particles is that at the moment when a  $B$ -particle jumps to a site which contains an  $A$ -particle, or vice versa, the  $A$ -particle turns into a  $B$ -particle. All  $B$ -particles recuperate (that is, turn back into  $A$ -particles) independently of each other at a rate  $\lambda$ . We assume that we start the system with  $N_A(x, 0-)$   $A$ -particles at  $x$ , and that the  $N_A(x, 0-)$ ,  $x \in \mathbb{Z}^d$ , are i.i.d., mean  $\mu_A$  Poisson random variables. In addition we start with one additional  $B$ -particle at the origin. We show that there is a critical recuperation rate  $\lambda_c > 0$  such that the  $B$ -particles survive (globally) with positive probability if  $\lambda < \lambda_c$  and die out with probability 1 if  $\lambda > \lambda_c$ .

### 1. Introduction

In [KSc], [KSb] we investigated the model discussed in the abstract, but without recuperation, that is, with  $\lambda = 0$  only. We heard of the present version from Ronald Meester and we also learned from him the conjecture that there would be a phase transition in  $\lambda$ , as is now confirmed by our principal theorem here.

Before formally stating our theorem we make some comments about the precise formulation of the model, and introduce some notation. First we define for  $\eta = A$  or  $B$

$$N_\eta(x, t) = \text{number of } \eta\text{-particles at the space-time point } (x, t).$$

Throughout we write  $\mathbf{0}$  for the origin. As stated in the abstract, we put  $N_A(x, 0-)$   $A$ -particles at  $x$  just before we start, with the  $\{N_A(x, 0-), x \in \mathbb{Z}^d\}$

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i.i.d. Poisson variables with mean  $\mu_A$ . We then introduce a  $B$ -particle at the origin and turn some of the particles at the origin instantaneously to  $B$ -particles, so that at time 0 we start with  $N_A(x, 0) = N_A(x, 0-)$   $A$ -particles at  $x \neq \mathbf{0}$  and  $N_B(\mathbf{0}, 0) \in [1, N_A(\mathbf{0}, 0-) + 1]$   $B$ -particles at  $\mathbf{0}$ . However, at any time  $t > 0$  an  $A$ -particle can turn into a  $B$ -particle only if the  $A$ -particle itself jumps at  $t$  or if some  $B$ -particle jumps to the position of the  $A$ -particle at time  $t$ . Thus, we are not saying that an  $A$ -particle turns into a  $B$ -particle whenever it coincides with a  $B$ -particle. We adopted the rule that a jump is required for the following reason. If we did not make this requirement, then  $B$ -particles could effectively not recover at a space-time point  $(x, t)$  with several  $B$ -particles present. Indeed, if one of them tried to turn back into an  $A$ -particle at time  $t$ , it would immediately become of type  $B$  again because it coincided with another  $B$ -particle. This creates some sort of singularity in the model which we are unable to handle at the moment (see, however, Remark 3 below). This is the reason for the requirement of a jump for a change from type  $A$  to type  $B$  at all strictly positive times  $t$ . Only at  $t = 0$  did we change some  $A$ -particles at  $\mathbf{0}$  to  $B$ -particles because they coincided with a  $B$ -particle (even though no jump occurred). The choice of the set of  $A$ -particles at  $\mathbf{0}$  which is turned into  $B$ -particles at time 0 will not influence our arguments. Note that because of the jump requirement there may be particles of both types at a single space-time point.

We have not attempted to give a formal proof of the existence of our process here as a strong Markov process on a suitable probability space. We did carry out such a proof for the model without recuperation in [KSb], and this indicates that such an existence proof for the present model is probably non-trivial, and in any case rather tedious. Probably one can build on the proof for the case without recuperation, because there are fewer  $B$ -particles in the model with recuperation than in the one without recuperation, as shown in Corollary 3 below. We merely mention that in [KSb] our basic state space for the process without recuperation was a subset of the collection of right continuous paths with left limits from  $[0, \infty)$  into

$$(1.1) \quad \Sigma := \prod_{k \geq 1} ((\mathbb{Z}^d \cup \partial_k) \times \{A, B\}).$$

The  $\partial_k$  are cemetery points which we can ignore here, since the process is defined such that it almost surely does not reach any of these points. The initial particles are ordered in some way as  $\rho_1, \rho_2, \dots$ . A typical point of  $\Sigma$  is written as  $\sigma = (\sigma'(k), \sigma''(k))_{k \geq 1}$ . For fixed  $k$ ,  $t \mapsto (\sigma'_t(k), \sigma''_t(k))$  is a path from  $[0, \infty)$  into  $\mathbb{Z}^d \times \{A, B\}$ . The value of this path at time  $t$  represents the position and type of  $\rho_k$  at time  $t$ . We often write  $\pi(t, \rho_k)$  and  $\eta(t, \rho_k)$  for the position and type of  $\rho_k$  at time  $t$ . Thus we have attached to each particle  $\rho$  a path  $t \mapsto \pi(t, \rho)$ . The quantity  $\{\pi_A(t, \rho) := \pi(t, \rho) - \pi(0, \rho)\}_{t \geq 0}$  gives the displacement at time  $t$  of  $\rho$  from its starting point. The paths  $\pi_A(\cdot, \rho)$

for the different  $\rho$  are all taken as independent copies of a continuous time simple random walk  $\{S_t\}_{t \geq 0}$  with jump rate  $D$  and starting point  $S_0 = \mathbf{0}$ . The type of  $\rho_k$  at time  $t$  is a complicated function of the initial types and the restrictions to  $[0, t]$  of all the paths  $\pi_A(\cdot, \rho)$ . More details of dependence of the types as functions of the paths can be found in Section 2 of [KSb].

In the case where recuperation is allowed, as in the present article, we further attach to each particle  $\rho$  a sequence of potential recuperation times  $r(1, \rho) < r(2, \rho) < \dots$ . The  $r(i, \rho)$  are the jump times of a rate  $\lambda$  Poisson process, and these processes are all independent of each other for different  $\rho$  and independent of the  $\pi(\cdot, \rho)$ . If  $\rho$  is of type  $B$  at a time  $t$ , then its type will turn back to  $A$  at the first  $r(i, \rho) \geq t$ . A great advantage of the assumption that the random walks are independent of the types is that the  $\pi(\cdot, \rho)$  and the  $r(i, \rho)$  can be determined once and for all at time 0. The actual evolution of the type of each particle over time is then a complicated function of all the paths and recuperation times for all particles. We shall make a few more comments about this function in the beginning of Section 2. We point out that another reason for our restriction to the case of equal random walks for the different types is that the basic monotonicity properties of the next section may fail if the random walks are different for the different types.

We say that the infection *survives* if

$$(1.2) \quad P\{\text{there are some } B\text{-particles at all times}\} > 0.$$

Since there cannot be any  $B$ -particles after time  $t$  if there are no  $B$ -particles at  $t$ , it follows that (1.2) is equivalent to

$$(1.3) \quad \lim_{t \rightarrow \infty} P\{\text{there are some } B\text{-particles at time } t\} > 0.$$

One may even replace  $\lim_{t \rightarrow \infty}$  by  $\liminf_{t \rightarrow \infty}$  in (1.3). Note that the survival in (1.2) or (1.3) is only global survival. Local survival in its strongest form would say that

$$(1.4) \quad \liminf_{t \rightarrow \infty} P\{N_B(\mathbf{0}, t) > 0\} > 0.$$

A weaker form of local survival would be that

$$(1.5) \quad P\{N_B(\mathbf{0}, t) > 0 \text{ for arbitrarily large } t\} > 0.$$

Clearly (1.4) implies (1.5), and this, in turn implies (1.2). We do not know how to prove that either of the forms (1.4) or (1.5) of local survival holds if  $\lambda$  is small enough. The infection is said to *die out* or to *become extinct* if it does not survive, i.e., if

$$(1.6) \quad P\{\text{there is some (random) } t \text{ such that} \\ \text{there are no } B\text{-particles after } t\} = 1.$$

Here is our principal result.

THEOREM 1. *There exists a  $0 < \lambda_c < \infty$  such that the infection survives if  $\lambda < \lambda_c$  and dies out if  $\lambda > \lambda_c$ .*

REMARK 1. The restriction to only one  $B$ -particle at time 0 is for convenience only. The theorem remains valid if we start with any finite number of  $B$ -particles at (nonrandom) positions.

REMARK 2. We already remarked that the theorem does not give local survival if  $\lambda$  is sufficiently small. Neither does it tell us anything about the location of the  $B$ -particles as a function of  $t$  on the event that the  $B$ -particles survive forever.

By a special argument one can show that (1.5) holds for  $d = 1$  and  $\lambda < \lambda_c$  on the event that the  $B$ -particles survive forever.

REMARK 3. The proof that there is survival for small  $\lambda > 0$  works even in the case in which an  $A$ -particle turns into a  $B$ -particle whenever it coincides with a  $B$ -particle, that is, if we do not require that the  $A$  or  $B$ -particle jumps before reinfection can occur after recuperation of a  $B$ -particle.

REMARK 4. A similar result for another variant of the model is obtained in [AMP]. This article considers the so-called frog model in which only the  $B$ -particles move and the  $A$ -particles stand still. In [AMP] time is taken discrete. It is assumed that each  $B$ -particle is removed from the system at its first recuperation. One could interpret this by means of the introduction of a third type of particles, namely immune ones which do not interact with any particles. When a particle recuperates from the infection it becomes immune. This results also in some conclusions which differ from the ones in the present paper. In particular, [AMP] shows that in their case there never is survival in dimension 1, if recuperation is allowed (i.e.,  $\lambda_c = 0$ , so that *there is no nontrivial phase transition in dimension 1*, in contrast to our model).

The fact that the  $A$ -particles can move in our model makes the analysis here much harder than in [AMP]. This also forces us to stick to Poisson initial conditions, while [AMP] can handle much more general initial conditions, as well as more general graphs as  $\mathbb{Z}^d$ .

We note that our proof of survival in Section 3 still goes through if the  $A$  and  $B$ -particles perform the same random walk and  $B$ -particles are immune after recuperation. In this case one also has extinction for large  $\lambda$  by Theorem 1 and monotonicity arguments as in Lemma 4 below. The system in which  $B$ -particles become immune lies stochastically below the system we are investigating here (in the sense of Lemma 4). Thus Theorem 1 remains valid if  $B$ -particles are immune after recuperation.

REMARK 5. The following version of the frog model can still be analyzed to some extent. Take time continuous, and assume that the  $A$ -particles cannot

move. Assume further that a  $B$ -particle turns any  $A$ -particle with which it coincides instantaneously into a  $B$ -particle.  $B$ -particles turn back to  $A$ -particles at a constant rate  $\lambda > 0$ , but these recuperated particles stay in the system and act as any original  $A$ -particle. For the initial state take the  $N_A(x, 0-)$  as i.i.d., mean  $\mu_A$  Poisson variables, and add one  $B$ -particle at  $\mathbf{0}$ . We have not constructed such a process, but we take it for granted that this process can be properly defined so as to justify the argument below.

[AMP] proved survival for the process in discrete time, in which the  $A$ -particles stand still,  $\lambda$  is small, and in which particles which recuperate are removed from the system, and  $d \geq 2$ . We expect that this also holds for the process just described. It is perhaps surprising, though, that the rules of the preceding paragraph imply that for large  $\mu_A$  the process *always survives*. More precisely, we show that if  $\mu_A >$  some  $\mu_{A,d}$  (which depends on the dimension  $d$  only), then the  $B$ -particles survive for all values of  $\lambda$ , so that *there is no phase transition*.

The key observation for proving this lack of a phase transition is that if there are several particles present at some space-time point  $(x, t)$ , then they are all of type  $A$  or all of type  $B$ . In the latter case, if one of the  $B$ -particles tries to recuperate, it is immediately reinfected by the other  $B$ -particles at the same location, and so, as long as there are at least two particles on one site, none of the particles at that site can change from type  $B$  to  $A$ . This shows that  $B$ -particles can turn back to  $A$ -particles only at sites with no other particle. Since the  $A$ -particles stand still, it follows that, at any fixed site, at most one  $B$ -particle can recuperate and stay of type  $A$  forever after.

We shall also use that for  $\mu_A >$  some  $\mu_{A,d}$  it holds

$$(1.7) \quad \sum_{\substack{C: C \text{ connected} \\ \mathbf{0} \in C}} P\left\{ \sum_{x \in C} N_A(x, 0-) < \frac{1}{2} \mu_A |C| \right\} < \infty.$$

This follows from standard large deviation estimates for the Poisson distribution, since  $\sum_{x \in C} N_A(x, 0-)$  has a Poisson distribution with mean  $\mu_A |C|$ , and from the fact that the number of connected sets  $C$  with  $\mathbf{0} \in C$  grows only exponentially in  $|C|$ . It follows from (1.7) and the Borel-Cantelli lemma that for  $\mu_A > \mu_{A,d}$ , almost surely there exists some random  $k_0$  such that for any connected set  $C \subset \mathbb{Z}^d$  which contains  $\mathbf{0}$  and with  $|C| \geq k_0$ ,

$$(1.8) \quad \sum_{x \in C} N_A(x, 0-) \geq \frac{1}{2} \mu_A |C|.$$

Assume now that there exist  $k_0$  *distinct* particles  $\rho_1, \dots, \rho_{k_0}$ , and space-time points  $(x_i, t_i)$ , such that  $\rho_i$  is at  $x_i$  at time  $t_i$  as a  $B$ -particle. (Some of the  $x_i$  or  $t_i$  with different  $i$  may have the same value.) Assume further that

$$(1.9) \quad A_0 := \{x_1, \dots, x_{k_0}\} \text{ is connected.}$$

Assume also that the infection dies out at some time  $t_\infty < \infty$ . Let  $C_0$  be the collection of sites visited by one of the  $\rho_i$  before the infection dies out, or more precisely

$$(1.10) \quad C_0 := \{x : \text{for some } 1 \leq i \leq k_0, \rho_i \text{ visits } x \text{ during } [t_i, t_\infty]\}.$$

$C_0$  is again a connected set, because each particle  $\rho_i$  moves by a simple random walk through a connected set. Next, let  $D_0$  be the collection of sites at which the  $k_0$  particles  $\rho_i$  are at time  $t_\infty$  (and hence also at  $t > t_\infty$ , because each  $\rho_i$  must have type  $A$  from the time of extinction of the infection on). Then, by the one but last paragraph,

$$|C_0| \geq |D_0| = k_0.$$

Now, let  $\zeta$  be some particle at some  $x \in C_0$  at time 0 (if such a particle exists). Then  $x$  is visited by some  $\rho_i$  at some time  $s_i \in [t_i, t_\infty]$ . Pick such an  $i$  and let  $s_i$  be the smallest time in  $[t_i, t_\infty]$  at which  $\rho_i$  is at  $x$ . We claim that  $\rho_i$  must have type  $B$  at time  $s_i$ . Indeed, if  $s_i = t_i$ , this is true by our assumption on  $\rho_i$  at  $(x_i, t_i)$ . If  $s_i > t_i$ , then  $\rho_i$  must jump to  $x$  at time  $s_i$ . But only  $B$ -particles do jump, so that our claim also holds in this case. Now, either  $\zeta$  has type  $B$  at some time during  $[0, s_i)$ , or is of type  $A$  and sits still at  $x$  during all of  $[0, s_i]$  and then it is turned into type  $B$  by  $\rho_i$  at  $s_i$ . In either case, the infection cannot die out before  $\zeta$  too recuperates for a last time. But, by (1.8) the number of particles in  $C_0$  at time 0 is at least

$$(1.11) \quad \sum_{x \in C_0} N_A(x, 0-) \geq \frac{1}{2} \mu_A |C_0| \geq \frac{1}{2} \mu_A k_0 \geq 2k_0,$$

provided we take  $\mu_A \geq \mu_{A,d} \geq 4$ . Thus we now have found at least  $2k_0$  particles which must recuperate during  $[0, t_\infty]$ . We can repeat the argument with the collection  $\rho_1, \dots, \rho_{k_0}$  replaced by the particles in  $C_0$ , and  $A_0$  replaced by  $A_1 := C_0$ .  $k_0$  is then replaced by some  $k_1 \geq 2k_0$ . By repeating this argument infinitely often we see that it is impossible for the infection to die out in finite time, if there is a  $k_0$  such that (1.8) holds for all connected  $C \supset \{\mathbf{0}\}$  with  $|C| \geq k_0$ , and particles  $\rho_i$ ,  $1 \leq i \leq k_0$ , as above. Here we have taken it for granted (without proof) that in any reasonable version of the process only finitely many  $B$ -particles can be formed in finite time. We apply the preceding remarks with  $A_0 = \{\mathbf{0}\}$  and a large non-random  $k_0$ . This shows that

$$\begin{aligned} &P\{\text{infection does not die out}\} \\ &\geq P\{(1.8) \text{ holds for all connected } C \supset \{\mathbf{0}\} \\ &\quad \text{with } |C| \geq k_0 \text{ and } N_A(\mathbf{0}, 0-) \geq k_0\} \\ &\geq P\{(1.8) \text{ holds for all connected } C \supset \{\mathbf{0}\} \\ &\quad \text{with } |C| \geq k_0\} P\{N_A(\mathbf{0}, 0-) \geq k_0\} \\ &> 0 \quad (\text{by the Harris-FKG inequality}). \end{aligned}$$

This argument is independent of the value of  $\lambda$  and therefore proves our claim that there is no phase transition. This ends Remark 5.

In the next section we begin with a monotonicity property which immediately implies that there exists a critical  $\lambda_c$  with the properties stated in Theorem 1, except that  $\lambda_c = 0$  or  $\infty$  is still not excluded. In Section 3 we then show that  $\lambda_c > 0$  and in Section 5 we show that  $\lambda_c < \infty$ . These two sections which show that there are nontrivial regions of survival and extinction, respectively, form the core of this paper. Section 4 is a kind of interlude in which we prove that the maximal number of jumps during  $[0, t]$  in a certain class of paths is at most  $O(t)$ . This estimate is crucial for the proof of extinction in Section 5.

Our methods are a combination of the multi-scale analysis of [KSa], [KSb] and percolation arguments. To show that the infection survives for small  $\lambda$  we introduce (in Section 3) a certain directed (dependent) percolation process with the property that if percolation occurs in this process, then the infection survives. We then show that percolation occurs for sufficiently small  $\lambda$  by showing that there is only a very small probability that the origin is separated from  $\infty$  by a distant separating set. To show that the infection dies out when  $\lambda$  is large we use a block argument (in Section 5). We show that with high probability, along “almost all” paths in space-time there have to be blocks which prevent the transmission of the infection. The paragraph following the statement of Proposition 24 in Section 5 gives some more details of this strategy.

A reader interested in the details of the proofs will have to refer to [KSa]–[KSb] a number of times.

Throughout this paper we make the following convention about constants.  $K_i$  will denote a strictly positive, finite constant, whose precise value is unimportant for our purposes. The value of the same  $K_i$  may be different in different formulas. We use  $C_i$  for constants whose value remains fixed throughout the paper. They will again have values in  $(0, \infty)$ . If necessary, we indicate on what quantities a constant depends at the time when it is first introduced. Throughout  $\|x\|$  denotes the  $\ell^\infty$  norm of the vector  $x = (x(1), \dots, x(d)) \in \mathbb{R}^d$ , i.e.,

$$(1.12) \quad \|x\| = \max_{1 \leq i \leq d} |x(i)|.$$

$$(1.13) \quad \mathcal{C}(m) = \{x : \|x\| \leq m\} = [-m, m]^d.$$

$\mathbf{0}$  will denote the origin (in  $\mathbb{Z}^d$  or  $\mathbb{R}^d$ );  $|C|$  usually denotes the cardinality of the set  $C$ .

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## 2. Two monotonicity properties

We repeat that *we assume that all particles perform independent copies of the same random walk*. In this section we show that increasing the recuperation rate decreases the number of infected particles. In addition we repeat a monotonicity property from [KSc] for the system without recuperation.

First some recapitulation of the notation used in [KSc], [KSb] for the construction of a suitable Markov process.  $\Sigma_0$  is a subset of  $\Sigma$  (defined in (1.1)) which serves as the state space for a strong Markov process  $\{Y_t\}_{t \geq 0}$  constructed as a suitable version of our infection process without recuperation. For our purposes here we do not have to know the exact definition of  $\Sigma_0$ , but we merely have to know that the initial conditions, as described by the Poisson variables  $N_A(x, 0-)$ , lie almost surely in  $\Sigma_0$  (by Proposition 4 of [KSc]), and that then the Markov chain takes values in  $\Sigma_0$  for all times, almost surely. Moreover, we have from Section 2 in [KSc] (see (2.18) there), that almost surely

$$(2.1) \quad \sup_{s \leq t} (\text{number of } B\text{-particles at time } s \text{ in the process } \{Y_t\}) < \infty.$$

$\Sigma_0$  will also be the state space for the infection process with recuperation. We write  $\{Y_t(\lambda)\}$  for the process with recuperation rate  $\lambda$ , even when  $\lambda = 0$ . The process  $\{Y_t(0)\}$  does not allow recuperation, but it is not the same as the process  $\{Y_t\}$  of [KSa], [KSc]. In the former process an  $A$ -particle turns into a  $B$ -particle only when one of these two particles jumps to the position of the other. In particular this process can have  $A$  and  $B$ -particles at the same site. In the process  $\{Y_t\}$  this is not possible, because an  $A$ -particle turns instantaneously to a  $B$ -particle when it coincides with a  $B$ -particle. The difference between these two processes, even though it is small, forces some extra work on us.



To motivate our construction for  $\{Y_t(\lambda)\}$  consider a particle  $\rho$  which is of type  $B$  at time  $s$  in the process  $\{Y_t(\lambda)\}$ , and which has changed type only finitely often in this process. Such a particle should have an analogue of a genealogical path as introduced in Proposition 4 in [KSc] in  $\{Y_t\}$ . Specifically, there should be space-time points  $(x_i, s_i)$  with  $1 \leq i \leq \ell$  for some  $\ell$ , and  $0 < s_1 < \dots < s_\ell < s$ , and particles  $\rho_i$  for  $0 \leq i \leq \ell + 1$  with  $\rho_{\ell+1} = \rho$ , such that at time  $s_i$ ,  $\rho_i$  jumps to the position of  $\rho_{i+1}$  or vice versa. Moreover, (with  $s_{\ell+1} = s$ )  $\rho_0$  should have type  $B$  at time 0, and  $\rho_i$  should have type  $B$  and not recuperate during  $[s_i, s_{i+1}]$  in  $\{Y_t(\lambda)\}$ . This last requirement was of course not present in [KSc], but nevertheless the backwards construction of the genealogical path from [KSc] works with only trivial modifications. To be more specific, start with  $\rho$  of type  $B$  at time  $s$  and find the time  $t_1 := \min\{u : \rho$  has type  $B$  in  $\{Y_t(\lambda)\}$  during  $[u, s]\} = \min\{u : \rho$  does not recuperate in  $\{Y_t(\lambda)\}$  during  $[u, s]\}$ . Then, either  $t_1 = 0$  or  $t_1 > 0$ . If  $t_1 = 0$  then  $\rho$  was of type  $B$  at time 0 and did not recuperate during  $[0, s]$  and we are done. If  $t_1 > 0$ , then there must have been some other particle  $\rho^{(1)}$  of type  $B$  in  $\{Y_t(\lambda)\}$ , and this  $\rho^{(1)}$  must have jumped to the position of  $\rho$ , or vice versa, at time  $t_1$ . We then define  $t_2 = \min\{u : \rho^{(1)}$  has type  $B$  in  $\{Y_t(\lambda)\}$  during  $[t_2, t_1]\}$ , etc., until we arrive, for some  $\ell$  at time  $t_\ell$  and a particle  $\rho^{(\ell+1)}$  which had type  $B$  in  $\{Y_t(\lambda)\}$  during  $[0, t_\ell]$ . The genealogical path for  $\rho$  in  $\{Y_t(\lambda)\}$  is then obtained by using the  $t_i$  and  $\rho^{(i)}$  in reverse order for the  $s_i$  and  $\rho_i$ . Note that if  $\rho$  is of type  $B$  at time  $s$  and has a genealogical path of times  $0 < s_1 < \dots < s_\ell < s_{\ell+1} = s$  and corresponding particles  $\rho_i$ , in the process  $\{Y_t(\lambda)\}$ , then  $\rho$  can also be regarded as a  $B$ -particle at time  $s$  in the process  $\{Y_t\}$ . Indeed, one easily shows by induction on  $i$  that each of the particles  $\rho_i$  must have type  $B$  at time  $s_i$  in  $\{Y_t\}$ . (Note that we are not saying that  $\rho_i$  changes type from  $A$  to  $B$  at time  $s_i$  in  $\{Y_t\}$ ; the argument here does not rule out that  $\rho_i$  is already of type  $B$  just before  $s_i$ , but this does not matter.)

With the motivation provided by the preceding paragraph we construct  $\{Y_t(\lambda)\}$  on the product of the probability space for  $\{Y_t\}$  with the probability space for all the recuperation processes  $\{r(i, \rho)\}$ . For a generic point  $\sigma = (\sigma'(k), \sigma''(k))$  in the state space  $\Sigma$  (see (1.1)) define  $\bar{\sigma}$  to be the point obtained from  $\sigma$  by taking  $\sigma''(k) = B$  for all  $k$  for which there is an  $\ell$  with  $\sigma'(\ell) = \sigma'(k)$  and  $\sigma''(\ell) = B$ . This means that  $\bar{\sigma}$  is obtained from  $\sigma$  by changing to  $B$  the type of all particles at a position which already has at least one  $B$ -particle. We now describe the process  $\{Y_t(\lambda)\}$  starting from a  $\sigma$  for which  $\bar{\sigma} \in \Sigma_0$ . In [KSc], [KSb] we defined the process  $\{Y_t\}$  starting from  $\bar{\sigma}$ . This begins with assigning to each particle  $\rho$  a random walk path  $\pi_A(\cdot, \rho)$  and then giving  $\rho$  the position  $\pi(t, \rho) = \pi(0, \rho) + \pi_A(t, \rho)$  at time  $t$ , where  $\pi(0, \rho)$  is just the initial position of  $\rho$ . We now assign to  $\rho$  the same positions  $\{\pi(t, \rho)\}_{t \geq 0}$  in  $\{Y_t(\lambda)\}$ . To complete the description we merely have to decide what type to assign to a particle  $\rho$  as a function of time in  $\{Y_t(\lambda)\}$ . If  $\rho$  has type  $A$  at time  $s$  in  $\{Y_t\}$  starting from  $\bar{\sigma}$ , then we also assign it type  $A$  at time  $s$  in  $\{Y_t(\lambda)\}$ .

In particular, since almost surely only finitely many particles meet a particle of type  $B$  during  $[0, s]$  in  $\{Y_t\}$  (by (2.1) and the fact that any particle which meets a  $B$ -particle before time  $s$  has type  $B$  at time  $s$  in  $\{Y_t\}$ ), this rule also assigns type  $A$  during  $[0, s]$  to all but finitely many particles in  $\{Y_t(\lambda)\}$ . Let  $\rho^{(1)}, \dots, \rho^{(m)}$  be the finitely many particles of type  $B$  at time  $s$  in  $\{Y_t\}$ . The particles which have type  $A$  at time  $s$  in  $\{Y_t\}$  have no influence at all on the types of the  $\rho^{(j)}$ ,  $1 \leq j \leq m$ , during  $[0, s]$ . We can therefore construct the types of the finitely many  $\rho^{(j)}$  in  $\{Y_t(\lambda)\}$  by changing types appropriately at the only finitely many times during  $[0, s]$  when one of these particles jumps to the position of another one, or when a recuperation event  $r(i, \rho^{(j)})$  occurs for some  $j \leq m$ . It is not hard to check that if  $0 \leq s_1 < s_2$ , then the restriction of the process so constructed on  $[0, s_2]$  to  $[0, s_1]$  agrees with the process constructed on  $[0, s_1]$ . Indeed the only difference between the two constructions on  $[0, s_1]$  could come from the particles which have type  $A$  at  $s_1$ , but type  $B$  at  $s_2$ . However, these particles have not interacted with any particle during  $[0, s_1]$ . We shall not discuss the construction of the process  $\{Y_t(\lambda)\}$  further, and in particular shall not verify that the above construction actually gives us a good version of  $\{Y_t(\lambda)\}$ .

The preceding construction provides also a coupling of the processes  $\{Y_t\}$  and  $\{Y_t(\lambda)\}$ . This coupling shows that  $\{Y_t\}$  has more  $B$ -particles than the  $\{Y_t(\lambda)\}$  process starting from  $\sigma$ , in the sense of the following lemma.

LEMMA 2. *Let  $\{Y_t(\lambda)\}$  and  $\{Y_t\}$  start at  $\sigma$  and  $\bar{\sigma}$ , respectively, with  $\bar{\sigma} \in \Sigma_0$ . In particular, each particle is at the same position at time 0 in both processes and each particle which has type  $B$  in  $\{Y_t(\lambda)\}$  at time 0 also has type  $B$  in  $\{Y_t\}$  at time 0. Then the coupling described above is such that any particle present at a space-time point  $(x, s)$  in one of the processes  $\{Y_t(\lambda)\}$  and  $\{Y_t\}$  is also present in the other. Moreover, if a particle at  $(x, s)$  has type  $B$  in  $\{Y_t(\lambda)\}$ , then it also has type  $B$  in  $\{Y_t\}$ .*

The lemma is immediate from the construction. The next lemma is very similar. It proves a monotonicity in the recuperation rate.

LEMMA 3. *Let  $0 \leq \lambda_1 \leq \lambda_2$  and let  $\{r_1(i, \rho)\}$  and  $\{r_2(i, \rho)\}$  be Poisson processes with the rates  $\lambda_1$  and  $\lambda_2$ , respectively. Assume that these are coupled such that for each  $\rho$*

$$(2.2) \quad \{r_1(i, \rho)\}_{i \geq 1} \subset \{r_2(i, \rho)\}_{i \geq 1}.$$

*Let  $\{Y_t(\lambda_j)\}$  be the infection process corresponding to the recuperation rate  $\lambda_j$ ,  $j = 1, 2$ , and assume that  $\{Y_t(\lambda_1)\}$  and  $\{Y_t(\lambda_2)\}$  are constructed from the same initial state  $\sigma$  and the same set of random walk paths  $\pi(\cdot, \rho)$ , but potential recuperation times  $r_1(i, \rho)$  and  $r_2(i, \rho)$ , respectively. Assume that  $\bar{\sigma} \in \Sigma_0$ . Then the processes  $\{Y_t(\lambda_1)\}$  and  $\{Y_t(\lambda_2)\}$  are coupled in such a way that any particle present at a space-time point  $(x, s)$  in one of the  $\{Y_t(\lambda_j)\}$  is*

also present in the other. Moreover, a.s. it holds for all  $s$  that if a particle at  $(x, s)$  has type  $B$  in  $\{Y_t(\lambda_2)\}$ , then it also has type  $B$  in  $\{Y_t(\lambda_1)\}$ .

*Proof.* Clearly any particle  $\rho$  present in one of the  $\{Y_t(\lambda_j)\}$  at  $(x, s)$  is also present at  $(x, s)$  in the other process since the position of any initial particle  $\rho$  at time  $s$  is  $\pi(s, \rho)$  in both processes.

We can now couple the process  $\{Y_t(\lambda_j)\}$  with a process  $\{Y_t\}$  which starts in  $\bar{\sigma}$ , as in Lemma 2. Then, by Lemma 2, the number of particles in  $\{Y_t(\lambda_j)\}$  and in  $\{Y_t\}$  at any space-time point is the same, and the number of  $B$ -particles in  $\{Y_t(\lambda_j)\}$  is no more than in  $\{Y_t\}$  at any space-time point. This implies that a.s., for  $j = 1$  and for  $j = 2$ ,

$$(2.3) \quad \sup_{s \leq t} (\text{number of particles at } (x, s) \text{ in } \{Y_t(\lambda_j)\}) < \infty$$

for all  $x \in \mathbb{Z}^d, t \geq 0$ ,

and that there are only finitely many  $B$ -particles in  $Y_t(\lambda_j)$  at any time  $t$  (by virtue of Lemma 2 of [KSc]). In particular, a.s. for all  $s$ , any  $B$ -particle at time  $s$  in  $\{Y_t(\lambda_j)\}$  has an analogue of a genealogical path as above.

Assume now that a particle  $\rho$  has type  $B$  at time  $s$  in  $\{Y_t(\lambda_2)\}$ . Let its genealogical path in  $\{Y_t(\lambda_2)\}$  be determined by the space-time points  $(x_i, s_i)$  and by the particles  $\rho_i$ . That means that there are space-time points  $(x_i, s_i)$  with  $1 \leq i \leq \ell$  for some  $\ell$ , and  $0 < s_1 < \dots < s_\ell < s$  and particles  $\rho_i$  for  $0 \leq i \leq \ell + 1$  with  $\rho_{\ell+1} = \rho$ , such that at time  $s_i$ ,  $\rho_i$  jumps to the position of  $\rho_{i+1}$  or vice versa. Moreover,  $\rho_0$  has type  $B$  at time 0, and  $\rho_i$  does not recuperate during  $[s_i, s_{i+1}]$  in  $\{Y_t(\lambda_2)\}$  (with  $s_{\ell+1} = s$ ). Note that, because  $\rho_i$  stays of type  $B$  in  $\{Y_t(\lambda_2)\}$  during  $[s_i, s_{i+1}]$ ,  $r_2(j, \rho_i) \notin [s_i, s_{i+1}]$  for all  $j$ . But then  $\rho_i$  does not recuperate during  $[s_i, s_{i+1}]$  in  $\{Y_t(\lambda_1)\}$  either, by virtue of (2.2). It then follows by induction on  $i$  that also in  $\{Y_t(\lambda_1)\}$ , each  $\rho_i$  is of type  $B$  at time  $s_i$  and stays of type  $B$  through time  $s_{i+1}$ . In particular  $\rho = \rho_{\ell+1}$  must have type  $B$  at time  $s$  in  $\{Y_t(\lambda_1)\}$ . □

A consequence of Lemma 3 is that if the infection dies out for some value  $\lambda^{(1)}$  of the recuperation rate, then it dies out for all larger recuperation rates. As already stated this shows that  $\lambda_c$  exists, but it may still have the value 0 or  $\infty$ .

We will also need another monotonicity property for  $\{Y_t(\lambda)\}$ . Basically this says that if we increase the number of  $B$ -particles in the initial state, then this will increase the number of  $B$ -particles at any later time. The analogue of this result for  $\{Y_t\}$  is in Lemma 14 of [KSc].

LEMMA 4. Let  $\lambda \geq 0$  and let  $\sigma^{(2)}$  be such that  $\bar{\sigma}^{(2)} \in \Sigma_0$ . Assume further that  $\sigma^{(1)}$  lies below  $\sigma^{(2)}$  in the following sense:

$$(2.4) \quad \text{for any site } x \in \mathbb{Z}^d, \text{ all particles present in } \sigma^{(1)} \text{ at } x \\ \text{are also present in } \sigma^{(2)} \text{ at } x,$$

and

$$(2.5) \quad \text{any particle which has type } B \text{ in } \sigma^{(1)} \text{ also has type } B \text{ in } \sigma^{(2)}.$$

Let  $\pi_A(\cdot, \rho)$  be the random walk paths associated to the various particles. Assume that the Markov processes  $\{Y_t^{(1)}(\lambda)\}$  and  $\{Y_t^{(2)}(\lambda)\}$  are constructed (as explained before Lemma 2) by means of the same set of paths  $\pi_A(\cdot, \rho)$  and the same recuperation processes  $\{r(1, \rho)\}$  for any  $\rho$  present in  $\sigma^{(1)}$ . Assume further that  $\{Y_t^{(i)}(\lambda)\}$  starts in  $\sigma^{(i)}$ ,  $i = 1, 2$ . Then, almost surely,  $\{Y_t^{(1)}(\lambda)\}$  and  $\{Y_t^{(2)}(\lambda)\}$  satisfy (2.4) and (2.5) for all  $t$ , with  $\sigma^{(i)}$  replaced by  $Y_t^{(i)}(\lambda)$ ,  $i = 1, 2$ . Moreover, almost surely

$$(2.6) \quad \sup_{s \leq t} (\text{number of particles at } (x, s) \text{ in } \{Y_t^{(i)}(0)\}) < \infty \\ \text{for all } x \in \mathbb{Z}^d, t \geq 0,$$

for  $i = 1$  and for  $i = 2$ .

*Proof.* It is clear that (2.4) holds with  $\sigma^{(i)}$  replaced by  $Y_t^{(i)}(\lambda)$ , that is,

$$(2.7) \quad \text{for any site } x \in \mathbb{Z}^d, \text{ and } t \geq 0, \text{ all particles present in } Y_t^{(1)} \text{ at } x \\ \text{are also present in } Y_t^{(2)} \text{ at } x.$$

By Lemma 14 in [KSc]  $\bar{\sigma}^{(2)} \in \Sigma_0$  implies that also  $\bar{\sigma}^{(1)} \in \Sigma_0$ . In the same way as in the second paragraph of the proof of Lemma 3 one now shows that a.s. (2.6) holds and that a.s. there are only finitely many  $B$ -particles in  $Y_t^{(i)}(\lambda)$  at any time  $t$ . Also a.s. for all  $s$  any  $B$ -particle at time  $s$  in  $\{Y_t^{(i)}(\lambda)\}$  has an analogue of a genealogical path.

To prove (2.5) with  $\sigma^{(i)}$  replaced by  $Y_s^{(i)}(\lambda)$ , assume that  $\rho$  has type  $B$  at time  $s$  in the first process, i.e., in  $\{Y_t^{(1)}(\lambda)\}$ . Then it has a genealogical path determined by space-time points  $(x_j, s_j)_{1 \leq j \leq \ell}$  for some  $\ell$ , and  $0 < s_1 < \dots < s_\ell < s$  and particles  $\rho_j$  for  $0 \leq j \leq \ell + 1$  with  $\rho_{\ell+1} = \rho$  and  $s_{\ell+1} = s$ , such that at time  $s_j$ ,  $\rho_j$  jumps to the position of  $\rho_{j+1}$  or vice versa. Moreover, all these  $\rho_j$  and  $\rho$  are present in  $\sigma^{(1)}$  (and hence are particles in  $\{Y_t^{(1)}\}$ ),  $\rho_0$  has type  $B$  at time 0, and  $\rho_j$  has type  $B$  and does not recuperate during  $[s_i, s_{i+1}]$  in  $\{Y_t^{(1)}(\lambda)\}$  (with  $s_{\ell+1} = s$ ). One then proves by induction on  $j$  that each  $\rho_j$ ,  $0 \leq j \leq \ell + 1$ , is also present and has type  $B$  during  $[s_j, s_{j+1}]$  in  $\{Y_t^{(2)}\}$ .

In particular,  $\rho = \rho_{\ell+1}$  is present and of type  $B$  at time  $s$  in  $\{Y_t^{(2)}\}$ . Thus, (2.5) holds.  $\square$

### 3. Survival for small $\lambda$

In this section we show that  $0 < \lambda_c \leq \infty$ . To introduce the directed percolation process which we promised in the introduction, we must describe certain blocks in  $\mathbb{Z}^{d+1}$ .  $C_0$  will be the same large integer as in [KSc] (see (4.18), (4.19) there). Without loss of generality we take  $C_0$  even. Also  $\gamma_0 \in (0, \infty)$  will be as in [KSc]. Many constants  $K_i$  and  $p_i$  will appear in the proof. These will all depend only on  $d, D, C_0, \gamma_0, \mu_A$ . All  $K_i$  and  $p_i$  are finite and strictly positive. These properties of the  $K_i, p_i$  will not be mentioned further. Throughout this section we think of  $p$  as fixed, and often suppress it in the notation; we shall see at the end of the proof of Lemma 12 that any large enough value of  $p$  will work for our purposes. For the time being we only need to know that  $p$  is an integer  $\geq 1$ . We also fix

$$q = 2d + 1$$

and define, for any positive integer  $r$ ,

$$\Delta_r = C_0^{6r}.$$

For  $\mathbf{i} = (i(1), \dots, i(d)) \in \mathbb{Z}^d$  and  $k \in \mathbb{Z}$  we take

$$(3.1) \quad \widehat{\mathcal{B}}_p(\mathbf{i}, k) = \prod_{s=1}^d [i(s)\Delta_p, (i(s) + 1)\Delta_p) \times [kp^q\Delta_p, (k + 1)p^q\Delta_p).$$

This definition is similar to that of the blocks  $\mathcal{B}_r(\mathbf{i}, k)$  used in [KSa]–[KSb], but there are obvious differences in the handling of the last coordinate in these definitions. We further define the *bottom* of the block  $\widehat{\mathcal{B}}_p(\mathbf{i}, k)$  as

$$(3.2) \quad \mathcal{Z}_p(\mathbf{i}, k) = \prod_{s=1}^d [(i(s) - 4d - 1)\Delta_p, (i(s) + 4d + 2)\Delta_p) \times \{kp^q\Delta_p\}.$$

The directed graph  $\mathcal{D}$  will be the graph with vertex set  $\mathbb{Z}^d \times \{-1, 0, 1, 2, \dots\}$ , and with a directed edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, \ell)$  if and only if  $\|\mathbf{i} - \mathbf{j}\| \leq 1$  and  $\ell = k + 1$ . (Recall that the first condition means  $|i(s) - j(s)| \leq 1$  for  $1 \leq s \leq d$ .) We also need the graph  $\mathcal{L}$ . It has vertex set  $\mathbb{Z}^{d+1}$  and an edge between  $v$  and  $w$  if and only if  $\|v - w\| = 1$ . Note that  $\mathcal{L}$  strictly contains  $\mathbb{Z}^{d+1}$  (viewed as a graph);  $\mathcal{L}$  also has “diagonal” edges. We shall call the edges of  $\mathcal{D}$  and  $\mathcal{L}$ ,  $\mathcal{D}$ -edges and  $\mathcal{L}$ -edges, respectively. We shall call  $(\mathbf{i}, k)$  a *parent* of  $(\mathbf{j}, k + 1)$  if there is a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$ .

For any set  $A$  in the vertex set of  $\mathcal{L}$  (i.e.,  $A \subset \mathbb{Z}^{d+1}$ ) we define the following pieces of its boundary:

$$\partial_{ext}A = \{v \in \mathbb{Z}^{d+1} : v \text{ is adjacent on } \mathcal{L} \text{ to some } w \in A, v \notin A, \text{ and there exists a path on } \mathbb{Z}^{d+1} \text{ from } v \text{ to } \infty \text{ which avoids } A\};$$

$$\partial_{ext}^+A := \{v \in \partial_{ext} : \text{there is some } w \in A \text{ such that the edge from } w \text{ to } v \text{ is a } \mathcal{D}\text{-edge}\};$$

$$\partial_{ext}^*A := \{v \in \partial_{ext} : v + e_{d+1} \in A\}.$$

Note that  $\partial_{ext}^+$  and  $\partial_{ext}^*$  are not disjoint in general. If  $A, S \subset \mathbb{Z}^{d+1}$ , then we say that  $S$  separates  $A$  from  $\infty$  on  $\mathbb{Z}^{d+1}$  if  $S \cap A = \emptyset$  and every path on  $\mathbb{Z}^{d+1}$  from  $A$  to  $\infty$  contains a point of  $S$ .

The next lemma is of a topological nature only.

LEMMA 5. *Let  $A \subset \mathbb{Z}^d \times \{0, 1, 2, \dots\}$  be a finite, non-empty,  $\mathcal{L}$ -connected set. Then*

$$(3.3) \quad \partial_{ext}A \text{ is } \mathbb{Z}^{d+1}\text{-connected and separates } A \text{ from } \infty \text{ on } \mathbb{Z}^{d+1}$$

and

$$(3.4) \quad |\partial_{ext}A| \leq 6|\partial_{ext}^+A|.$$

*Proof.* Relation (3.3) is just a special case of Lemma 2.23 in [Kb] (with  $d$  replaced by  $d + 1$ ). [Kb] does not state the fact that  $\partial_{ext}A$  separates  $A$  from  $\infty$  in the generality of the present lemma. However, the proof on the top of p. 144 of [Kb] shows easily that the separation property in (3.3) holds.

To prove (3.4), assume that  $v \in \partial_{ext}A$ . Then  $v$  is adjacent on  $\mathcal{L}$  to some  $w \in A$  and there exists some path  $\pi$  from  $v$  to  $\infty$  on  $\mathbb{Z}^{d+1}$  which is disjoint from  $A$ . We distinguish three main cases according to the value of  $v(d+1) - w(d+1)$ , where  $v(d+1)$  is the last coordinate of  $v$ ; the last two cases are split into two subcases.

*Case (a):*  $v(d+1) = w(d+1) + 1$ . In this case the edge from  $v$  to  $w$  is a  $\mathcal{D}$ -edge, so that  $v \in \partial_{ext}^+A$ . Thus the number of vertices  $v \in \partial_{ext}A$  which are in case (a) is at most  $|\partial_{ext}^+A|$ .

*Case (b):*  $v(d+1) = w(d+1)$ .

*Subcase (bi):*  $v + e_{d+1} \notin A$ . Here we abuse notation somewhat.  $e_j$  denotes the  $j$ -th coordinate vector and the  $s$ -th component of  $v + e_{d+1}$  equals  $v(s)$  if  $s \leq d$  and equals  $v(d+1) + 1$  if  $s = d + 1$ . In this subcase, the path on  $\mathbb{Z}^{d+1}$  consisting of the edge from  $v + e_{d+1}$  to  $v$  followed by  $\pi$  is a path on  $\mathbb{Z}^{d+1}$  from  $v + e_{d+1}$  to  $\infty$  which is disjoint from  $A$ . Moreover,  $(v + e_{d+1})(d+1) = w(d+1) + 1$  and the edge from  $w$  to  $v + e_{d+1}$  is a  $\mathcal{D}$ -edge. Thus  $v + e_{d+1} \in \partial_{ext}^+A$  and again, the number of vertices  $v \in \partial_{ext}A$  which are in case (bi) is at most  $|\partial_{ext}^+A|$ .

*Subcase (bii):*  $v + e_{d+1} \in A$ . Then the edge from  $v + e_{d+1}$  to  $v$  goes from a point of  $A$  to a point of  $\partial_{ext}A$ , but the last coordinate decreases by one along this edge. Thus,  $v \in \partial_{ext}^*A$  in this case. Thus the number of vertices  $v \in \partial_{ext}A$  which are in case (bii) is at most  $|\partial_{ext}^*A|$ . To complete the handling of this subcase we prove that in general

$$(3.5) \quad |\partial_{ext}^*A| \leq |\partial_{ext}^+A|$$

for any finite  $A \subset \mathbb{Z}^d \times \{0, 1, \dots\}$ . To see (3.5) consider any line parallel to the last coordinate axis of the form  $\{v_0 + ne_{d+1} : n \in \mathbb{Z}\}$ . The points of this line are in the unbounded component of  $\mathbb{Z}^{d+1} \setminus A$  for large  $n$  both in the positive and negative direction. Therefore, as one lets  $n$  run from  $-\infty$  to  $+\infty$ , there are as many transitions from the unbounded component in  $\mathbb{Z}^{d+1}$  of  $\mathbb{Z}^{d+1} \setminus A$  to  $A$  as there are transitions from  $A$  to the unbounded component of  $\mathbb{Z}^{d+1} \setminus A$ . The former transitions go from a vertex  $v$  outside  $A$  to a vertex in  $A$  by adding  $e_{d+1}$ , and therefore occur for  $v \in \partial_{ext}^*$ . The latter transitions are along a  $\mathcal{D}$ -edge from a vertex of  $A$  to a vertex  $v$  outside  $A$  and therefore occur when  $v + e_{d+1} \in \partial_{ext}^+$ . The numbers of the two types of transitions are equal, and this holds for any choice of  $v_0$ . (3.5) follows.

*Case (c):*  $v(d+1) = w(d+1) - 1$ . Again this has the subcases (ci) with  $v + e_{d+1} \notin A$  and (cii) with  $v + e_{d+1} \in A$ . In case (ci) one easily checks (by the argument for case (bi)) that  $\tilde{v} := v + e_{d+1} \in \partial_{ext}A$ , and that  $\tilde{v}$  is in case (b). Thus, by the results for case (b) the number of vertices  $v \in \partial_{ext}A$  which are in case (ci) is at most  $2|\partial_{ext}^+A|$ . Finally, if  $v$  is in subcase (cii), then replace  $w$  by  $\tilde{w} = v + e_{d+1}$ . In this situation,  $v$  is adjacent on  $\mathcal{L}$  to  $\tilde{w} \in A$  and therefore  $v$  lies in  $\partial_{ext}^*$ . (3.5) therefore shows that also the number of vertices  $v \in \partial_{ext}A$  which are in case (cii) is at most  $|\partial_{ext}^+A|$ . The inequality (3.4) follows by adding the contributions of the various cases.  $\square$

We can now set up our percolation problem on the graph  $\mathcal{D}$ . We define  $m(\mathbf{i}) = m_p(\mathbf{i}) \in (\mathbb{Z} + \frac{1}{2})^d \Delta_p$  as the point with components

$$m(\mathbf{i})(s) = (i(s) + 1/2)\Delta_p, \quad 1 \leq s \leq d.$$

$m(\mathbf{i})$  is in some sense the midpoint of  $\prod_{s=1}^d [(i(s)\Delta_p, (i(s) + 1)\Delta_p)$ , which constitutes the spatial part of  $\tilde{\mathcal{B}}_p(\mathbf{i}, k)$ .  $m(\mathbf{i})(s)$  is an integer because we took  $C_0$  even. For purposes of the proof of survival of the infection, it turns out to be convenient to change the initial conditions of the  $B$ -particles slightly. *For the rest of this section we will assume that we do not add a  $B$ -particle at the origin at time 0, but instead add a  $B$ -particle at  $m(\mathbf{0})$ .* Thus we take the state at time 0 to satisfy

$$(3.6) \quad N_A(x, 0) = N_A(x, 0-) \text{ if } x \neq m(\mathbf{0}),$$

$$(3.7) \quad N_A(x, 0) + N_B(x, 0) = N_A(x, 0-) + 1 \text{ if } x = m(\mathbf{0}),$$

$$(3.8) \quad N_B(x, 0) = 0 \text{ if } x \neq m(\mathbf{0}),$$

and

$$(3.9) \quad 1 \leq N_B(x, 0) \leq N_A(m(\mathbf{0}), 0-) + 1 \text{ if } x = m(\mathbf{0}).$$

Clearly (1.2) holds with the original initial condition if and only if it holds in this modified system. Thus it suffices for showing  $\lambda_c > 0$  that

$$(3.10) \quad P\{\text{there are } B\text{-particles at all times in the system} \\ \text{which starts with (3.6)–(3.9)}\} > 0.$$

It will be necessary in the proofs of Lemma 6 and 7 to consider initial conditions in which a  $B$ -particle is added at time 0 at a finite number of sites  $m(\mathbf{c}_1), \dots, m(\mathbf{c}_r)$ . In this situation  $m(\mathbf{0})$  in (3.6)–(3.9) has to be replaced by  $m(\mathbf{c}_1), \dots, m(\mathbf{c}_r)$ . Till the end of Lemma 7 we shall allow this, but will indicate the location of the initial particles in the notation only where it is crucial.

We further define

$$(3.11) \quad t(k) = t_p(k) = kp^q \Delta_p,$$

and

$$(3.12) \quad Z_p(\mathbf{i}) = \prod_{s=1}^d [(i(s) - 4d - 1)\Delta_p, (i(s) + 4d + 2)\Delta_p] \subset \mathbb{Z}^d,$$

so that  $Z_p(\mathbf{i}, k) = Z_p(\mathbf{i}) \times \{t(k)\}$ . We also define  $x(\mathbf{i}, k) \in \mathbb{Z}^d$  as the nearest (in the  $\ell^\infty$  sense on  $\mathbb{Z}^d$ ) site to  $m(\mathbf{i})$  which contains a  $B$ -particle at time  $t(k) = kp^q \Delta_p$  in our infection process  $\{Y_t(\lambda)\}$ . If there are several possible choices for  $x(\mathbf{i}, k)$ , then we use some deterministic rule to break the tie. If there are no  $B$ -particles in  $\{Y_t(\lambda)\}$  at time  $t_k$ , then we leave  $x(\mathbf{i}, k)$  undefined. If a  $B$ -particle is added at  $m(\mathbf{c})$  at time 0, then we take  $x(\mathbf{c}, 0) = m(\mathbf{c})$ . We call the vertex  $(\mathbf{i}, k)$  of  $\mathcal{D}$  *active* (or more explicitly  $\lambda$ -*active*) if there is a site  $x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  which is occupied by at least one  $B$ -particle at time  $t(k)$  in our infection process with recuperation  $\{Y_t(\lambda)\}$  (see (1.13) for  $\mathcal{C}$ ). By convention, if a  $B$ -particle is added at  $m(\mathbf{c})$  at time 0, the vertex  $(\mathbf{c}, 0)$  is active.

We now want to define when certain  $\mathcal{D}$ -edges are open. To this end we first define the  $Z_p(\mathbf{i}, k)$ -*process started at*  $(x, t(k))$  for any  $x \in Z_p(\mathbf{i})$ . This process is defined only from time  $t(k)$  on and it will use only particles which are in  $Z_p(\mathbf{i})$  at time  $t(k)$ . Also, we only define this process if  $x$  is occupied by some particle at time  $t(k)$ . To define this process we first reset the types of the particles in  $Z_p(\mathbf{i})$  at time  $t(k)$ . All particles in  $Z_p(\mathbf{i}) \setminus \{x\}$  are given type  $A$ . One particle at  $x$  is given type  $B$ . Denote this particle by  $\rho(x, t(k))$ . All other particles at  $x$  (if any) are given type  $A$ . If there are  $B$ -particles at  $(x, t(k))$ , then  $\rho(x, t(k))$  is chosen from these  $B$ -particles, but apart from



this restriction  $\rho(x, t(k))$  can be selected from the particles at  $(x, t(k))$  in any way which does not depend on the future paths of the particles in  $Z_p(\mathbf{i})$  at time  $t(k)$ . The  $Z_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$  is then the evolution of the particles which are in  $Z_p(\mathbf{i})$  at time  $\{t(k)\}$  with the reset types according to the rules for  $\{Y_t(0)\}$ , that is, there is *no recuperation*, but we still insist that an *A*-particle turns into a *B*-particle only if it jumps onto a *B*-particle or a *B*-particle jumps onto it. Note that in this process all particles outside  $Z_p(\mathbf{i})$  at time  $t(k)$  are ignored.

We now say that the  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  is *open* if the following three events (3.13)–(3.15) occur.

$$(3.13) \quad (\mathbf{i}, k) \text{ is active.}$$

$$(3.14) \quad A(\mathbf{i}, k, \mathbf{j}) := \{ \text{the } Z_p(\mathbf{i}, k)\text{-process started at } (x(\mathbf{i}, k), t(k)) \\ \text{has at least one } B\text{-particle in } m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p) \\ \text{at time } t(k + 1) \}$$

(see Figure 1). If  $A(\mathbf{i}, k, \mathbf{j})$  occurs, then there exists in  $\{Y_t(0)\}$  a genealogical path from some *B*-particle at  $(x(\mathbf{i}, k), t(k))$  to some particle in  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$ . Among all such paths choose the first one in some deterministic ordering of such paths. Let this be determined by the times  $s_i, 1 \leq i \leq \ell$ , and particles  $\rho_0, \dots, \rho_\ell$ , in the sense that  $t(k) < s_1 < \dots < s_\ell < t(k + 1)$ ,  $\rho_0$  is some *B*-particle at  $(x(\mathbf{i}, k), t(k))$  and  $\rho_\ell$  is located in  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  at time  $t(k + 1)$ ; moreover, at time  $s_i, 1 \leq i \leq \ell$ , one of  $\rho_i$  and  $\rho_{i-1}$  jumps to the position of the other. All the particles  $\rho_i, 0 \leq i \leq \ell$ , are in  $Z_p(\mathbf{i})$  at time  $t(k)$ . Note that by our definition of the  $Z_p(\mathbf{i}, k)$ -process, the path here is chosen without reference to the recuperation events. The last required event for the edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  to be open is

$$(3.15) \quad B(\mathbf{i}, k, \mathbf{j}, \lambda) := \{ \text{with } s_i \text{ and } \rho_i \text{ as in the preceding lines, the particle } \rho_i \\ \text{has no recuperation event in } \{Y_t(\lambda)\} \text{ during } [s_i, s_{i+1}], \\ \text{that is, } r(h, \rho_i) \notin [s_i, s_{i+1}] \text{ for all } h \text{ and } 0 \leq i \leq \ell \\ \text{(with } s_{\ell+1} = t(k + 1)) \}.$$

Note that this definition applies only if there exists a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$ , that is if  $\mathbf{j} = \mathbf{i}$  or  $\mathbf{j} = \mathbf{i} \pm e_s$  for some  $s \in \{1, \dots, d\}$ . If any of the three conditions (3.13)–(3.15) fails, then the  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  is called *closed*.

By definition of “active”, the infection survives if there are with positive probability infinitely many active sites. The next lemma is a tool for finding active sites.

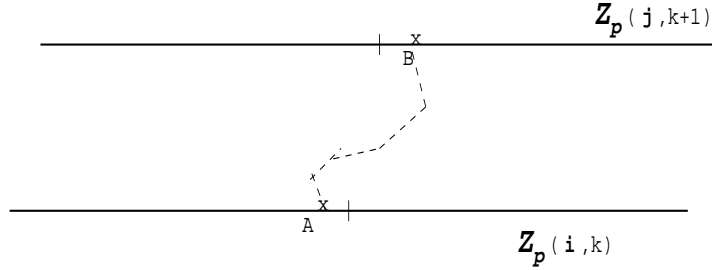


FIGURE 1. Relative location of the sets  $Z_p(i, k), Z_p(j, k + 1)$  for  $d = 1$ , where  $(i, k)$  is a parent of  $(j, k + 1)$ . The points marked by a small vertical bar are  $(m(i), t(k))$  and  $(m(j), t(k + 1))$ , respectively. The point  $A$ , marked by an  $x$  is  $(x(i, k), t(k))$ .  $B$ , also marked by an  $x$ , is the endpoint in  $(m(j) + \mathcal{C}(\frac{1}{8}\Delta_p), t(k + 1))$  of the dashed curve. This curve represents the genealogical path along which the infection is transmitted in the  $Z_p(i, k)$ -process started at  $A$ , so that  $A(i, k, j)$  occurs.

LEMMA 6. *Start the infection process by adding a  $B$ -particle at  $m(c_1), \dots, m(c_r)$  at time 0. If some vertex  $(i, k)$  is active and the  $\mathcal{D}$ -edge from  $(i, k)$  to  $(j, k + 1)$  is open, then  $(j, k + 1)$  is also active.*

*Proof.* By assumption (3.13),

$$x(i, k) \in m(i) + \mathcal{C}(\frac{1}{8}\Delta_p) \subset Z_p(i)$$

(this is even true for  $(i, k) = (c_j, 0)$ , since we interpret  $x(c_j, 0)$  as  $m(c_j)$ ). Now apply Lemma 4 with  $\sigma^{(2)}$  the true state of the process  $\{Y_t(\lambda)\}$  at time  $t(k)$  and  $\sigma^{(1)}$  the state obtained by resetting the type of the particles in  $Z_p(i, k)$  and ignoring the particles outside  $Z_p(i, k)$  to form the  $Z_p(i, k)$ -process started at  $(x(i, k), t(k))$ . This procedure only involves removing particles and changing the type of some particles from  $B$  to  $A$ . Indeed, according to our construction, the one particle at  $x(i, k)$  which is given type  $B$  at  $(x(i, k), t(k))$  is chosen from the particles which already had type  $B$  at time  $t(k)$  in  $\{Y_t(\lambda)\}$ . Therefore, (2.4) and (2.5) hold for these choices of  $\sigma^{(1)}$  and  $\sigma^{(2)}$ . If we now let the particles in the full system (i.e., the collection of all the particles, including the ones outside  $Z_p(i, k)$ ) develop till time  $t(k + 1)$  according to the rules for  $\{Y_t(0)\}$ , then Lemma 4 tells us that at time  $t(k + 1)$  the  $Z_p(i, k)$ -process started at  $(x(i, k), t(k))$  will still lie below the full process. Moreover, by assumption (3.14), the  $Z_p(i, k)$ -process started at  $(x(i, k), t(k))$  has at least one  $B$ -particle in  $m(j) + \mathcal{C}(\frac{1}{8}\Delta_p)$  at time  $t(k + 1)$ .

The last few lines tell us that there will be some  $B$ -particles in  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  at time  $t(k+1)$  in the full process, if it develops according to the rules for  $\{Y_t(0)\}$  (i.e., without recuperation) during  $[t(k), t(k+1)]$ . In fact we can choose times  $s_i$  and particles  $\rho_i$  as in the lines between (3.14) and (3.15). Then  $\rho_i$  has type  $B$  during  $[s_i, s_{i+1}]$ ,  $0 \leq i \leq \ell$ , and in particular  $\rho_\ell$  will have type  $B$  at  $t(k+1)$ , if we suppress recuperation during  $[t(k), t(k+1)]$ . Finally we note that by induction on  $i$ , the occurrence of  $B(\mathbf{i}, k, \mathbf{j}, \lambda)$  implies that  $\rho_i$  still has type  $B$  during  $[s_i, s_{i+1}]$  even if recuperation is allowed. Indeed, if  $\rho_i$  is of type  $B$  at time  $s_i$  in the process  $\{Y_t(\lambda)\}$ , then it will keep type  $B$  during  $[s_i, s_{i+1}]$ , even when recuperation is suppressed, since it has no recuperation during this interval anyway if  $B(\mathbf{i}, k, \mathbf{j}, \lambda)$  occurs. Then  $\rho_i$  and  $\rho_{i+1}$  coincide at time  $s_{i+1}$ , and therefore  $\rho_{i+1}$  will be of type  $B$  at  $s_{i+1}$  even in  $\{Y_t(\lambda)\}$ . In particular  $\rho_{\ell+1}$  has type  $B$  at time  $t(k+1)$ .

We conclude from the last two paragraphs that even in the process  $\{Y_t(\lambda)\}$  there is a  $B$  particle in  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  at time  $t(k+1)$ . In particular,  $x(\mathbf{j}, t(k+1))$ , the position of the nearest  $B$ -particle to  $m(\mathbf{j})$  at time  $t(k+1)$  in  $\{Y_t(\lambda)\}$ , must lie in  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$ . □

Again assume that we add a  $B$ -particle at  $m(\mathbf{c}_1), \dots, m(\mathbf{c}_r)$  at time 0. Now define the open cluster of the set  $\{(\mathbf{c}_1, 0), \dots, (\mathbf{c}_r, 0)\}$  on  $\mathcal{D}$  as

$$(3.16) \quad \mathfrak{C} = \mathfrak{C}(\mathbf{c}_1, \dots, \mathbf{c}_r) = \{v \in \mathcal{D} : \exists \text{ path } v_0, v_1, \dots, v_n = v \text{ from some } v_0 = (\mathbf{c}_j, 0) \\ \text{to } v \text{ on } \mathcal{D} \text{ for some } n \text{ such that each edge } \{v_i, v_{i+1}\}, \\ 0 \leq i \leq n-1, \text{ is open}\}.$$

We always include each  $(\mathbf{c}_j, 0)$  in  $\mathfrak{C}(\mathbf{c}_1, \dots, \mathbf{c}_r)$ . From Lemma 6 it follows that each vertex  $(\mathbf{i}, k)$  in  $\mathfrak{C}$  must be active. Thus, if  $\mathfrak{C}$  is infinite with positive probability, then the infection survives. If  $\mathfrak{C}$  is finite (and nonempty, since it contains  $(\mathbf{c}_j, 0)$ ), then  $\partial_{ext}\mathfrak{C}$  is  $\mathbb{Z}^{d+1}$ -connected and separates  $\mathfrak{C}$  from  $\infty$  on  $\mathbb{Z}^{d+1}$ , by Lemma 5. Moreover, each  $\mathcal{D}$ -edge from some vertex  $(\mathbf{i}, k) \in \mathfrak{C}$  to some vertex  $(\mathbf{j}, k+1) \in \partial_{ext}^+\mathfrak{C}$  must be closed. This is true by definition, because if  $(\mathbf{i}, k) \in \mathfrak{C}$  and the edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k+1)$  were open, then also  $(\mathbf{j}, k+1)$  would belong to  $\mathfrak{C}$ . These observations indicate that it will be useful to have an upper bound for the probability that the edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k+1)$  is closed. To derive such a bound we generalize the definitions of the events  $A$  and  $B$  in (3.14) and (3.15). For  $x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  we define

$$(3.17) \quad \tilde{A}(x, t(k), \mathbf{j}) := \{x \text{ is not occupied at time } t(k)\} \cup \{x \text{ is occupied at } t(k) \\ \text{and the } \mathcal{Z}_p(\mathbf{i}, k)\text{-process started at } (x, t(k)) \text{ has at least} \\ \text{one } B\text{-particle in } m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p) \text{ at time } t(k+1)\}.$$

If  $\tilde{A}(x, t(k), \mathbf{j})$  occurs and  $x$  is occupied at time  $t(k)$ , then we define  $\tilde{B}(x, t(k), \mathbf{j}, \lambda)$  as follows. First we reset the types of the particles in  $\mathcal{Z}_p(\mathbf{i}, k)$  to form the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ . We then pick a genealogical path in this process from the unique  $B$ -particle (after resetting) at  $(x, t(k))$  to a vertex in  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  at time  $t(k + 1)$ . As in the lines between (3.14) and (3.15) let this path be determined by times  $s_1, \dots, s_\ell$  and particles  $\rho_0, \dots, \rho_\ell$ , which have to be in  $Z_p(\mathbf{i})$  at time  $t(k)$ . We then also define (with  $s_{\ell+1} = t(k + 1)$ )

(3.18)

$$\tilde{B}(x, t(k), \mathbf{j}, \lambda) := \{x \text{ is not occupied at time } t(k)\} \cup \{(x, t(k)) \text{ is occupied, and for } 0 \leq i \leq \ell \text{ the particle } \rho_i \text{ has no recuperation event in } \{Y_t(\lambda)\} \text{ during } [s_i, s_{i+1}]\}.$$

We note that many choices have to be specified before these definitions become unambiguous. Ties may have to be broken in the choice of  $x(\mathbf{i}, k)$ , a particle has to be singled out at  $(x(\mathbf{i}, k), t(k))$  to have type  $B$  in the  $\mathcal{Z}_p(\mathbf{i}, k)$  process started at  $(x(\mathbf{i}, k), t(k))$ , and a genealogical path has to be chosen in the definitions of  $B(\mathbf{i}, k, \mathbf{j}, \lambda)$  and  $\tilde{B}(x, t(k), \mathbf{j}, \lambda)$ . We shall not write out explicit rules for making these choices, but merely repeat the general principle, that we make such choices by ordering all possibilities for a particular choice before any process  $\{Y_t(\lambda)\}$  is started and then pick the first possibility in such an ordering at the time when the choice has to be made. Making the list of possibilities does not involve probability. We have to make sure that in ordering the possibilities for  $x(\mathbf{i}, k)$  and a  $B$ -particle at  $(x(\mathbf{i}, k), t(k))$  we do not use any information about our process and its recuperation events in the future (i.e., after time  $t(k)$ ), and that in the choice of a genealogical path for  $B(\mathbf{i}, k, \mathbf{j}, \lambda)$  and  $\tilde{B}(x, t(k), \mathbf{j}, \lambda)$  we use no information on the recuperation events during  $(t(k), \infty)$ .

Once  $x(\mathbf{i}, k)$  has been determined we make our choices for  $\tilde{A}(x, t(k), \mathbf{j})$  and  $\tilde{B}(x, t(k), \mathbf{j}, \lambda)$  at  $x = x(\mathbf{i}, k)$  in such a way that, on the event  $\{(i, k) \text{ is active}\}$ ,  $A(\mathbf{i}, k, \mathbf{j}) = \tilde{A}(x(\mathbf{i}, k), t(k), \mathbf{j})$  and  $B(\mathbf{i}, k, \mathbf{j}, \lambda) = \tilde{B}(x(\mathbf{i}, k), t(k), \mathbf{j}, \lambda)$ . Finally, if there exists a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  and if  $(\mathbf{i}, k)$  is active, we set

$$(3.19) \quad C(\mathbf{i}, k, \mathbf{j}, \lambda) := \bigcup_{x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)} \left[ \tilde{A}(x, t(k), \mathbf{j}) \cap \tilde{B}(x, t(k), \mathbf{j}, \lambda) \right]^c.$$

If there is no  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$ , or  $(\mathbf{i}, k)$  is not active, then we set

$$C(\mathbf{i}, k, \mathbf{j}, \lambda) := \emptyset.$$

Finally, let  $\mathfrak{C}_0$  be some finite subset of vertices of  $\mathcal{D}$ . The vertices of  $\mathcal{D}$  can also be thought of as vertices of  $\mathbb{Z}^{d+1}$ , so that we can also think of  $\mathfrak{C}_0$  as a subset of  $\mathbb{Z}^{d+1}$ . We shall continue to denote generic vertices of  $\mathbb{Z}^{d+1}$

as  $(\mathbf{i}, k)$  with  $\mathbf{i} \in \mathbb{Z}^d$  and  $k \in \mathbb{Z}$ , because the last coordinate always plays the special role of time. We call a subset  $S$  of  $\mathbb{Z}^{d+1}$  a  $\mathfrak{C}_0$ -barrier if  $S$  is  $\mathbb{Z}^{d+1}$ -connected, separates  $\mathfrak{C}_0$  from  $\infty$  on  $\mathbb{Z}^{d+1}$  and satisfies the following condition:

$$(3.20) \quad S \text{ contains at least } |S|/6 \text{ vertices } (\mathbf{j}, k + 1) \text{ (with } k \in \mathbb{Z} \text{ arbitrary) which have a parent } (\mathbf{i}, k) \text{ such that the edge from } (\mathbf{i}, k) \text{ to } (\mathbf{j}, k + 1) \text{ is a } \mathcal{D}\text{-edge for which } C(\mathbf{i}, k, \mathbf{j}, \lambda) \text{ occurs.}$$

For  $\mathfrak{C}_0$  as above we define the quantity

$$(3.21) \quad \Upsilon(p, \lambda, \mathfrak{C}_0) = \sum_{n \geq 1} P\{\text{there exists a } \mathfrak{C}_0\text{-barrier } S \text{ with } |S| = n\},$$

where the probability is calculated in the  $\{Y_t(\lambda)\}$ -process which starts with a  $B$ -particle added to each  $m(\mathbf{c}_i, 0)$  in  $\mathfrak{C}_0$ . (Recall that  $p$  is the parameter which appears in the size of the blocks  $\widehat{\mathcal{B}}$ .)

LEMMA 7. *Assume that there is a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  and that  $(\mathbf{i}, k)$  is active. Then*

$$(3.22) \quad \{\text{the edge from } (\mathbf{i}, k) \text{ to } (\mathbf{j}, k + 1) \text{ is closed}\} \subset C(\mathbf{i}, k, \mathbf{j}, \lambda).$$

*If there exist  $p_0 \in \{1, 2, \dots\}$ ,  $\lambda_0 \in (0, \infty)$  and a  $\mathbb{Z}^d$ -connected set  $\{\mathbf{c}_1, \dots, \mathbf{c}_r\} \subset \mathbb{Z}^d$  such that, with  $\mathfrak{C}_0 = \{(\mathbf{c}_1, 0), \dots, (\mathbf{c}_r, 0)\}$ , it holds*

$$(3.23) \quad \Upsilon(p_0, \lambda_0, \mathfrak{C}_0) < 1,$$

*then*

$$(3.24) \quad P\{\text{infection survives in the process } \{Y_t(\lambda_0)\} \text{ which starts with a } B\text{-particle added at } m(\mathbf{0})\} > 0,$$

*and consequently  $\lambda_c \geq \lambda_0$ .*

*Proof.* Let there be a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  and let  $(\mathbf{i}, k)$  be active. By definition of an open edge, the edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$  can be closed only if  $A(\mathbf{i}, k, \mathbf{j}) \cap B(\mathbf{i}, k, \mathbf{j}, \lambda) = \widetilde{A}(x(\mathbf{i}, k), t(k), \mathbf{j}) \cap \widetilde{B}(x(\mathbf{i}, k), t(k), \mathbf{j}, \lambda)$  fails. In addition  $x(\mathbf{i}, k) \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  if (3.13) holds. Thus the inclusion (3.22) holds.

To prove (3.24), assume that (3.23) holds and fix  $p_0, \lambda_0$  and  $\mathbf{c}_1, \dots, \mathbf{c}_r$  such that  $\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$  is a finite  $\mathbb{Z}^d$ -connected set and such that (3.23) holds. Now consider the process  $\{Y_t(\lambda_0)\}$  which has recuperation rate  $\lambda_0$ , but start it by adding at time 0 a  $B$ -particle to each site  $m(\mathbf{c}_j)$ ,  $1 \leq j \leq r$ . Then all  $(\mathbf{c}_j, 0)$  are active. Let  $\mathfrak{C}$  be given by (3.16) and view it as a subset of  $\mathbb{Z}^{d+1}$ . This cluster is the open cluster of  $\mathfrak{C}_0$ . It contains  $\mathfrak{C}_0$  and is  $\mathcal{L}$ -connected (see the lines following (3.2) for  $\mathcal{L}$ ). Moreover, by Lemma 6, each vertex in  $\mathfrak{C}$  must be  $\lambda_0$ -active. Write  $S$  for  $\partial_{ext}\mathfrak{C}$ . If  $\mathfrak{C}$  is finite, then  $S$  is  $\mathbb{Z}^{d+1}$ -connected and

separates  $\mathfrak{C}_0$  from infinity, by Lemma 5. Moreover, as observed after (3.16), each  $\mathcal{D}$ -edge from some vertex  $(\mathbf{i}, k) \in \mathfrak{C}$  to some vertex  $(\mathbf{j}, k + 1) \in \partial_{ext}^+ \mathfrak{C}$  must be closed. By virtue of (3.4),  $S$  then must contain at least  $|S|/6$  vertices  $(\mathbf{j}, k + 1)$  which have a parent  $(\mathbf{i}, k) \in \mathfrak{C}$  with a closed  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$ , and therefore such that  $C(\mathbf{i}, k, \mathbf{j}, \lambda_0)$  occurs (by (3.22) and the fact that  $(\mathbf{i}, k)$  is active). Thus  $S$  must have property (3.20). This implies that

$$\begin{aligned} &P\{\mathfrak{C} \text{ is finite in the process } \{Y_t(\lambda_0)\} \text{ which starts with} \\ &\quad \text{a } B\text{-particle added to } m(\mathbf{c}_j) \text{ for each } 1 \leq j \leq r\} \\ &\leq \Upsilon(p_0, \lambda_0, \mathfrak{C}_0) < 1, \end{aligned}$$

or equivalently,

$$\begin{aligned} (3.25) \quad &P\{\text{the infection survives in } \{Y_t(\lambda_0)\} \text{ if one adds at time 0 a} \\ &\quad B\text{-particle at each } m(\mathbf{c}_j), 1 \leq j \leq r\} \\ &\geq P\{\mathfrak{C} \text{ is infinite in } \{Y_t(\lambda_0)\} \text{ if one adds at time 0 a} \\ &\quad B\text{-particle at each } m(\mathbf{c}_j), 1 \leq j \leq r\} > 0. \end{aligned}$$

It remains to show that the probability of survival of the infection remains strictly positive if we add a  $B$ -particle at time 0 only at  $m(\mathbf{0})$ . In fact, this statement is still ambiguous, because, so far, we have only mentioned the locations where we add a  $B$ -particle at time 0, but we haven't specified how many particles we turn into  $B$ -particles at these locations at time 0. To discuss this we remind the reader that  $\bar{\sigma}$  was defined before Lemma 2 as the state obtained from  $\sigma$  by changing all  $A$ -particles which coincide with a  $B$ -particle in the state  $\sigma$  to  $B$ -particles. Now if we start in a random state  $\sigma$  obtained by choosing  $N_A(x, 0-)$   $A$ -particles at  $x$  for i.i.d. Poisson variables  $N_A(x, 0-)$ , and adding finitely many  $B$ -particles to the system, then  $\bar{\sigma}$  lies a.s. in  $\Sigma_0$  (by Proposition 4 in [KSc]). Lemma 4 therefore shows that the more particles we turn into  $B$ -particles at time 0, the more likely survival is. Therefore, the strongest conclusion to prove is that

$$(3.26) \quad P\{\text{the infection survives if one adds at time 0 a } B\text{-particle at } m(\mathbf{0}) \text{ only, and turns no } A\text{-particles to type } B\} > 0.$$

And the weakest statement to start from is

$$(3.27) \quad P\{\text{the infection survives if one adds at time 0 a } B\text{-particle at each } m(\mathbf{c}_j), 1 \leq j \leq r, \text{ and turns all particles at these sites to } B\text{-particles}\} > 0.$$

We shall prove (3.26) from (3.27). Our argument for this is inspired by the proof at the bottom of p. 79 in [D]. For simplicity we take the  $\mathbf{c}_j, 1 \leq j \leq r$ , distinct. Only minor modifications are needed if some pairs of the  $\mathbf{c}_j$  can be equal. Assume that we add a  $B$ -particle  $\zeta_j$  at  $m(\mathbf{c}_j)$  for  $1 \leq j \leq r$  and

that we turn all  $A$ -particles at these sites to  $B$ -particles at time 0. Consider a sample point in the process  $\{Y_t(\lambda_0)\}$  with these initial conditions in which the infection survives. If  $v = (x, t)$  is a space-time point occupied by a  $B$ -particle in this process, then it has a genealogical path starting at some  $B$ -particle at one of the  $(m(\mathbf{c}_j), 0)$ . Thus, there exist times  $0 < s_1 < \dots < s_\ell < t$  and particles  $\rho_0, \rho_1, \dots, \rho_\ell$  such that  $\rho_0$  is a  $B$ -particle at some  $(\mathbf{c}_j, 0)$ ,  $\rho_\ell$  is at  $v$  at time  $t$ ,  $\rho_i$  has type  $B$  during  $[s_i, s_{i+1}]$  and has no recuperation during this interval. Consider instead the process starting with a particle  $\zeta_j$  added at each  $(m(\mathbf{c}_j), 0)$ , but now with only the particle  $\rho_0$  of type  $B$  and all other particles of type  $A$ . As before, induction on  $i$  shows that  $\rho_i$  is of type  $B$  during  $[s_i, s_{i+1}]$ . In particular, there will be a  $B$ -particle at  $(x, t)$  in this process with modified starting types. It follows from this same argument that for any choice of the  $a_k \geq 0, b_k \geq 0, 1 \leq k \leq r$ , and with  $\delta(k, j) = 1$  or 0 according as  $k = j$  or  $k \neq j$ ,

$$\begin{aligned}
 (3.28) \quad & P\{\text{infection survives} \mid N_A(m(\mathbf{c}_k), 0) = a_k, \\
 & \qquad \qquad \qquad N_B(m(\mathbf{c}_k), 0) = b_k, 1 \leq k \leq r\} \\
 & \leq \sum_{j=1}^r b_j P\{\text{infection survives} \mid N_A(m(\mathbf{c}_k), 0) = a_k + b_k - \delta(k, j), \\
 & \qquad \qquad \qquad N_B(m(\mathbf{c}_j), 0) = \delta(k, j), 1 \leq k \leq r\},
 \end{aligned}$$

If (3.27) holds, then the left hand side of (3.28) must be strictly positive for some choice of  $a_k \geq 0, b_k \geq 1$ . Consequently for some  $j$  and choice of the  $a_k, b_k$ ,

$$\begin{aligned}
 & P\{\text{infection survives if one adds a } B\text{-particle at } (m(\mathbf{c}_j), 0) \text{ only,} \\
 & \qquad \qquad \qquad \text{but changes no } A\text{-particles to } B\text{-particles}\} \\
 & \geq P\{\text{infection survives} \mid N_A(m(\mathbf{c}_k), 0) = a_k + b_k - \delta(k, j), \\
 & \qquad \qquad \qquad N_B(m(\mathbf{c}_k), 0) = \delta(k, j)\} \\
 & \quad \times P\{N_A(m(\mathbf{c}_k), 0) = a_k + b_k - \delta(k, j), 1 \leq k \leq r\} \\
 & > 0.
 \end{aligned}$$

By translation invariance we may assume  $c_j = \mathbf{0}$ , so that (3.26) follows. □

To conclude the proof of  $\lambda_c > 0$  we shall now establish that (3.23) holds for suitable  $p_0, \lambda_0, \mathbf{c}_0$ . Its proof has much in common with that of Proposition 3 in [KSc]. First some more notation and definitions. For purposes of comparison it is useful to couple our system with the system in which there are no  $B$ -particles and in which all original  $A$ -particles move forever without interaction. In this system, which we shall denote by  $\mathcal{P}^*$ , an  $A$ -particle  $\rho$  which starts at  $z$  will have position  $z + \pi_A(t, \rho)$  for all  $t$ . We write  $N^*(x, s)$  for the number

of particles at the space-time point  $(x, s)$  in the system  $\mathcal{P}^*$ .  $N^*(x, 0)$  is taken equal to  $N_A(x, 0-)$ , the initial number of  $A$ -particles at  $x$ . No initial  $B$ -particles are introduced in  $\mathcal{P}^*$  and all particles are of type  $A$  at all times in  $\mathcal{P}^*$ . Note that  $N^*(x, s)$  is independent of the recuperation rate  $\lambda$ , because it depends only on the paths, and not the types, of the particles. It is easy to see that

$$(3.29) \quad N^*(x, s) = N_A(x, s) + N_B(x, s) - [\text{number of } B\text{-particles introduced (in } \{Y_t(\lambda)\} \text{) at time 0 which are at } x \text{ at time } s].$$

In particular

$$(3.30) \quad N^*(x, s) \leq N_A(x, s) + N_B(x, s).$$

Next we define, for  $x = (x(1), \dots, x(d)) \in \mathbb{Z}^d$  and  $v \in [0, \infty)$ ,

$$(3.31) \quad \mathcal{Q}_p(x) = \prod_{s=1}^d [x(s), x(s) + C_0^p],$$

and

$$(3.32) \quad U_p(x, v) = \sum_{y \in \mathcal{Q}_p(x)} N^*(y, v) = \sum_{\substack{y: x(s) \leq y < x(s) + C_0^p \\ 1 \leq s \leq d}} N^*(y, v).$$

We call the bottom  $Z_p(\mathbf{i}, k) = Z_p(\mathbf{i}) \times \{t_p(k)\}$  *good* if

$$(3.33) \quad U_p(x, t_p(k)) \geq \gamma_0 \mu_A C_0^{dp} \text{ for all } x \text{ for which } \mathcal{Q}_p(x) \subset Z_p(\mathbf{i}),$$

where  $\gamma_0$  is the constant introduced in (4.10), (4.16) and (4.17) of [KSc]. We also need the following technical estimate of some random walk probabilities.

LEMMA 8. *There exists a  $p_1 = p_1(d, D) < \infty$  such that for all  $p \geq p_1$ ,  $\Delta_p \leq u \leq p^q \Delta_p$ ,  $x \in \prod_{s=1}^d [(i(s) - 4d)\Delta_p, (i(s) + 4d + 1)\Delta_p)$ , the event  $\{Z_p(\mathbf{i}, k) \text{ is good}\}$  implies*

$$(3.34) \quad \sum_{y \in Z_p(\mathbf{i})} [N_A(y, t(k)) + N_B(y, t(k))] P\{y + S_u = x\} \geq \frac{3}{4} \gamma_0 \mu_A.$$

*Proof.* This is nearly a copy of the proof of Lemma 5 in [KSc]. We introduce the blocks

$$\mathcal{M}(\ell) := \prod_{s=1}^d [\ell(s)C_0^p, (\ell(s) + 1)C_0^p].$$

In our previous notation  $\mathcal{M}(\ell) = \mathcal{Q}_p(z)$  with  $z(s) = \ell(s)C_0^p$ . These blocks have edge length only  $C_0^p$ , and the set  $Z_p(\mathbf{i})$  is a disjoint union of  $(8d + 3)^d C_0^{5dp}$  of these smaller blocks. Let

$$\Lambda = \Lambda(\mathbf{i}, p) = \{\ell \in \mathbb{Z}^d : \mathcal{M}(\ell) \subset Z_p(\mathbf{i})\}.$$



Also, for each  $\ell \in \Lambda$ , let  $y_\ell \in \mathcal{M}(\ell)$  be such that

$$P\{y_\ell + S_u = x\} = \min_{y \in \mathcal{M}(\ell)} P\{y + S_u = x\}.$$

Then the left hand side of (3.34) equals

$$(3.35) \quad \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} [N_A(y, t(k)) + N_B(y, t(k))] P\{y + S_u = x\} \\ \geq \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} N^*(y, t(k)) P\{y_\ell + S_u = x\}.$$

Since  $\mathcal{Z}_p(\mathbf{i}, k)$  is assumed to be good, we have

$$\sum_{y \in \mathcal{M}(\ell)} N^*(y, t(k)) = U_p(\mathcal{L}C_0^p, t(k)) \geq \gamma_0 \mu_A C_0^{dp} = \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A.$$

We can therefore continue (3.35) to obtain that the left hand side of (3.34) is at least

$$(3.36) \quad \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A P\{y_\ell + S_u = x\} \\ \geq \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A P\{y + S_u = x\} \\ - \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A \left| P\{y_\ell + S_u = x\} - P\{y + S_u = x\} \right|.$$

Now, since  $x \in \prod_{s=1}^d [(i(s) - 4d)\Delta_p, (i(s) + 4d + 1)\Delta_p)$ , the first multiple sum in the right hand side of (3.36) is at least

$$(3.37) \quad \sum_{w \in [-\Delta_p, \Delta_p]^d} \gamma_0 \mu_A P\{S_u = w\} \\ = \gamma_0 \mu_A [1 - P\{S_u \notin [-\Delta_p, \Delta_p]^d\}] \\ \geq \gamma_0 \mu_A [1 - K_2 \exp[-K_3^{-1} p^{-q} \Delta_p]]$$

for some constants  $K_2(d, D), K_3(d, D)$ . In the last inequality we used simple large deviation estimates for  $S_u$  (see for instance (2.40) in [KSa]) and the fact that  $u \leq p^q \Delta_p$ .

The second multiple sum in the right hand side of (3.36) is at most

$$(3.38) \quad \gamma_0 \mu_A \sum_{v \in \mathbb{Z}^d} \sup_{w: \|w-v\| \leq dC_0^p} |P\{S_u = v\} - P\{S_u = w\}|.$$

This sum has already been estimated in the proofs of Lemmas 6 and 12 of [KSa] (see in particular (5.26) there). It is at most  $K_4 \gamma_0 \mu_A C_0^p [\log u]^d u^{-1/2}$  for some constant  $K_4(d, D)$ .

For some  $p_1(d, D)$  and all  $p \geq p_1$  we finally have from (3.37) and (3.38) that the left hand side of (3.34) is at least

$$\gamma_0 \mu_A \left[ 1 - K_2 \exp[-K_3^{-1} p^{-q} \Delta_p] - K_4 C_0^p [\log \Delta_p]^d \Delta_p^{-1/2} \right] \geq \frac{3}{4} \gamma_0 \mu_A. \quad \square$$

We define the  $\sigma$ -fields

$\mathcal{H}(\mathbf{i}, k) = \mathcal{H}_p(\mathbf{i}, k) = \sigma$ -field generated by the positions and types of all particles at time 0, by all paths  $\pi(\cdot, \rho)$  during  $[0, t_p(k)]$  and by the paths for all times of all particles outside  $Z_p(\mathbf{i})$  at time  $t_p(k)$ , and by all recuperation times  $r(i, \rho)$  during  $[0, t_p(k)]$ .

We note that all  $N_A(x, 0), N_B(x, 0)$  and the types of all particles at time  $t(k)$  are  $\mathcal{H}(\mathbf{i}, k)$ -measurable. Also the event that a given  $x$  is occupied at time  $t(k)$  belongs to  $\mathcal{H}(\mathbf{i}, k)$ . The next lemma contains the crucial estimate for establishing (3.23). It proves that in the reset processes  $\mathcal{Z}_p(\mathbf{i}, k)$  with sufficiently large  $p$ , the infection will with high probability be transmitted, in a certain sense, along a given  $\mathcal{D}$ -edge. Recall that recuperation is ignored in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process, so this is very similar to showing that the infection spreads with a certain minimal speed if recuperation is not possible, as done in [KSc]. For the infection to reach a certain cube  $\mathcal{C}$  (of size  $\Delta_p/8$ ) we define (as in [KSc]) a random path along which a “distinguished”  $B$ -particle has a “drift towards  $\mathcal{C}$ .” We use a corresponding martingale to show that with high probability the distinguished  $B$ -particle has to follow the drift and will reach  $\mathcal{C}$ .

LEMMA 9. Assume that there is a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1)$ . There exists a constant  $p_2$  (independent of  $\mathbf{i}, k, \mathbf{j}, \lambda$ ) such that on the event

$$(3.39) \quad \{\mathcal{Z}_p(\mathbf{i}, k) \text{ is good}\},$$

it holds

$$(3.40) \quad \sum_{x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8} \Delta_p)} P\{\tilde{A}(x, t(k), \mathbf{j}) \text{ fails} \mid \mathcal{H}(\mathbf{i}, k)\} \leq \Delta_p^{-1} \text{ for } p \geq p_2.$$

*Proof.* Note that  $\{\mathcal{Z}_p(\mathbf{i}, k) \text{ is good}\} \in \mathcal{H}(\mathbf{i}, k)$ , because this event is defined in terms of the initial conditions, and paths during  $[0, t(k)]$  only. We divide the proof into 6 steps. A number of constants  $p_i$  and  $K_i$  will appear in this proof. These all depend only on  $d, D, C_0$  and  $\gamma_0 \mu_A$ . We shall not make any further mention of this.

Step (i). We begin with a lower bound for the number of particles in certain intervals in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$  for some  $x \in$

$m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)$ . To this end we define

$$\begin{aligned} \tilde{K}_p(z, v) &= \tilde{K}_p(z, v; x, t(k)) \\ &= [\text{total number of particles in } \prod_{s=1}^d [z(s), z(s) + K_5p] \\ &\quad \text{at time } v \text{ in the } \mathcal{Z}_p(\mathbf{i}, k)\text{-process started at } (x, t(k))], \end{aligned}$$

for a constant  $K_5$  to be chosen soon. We are interested in space time points  $(z, v)$  satisfying

$$(3.41) \quad z \in \prod_{s=1}^d [(i(s) - 4d)\Delta_p, (i(s) + 4d + 1)\Delta_p - K_5p] \text{ and} \\ v \in [t_p(k) + \Delta_p, t_p(k + 1)], \quad v \in \mathbb{Z}.$$

In this step we shall prove that we can choose  $K_5$  and  $p_3$  such that on the event (3.39) and for  $p \geq p_3$

$$(3.42) \quad P\{\tilde{K}_p(z, v) < \frac{1}{2}\gamma_0\mu_A [K_5p]^d \text{ for some } (z, v) \\ \text{satisfying (3.41) } | \mathcal{H}(\mathbf{i}, k)\} \leq \Delta_p^{-d-1}.$$

Note that we are only interested in numbers of particles in (3.42), irrespective of their types. To see (3.42), fix some  $(z, v)$  in the set (3.41). Now note that if the  $N_A(y, t_p(k)) + N_B(y, t_p(k))$  for  $y \in Z_p(\mathbf{i})$  are given, then the conditional distribution of  $\tilde{K}_p(z, v)$  for any fixed  $(z, v)$ , given  $\mathcal{H}(\mathbf{i}, k)$ , equals the distribution of  $\sum_{y,n} X(y, n)$ , where the  $X(y, n)$  are independent binomial variables with

$$\begin{aligned} P\{X(y, n) = 1\} &= 1 - P\{X(y, n) = 0\} \\ &= P\{y + S_{v-t(k)} \in \prod_{s=1}^d [z(s), z(s) + K_5p]\} \\ &= \sum_{w \in \prod_{s=1}^d [z(s), z(s) + K_5p]} P\{S_{v-t(k)} = w - y\}, \end{aligned}$$

and  $\sum_{y,n}$  runs over  $y \in Z_p(\mathbf{i})$  and, for given  $y$ , over  $1 \leq n \leq N_A(y, t(k)) + N_B(y, t(k))$ . By Lemma 8, on (3.39),

$$\sum_{y,n} P\{X(y, n) = 1\} \geq \frac{3}{4}\gamma_0\mu_A \left| \prod_{s=1}^d [z(s), z(s) + K_5p] \right| \geq \frac{3}{4}\gamma_0\mu_A [K_5p]^d,$$

provided  $p \geq p_1$ . Standard large deviation arguments now give that

$$(3.43) \quad P\{\tilde{K}_p(z, v) < \frac{1}{2}\gamma_0\mu_A [K_5p]^d | \mathcal{H}(\mathbf{i}, k)\}$$

is at most

$$\exp \left[ -\frac{1}{8}\theta_0\gamma_0\mu_A [K_5 p]^d \right],$$

for any  $\theta_0 > 0$  which satisfies  $(3/4)(1 - \exp[-\theta_0]) \geq (5/8)\theta_0$  (compare (4.28) and the lines preceding it in [KSc]). Thus we can choose  $K_5 = K_5(d, D, C_0, \gamma_0\mu_A)$  and  $p_3 \geq p_1$  such that (3.43) is at most  $(8d + 1)^{-d} p^{-q} \Delta_p^{-2d-3}$  for  $p \geq p_3$ . Since (3.41) allows no more than  $(8d + 1)^d p^q \Delta_p^{d+1}$  possible choices for  $(z, v)$ , (3.42) then follows.

*Step (ii).* In this step we largely imitate Lemma 9 of [KSc]. We define a path  $\vartheta(\cdot, \mathbf{j})$  and use it to construct a martingale which shows that  $\vartheta(\cdot, \mathbf{j})$  has a drift towards  $m(\mathbf{j})$ .  $\vartheta(s, \mathbf{j})$  will be chosen for  $t(k) \leq s \leq t(k + 1)$  according to the rules (i)–(v) below. In general, these rules do not determine  $\vartheta$  uniquely. Throughout this proof  $x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  is a fixed vertex, occupied at time  $t(k)$ , and we are working in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ . In particular, we do not allow recuperation and only consider particles which are in  $\mathcal{Z}_p(\mathbf{i})$  at time  $t(k)$ , and the types of the particles refer to the types they have in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ . Here are rules (i)–(v):

- (i)  $\vartheta(t(k), \mathbf{j}) = x$ ;
- (ii) for all  $s \in [t(k), t(k + 1)]$  there is a distinguished  $B$ -particle,  $\widehat{\rho}(s)$  say, at  $\vartheta(s, \mathbf{j})$ ;  $\widehat{\rho}(s)$  is a particle in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ ; at time  $t(k)$ ,  $\widehat{\rho}(t(k))$  is the unique  $B$ -particle at  $x$  in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ ;
- (iii)  $s \mapsto \vartheta(s, \mathbf{j})$  can jump only at times when  $\widehat{\rho}(s-)$  jumps away from  $\vartheta(s, \mathbf{j})$  and  $\vartheta(\cdot, \mathbf{j})$  is constant between such jumps;
- (iv) if  $\widehat{\rho}(s-)$  jumps from  $\vartheta(s-, \mathbf{j}) = w$  to  $w'$  at some time  $s$ , and if this was the only  $B$ -particle at  $w$  at time  $s-$ , then  $\vartheta(\cdot, \mathbf{j})$  also jumps to  $w'$  at time  $s$  (so that  $\vartheta(s, \mathbf{j}) = w'$ ), and  $\widehat{\rho}(s) = \widehat{\rho}(s-)$ , the particle which jumped at time  $s$ ;
- (v) if  $\widehat{\rho}(s-)$  jumps from  $\vartheta(s-, \mathbf{j}) = w$  to  $w'$  at some time  $s$  such that there is at least one other  $B$ -particle  $\rho'$  at  $w$  at time  $s$ , then  $\vartheta(\cdot, \mathbf{j})$  jumps to  $w'$  at time  $s$  if and only if  $\|w' - m(\mathbf{j})\|_2 < \|w - m(\mathbf{j})\|_2$ , and in this case again  $\widehat{\rho}(s) = \widehat{\rho}(s-)$ ; if, however,  $\|w' - m(\mathbf{j})\|_2 \geq \|w - m(\mathbf{j})\|_2$ , then  $\vartheta(\cdot, \mathbf{j})$  does not jump at time  $s$  and we take  $\widehat{\rho}(s) = \rho'$ .

Note that rules (iv) and (v) depend on whether there is another  $B$ -particle than  $\widehat{\rho}$  at  $\vartheta(s-, \mathbf{j})$ . In [KSc] any particle at the same space-time point as  $\widehat{\rho}$  automatically had type  $B$ , but this is not the case in the present setup,

because a jump is required before an  $A$ -particle can turn into a  $B$ -particle. This fact will necessitate a few extra remarks in the next step.

As in [KSc] (4.42), (4.43) we now define

$$\begin{aligned} I_1(u) &= I[N_B(\vartheta(u, \mathbf{j}), u) = 1] \\ &= I[\widehat{\rho}(u) \text{ is the only } B\text{-particle present at } (\vartheta(u, \mathbf{j}), u)]; \\ I_{\geq 2}(u) &= I[N_B(\vartheta(u, \mathbf{j}), u) \geq 2]; \end{aligned}$$

and with  $e_{d+i} = -e_i$  for  $1 \leq i \leq d$ ,

$$\begin{aligned} (3.44) \quad \Gamma_1(u) &= \frac{1}{2d} \sum_{i=1}^{2d} [\|\vartheta(u, \mathbf{j}) + e_i - m(\mathbf{j})\|_2 - \|\vartheta(u, \mathbf{j}) - m(\mathbf{j})\|_2]; \\ \Gamma_{\geq 2}(u) &= \frac{1}{2d} \sum^* [\|\vartheta(u, \mathbf{j}) + e_i - m(\mathbf{j})\|_2 - \|\vartheta(u, \mathbf{j}) - m(\mathbf{j})\|_2], \end{aligned}$$

where  $\sum^*$  is the sum over those  $i \in \{1, \dots, 2d\}$  for which

$$\|\vartheta(u, \mathbf{j}) + e_i - m(\mathbf{j})\|_2 - \|\vartheta(u, \mathbf{j}) - m(\mathbf{j})\|_2 < 0.$$

Finally we define, for  $t \geq t(k)$ ,

$$\begin{aligned} (3.45) \quad M(t) = M(t, \mathbf{j}) &:= \|\vartheta(t, \mathbf{j}) - m(\mathbf{j})\|_2 \\ &\quad - D \int_{t(k)}^t [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)] du \end{aligned}$$

(recall that  $D$  is the jump rate of all particles). The result of this step is that  $M(t)$  is a right continuous martingale under the measure which governs the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ , conditioned on  $\mathcal{H}(\mathbf{i}, k)$ , or conditioned on the

$$(3.46) \quad N_A(y, t(k)) + N_B(y, t(k)) \text{ for } y \in Z_p(\mathbf{i}).$$

The proof is essentially the same as that of Lemma 9 in [KSc], so we leave this to the reader. We merely remark that the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$  depends only on the variables in (3.46), the paths on  $[t(k), \infty)$  of the particles in  $Z_p(\mathbf{i})$  at time  $t(k)$ , and lastly, on the choice of which particle at  $(x, t(k))$  is given type  $B$ . However, changing this choice from one particle to another amounts to interchanging the roles of two particles at  $(x, t(k))$ . Since all particles move and recuperate in the same way, such an interchange does not influence the distribution of  $M(t)$  for later  $t$ . Thus, conditioning on  $\mathcal{H}(\mathbf{i}, k)$  or on the variables in (3.46) gives the same distribution for  $\{M(s)\}_{t(k) \leq s \leq t(k+1)}$ .

*Step (iii).* In this step we start on a lower bound for

$$(3.47) \quad Z(t) = Z(t, k) := \int_{t(k)}^{t(k)+t} I_{\geq 2}(u) du$$

and an upper bound for

$$\int_{t(k)}^{t(k)+t} [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)] du.$$

These bounds are essentially the same as in [KSc]. Following [KSc] we define for an integer  $L \geq 2$

$$(3.48) \quad \beta(L, d) = \begin{cases} 1 & \text{if } d = 1 \\ [\log L]^{-1} & \text{if } d = 2 \\ L^{2-d} & \text{if } d \geq 3, \end{cases}$$

$$(3.49) \quad \mathcal{E}_n = \mathcal{E}_n(\mathbf{j}, k) = \{ \text{there is some particle } \rho' \neq \widehat{\rho}(t(k) + 3L^2(n-1)) \\ \text{of the } \mathcal{Z}_p(\mathbf{i}, k)\text{-process started at } (x, t(k)) \text{ in} \\ \vartheta(t(k) + 3L^2(n-1), \mathbf{j}) + [-L, L]^d \text{ at time } t(k) + 3L^2(n-1) \},$$

and

$$J_n = I \left[ \text{at some time } u \in (t(k) + 3L^2(n-1), t(k) + L^2(3n-1)), \right. \\ \left. \widehat{\rho}(u) \text{ coincides with another } B\text{-particle} \right. \\ \left. \text{in the } \mathcal{Z}_p(\mathbf{i}, k)\text{-process started at } (x, t(k)) \right].$$

The only differences of any consequence with the definitions just before Lemma 11 of [KSc] is the insistence in the definition of  $J_n$  that  $\widehat{\rho}$  coincide with another  $B$ -particle, and that this be a particle in the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ . A  $B$ -particle is necessary because it is also possible for an  $A$  and  $B$ -particle to be at the same space-time point. The particle is required to belong to the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ , because these are the only particles under consideration at the moment. Despite these differences we have just as in Lemma 11 of [KSc] that for all  $L \geq 1$

$$(3.50) \quad E\{J_n \mid \mathcal{F}_{3L^2(n-1)}\} \geq K_7\beta(L, d) \text{ on the event } \mathcal{E}_n,$$

where this time, for  $s \geq t(k)$ ,

$$\mathcal{F}_s = \sigma\text{-field generated by } \mathcal{H}(\mathbf{i}, k) \text{ and the paths of} \\ \text{all particles in } \mathcal{Z}_p(\mathbf{i}, k) \text{ on } [t(k), s],$$

and the expectation is with respect to the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ ;  $K_7$  is some constant which depends on  $d, D$  only. To prove (3.50) we first observe that there is no loss of generality to assume that  $\widehat{\rho}$  and  $\rho'$  are at different locations at time  $3L^2(n-1) + 1$  on the event  $\mathcal{E}_n$ . This is so because there is a probability of at least  $e^{-D}(1 - e^{-D})$  that  $\widehat{\rho}$  does not jump, and  $\rho'$  has one jump in  $[3L^2(n-1), 3L^2(n-1) + 1)$ . If  $\widehat{\rho}$  and  $\rho'$  are at distinct sites, then  $\rho'$  will have to have type  $B$  no later than the first time when these

particles get together. With this observation the proof of (3.50) is the same as the proof of Lemma 11 in [KSc].

This time we take

$$(3.51) \quad L = \lceil K_5 p \rceil \text{ and } \bar{t} = p^q \Delta_p = t(k+1) - t(k).$$

These values for  $L$  and  $\bar{t}$  will remain fixed for the rest of this proof. Without loss of generality we take  $p_3$  so large that  $L \geq 1$  for  $p \geq p_3$ . Still following [KSc] we define

$$V(\bar{t}, L) = V(\bar{t}, L, \mathbf{j}, k) := \sum_{1 + \Delta_p / (3L^2) \leq n \leq [t(k+1) - t(k)] / (3L^2)} I[\mathcal{E}_n(\mathbf{j}, k)].$$

We further define the event

$$(3.52) \quad D(x, t(k), \mathbf{j}) := \{ \|\vartheta(u, \mathbf{j}) - m(\mathbf{j})\|_2 \geq \frac{1}{16} \Delta_p \text{ for all } u \in [t(k), t(k+1)] \}.$$

This event was not used in [KSc]. Nevertheless, with  $Z$  given by (3.47), we can essentially copy the proof of Lemma 12 in [KSc]. We use that

$$E\{J_n \mid \mathcal{F}_{3L^2(n-1)}\} \geq K_7 \beta(L, d) I[\mathcal{E}_n(\mathbf{j}, k)],$$

and consequently

$$E\left\{ \min\left\{1, \int_{3L^2(n-1)}^{3L^2 n} I_{\geq 2}(u) du\right\} \mid \mathcal{F}_{3L^2(n-1)}\right\} \geq e^{-2D} K_7 \beta(L, d) I[\mathcal{E}_n(\mathbf{j}, k)],$$

as in [KSc]. This yields for  $p \geq p_3$ ,  $0 < \varepsilon \leq 1$  and  $K_3$  a universal constant

$$(3.53) \quad \begin{aligned} &P\{Z(\bar{t}) \leq \varepsilon \beta(L, d) L^{-2\bar{t}} \text{ and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\ &\leq P\left\{V(\bar{t}, L) \leq \frac{2\varepsilon}{K_7} e^{2D} L^{-2\bar{t}} \text{ and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\right\} \\ &\quad + 2 \exp\left[-\frac{K_3}{3} \varepsilon^2 \beta^2(L, d) L^{-2\bar{t}}\right]. \end{aligned}$$

Next we recall for the reader the bounds (4.67) and (4.68) of [KSc]. These bounds say that for  $\vartheta, x \in \mathbb{Z}^d$

$$(3.54) \quad \frac{1}{2d} \sum_{i=1}^{2d} [\|\vartheta + e_i - x\|_2 - \|\vartheta - x\|_2] \leq \frac{K_{12}}{\|\vartheta - x\|_2 + 1},$$

and, with  $\sum^*$  as in (3.44),

$$(3.55) \quad \frac{1}{2d} \sum^* [\|\vartheta + e_i - x\|_2 - \|\vartheta - x\|_2] \leq -K_{13} + \frac{K_{12}}{\|\vartheta - x\|_2 + 1}$$

for some constants  $K_{12}, K_{13}$  which depend on  $d$  only. Moreover the left hand sides of (3.54) and (3.55) are at most 1 in absolute value.

It is immediate from (3.54), (3.55) and (3.44) that on  $D(x, t(k), \mathbf{j})$  it holds

$$\begin{aligned}
 (3.56) \quad & \int_{t(k)}^{t(k)+\bar{t}} [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)] du \\
 & \leq [t(k+1) - t(k)] \frac{16K_{12}}{\Delta_p} - K_{13}Z(t(k+1) - t(k)) \\
 & = 16K_{12}p^q - K_{13}Z(t(k+1) - t(k)).
 \end{aligned}$$

*Step (iv).* Here we use the martingale  $M(\cdot)$  to estimate  $P\{D(x, t(k), \mathbf{j}) \mid \mathcal{H}(\mathbf{i}, k)\}$  in terms of the distribution of  $V(t, L)$ . To this end we note first that

$$\begin{aligned}
 (3.57) \quad M(t(k)) &= \|\vartheta(t(k), \mathbf{j}) - m(\mathbf{j})\|_2 = \|x - m(\mathbf{i}) + m(\mathbf{i}) - m(\mathbf{j})\|_2 \\
 &\leq \sqrt{d} \frac{1}{8} \Delta_p + \|\mathbf{i} - \mathbf{j}\|_2 \Delta_p \leq \sqrt{d} \frac{9}{8} \Delta_p,
 \end{aligned}$$

where we used the fact that there is a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k+1)$ . On the other hand, on the event  $D(x, t(k), \mathbf{j})$  we have from (3.56) that

$$\begin{aligned}
 (3.58) \quad M(t(k+1)) &\geq -D \int_{t(k)}^{t(k)+\bar{t}} [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)] du \\
 &\geq -16DK_{12}p^q + K_{13}DZ(t(k+1) - t(k)).
 \end{aligned}$$

Further we have the martingale inequality (4.55) of [KSc]: for some constants  $K_{14} - K_{16}$  which depend on  $D$  only, we have for all  $a \geq 2 + 2D$ ,  $0 \leq b \leq 1$  and  $T \geq t(k)$ ,

$$\begin{aligned}
 (3.59) \quad & P\left\{ \sup_{t(k) \leq s \leq T} |M(s) - M(t(k))| \geq a + b(T - t(k)) \mid \mathcal{H}(\mathbf{i}, k) \right\} \\
 & \leq K_{14} \exp[-K_{15}(T - t(k))] + 2 \exp[-K_{16}ab].
 \end{aligned}$$

For  $a = \Delta_p/2$ ,  $b = \Delta_p/[2(t(k+1) - t(k))]$  and  $T = t(k+1)$  this implies

$$(3.60) \quad P\{|M(t(k+1)) - M(t(k))| \geq \Delta_p \mid \mathcal{H}(\mathbf{i}, k)\} \leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right],$$



provided  $p \geq p_4$  for some constant  $p_4 \geq p_3$ . Combined with (3.57) and (3.58) this gives

(3.61)

$$\begin{aligned}
 & P\{D(x, t(k), \mathbf{j}) \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \leq P\{M(t(k+1)) - M(t(k)) \geq \Delta_p \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \quad + P\{-16DK_{12}p^q + K_{13}DZ(t(k+1) - t(k)) \leq M(t(k)) + \Delta_p \\
 & \quad \quad \quad \text{and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] \\
 & \quad + P\{Z(t(k+1) - t(k)) \leq \frac{1}{K_{13}D}[(\sqrt{d}\frac{9}{8} + 1)\Delta_p + 16DK_{12}p^q] \\
 & \quad \quad \quad \text{and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] + P\{Z(t(k+1) - t(k)) \leq \frac{3\sqrt{d}}{K_{13}D}\Delta_p \\
 & \quad \quad \quad \text{and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\},
 \end{aligned}$$

for  $p \geq$  some  $p_5$ . Finally we note that  $\beta(L, d) \geq L^{1-d}$  (see (3.48)), so that for

$$\varepsilon = \varepsilon(d, D) = \min\left[\frac{K_7}{15e^{2D}}, 1\right]$$

and  $p \geq$  a suitable constant  $p_6$ ,

$$\varepsilon \frac{\beta(L, d)}{L^2} p^q \Delta_p \geq \frac{3\sqrt{d}}{K_{13}D} \Delta_p \text{ (recall that } q = 2d + 1 \text{ and } L = \lceil K_5 p \rceil).$$

Thus, by (3.53), we can continue (3.61) to obtain

$$\begin{aligned}
 (3.62) \quad & P\{D(x, t(k), \mathbf{j}) \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] \\
 & \quad + P\{V(\bar{t}, L) \leq \frac{2}{15}L^{-2}\bar{t} \text{ and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \quad + 2 \exp\left[-\frac{K_3}{3}\varepsilon^2\beta^2(L, d)L^{-2}\bar{t}\right].
 \end{aligned}$$

*Step (v).* In this step we shall estimate the probability in the right hand side of (3.62). This will be done by using the following direct consequence of the definitions of  $\tilde{K}$  and of  $\mathcal{E}_n$ :

$$\tilde{K}_p(\vartheta(t(k) + 3L^2(n-1), \mathbf{j}), t(k) + 3L^2(n-1)) \geq 2$$

implies that  $\mathcal{E}_n$  occurs (recall that  $L = \lceil K_5 p \rceil$ ). This will allow us to use the bound (3.42) on the probability that  $\tilde{K}$  is ‘small’. We turn to the details.

Take  $p_6$  so large that for all  $p \geq p_6$

$$\frac{1}{2}\gamma_0\mu_A(K_5p)^d \geq 2.$$

Assume now that the events

$$(3.63) \quad \{\tilde{K}_p(z, v) \geq \frac{1}{2}\gamma_0\mu_A[K_5p]^d \text{ for all } (z, v) \text{ satisfying (3.41)}\}$$

and

$$(3.64) \quad \{\vartheta(u, \mathbf{j}) \in \prod_{s=1}^d [(i(s) - 4d)\Delta_p, (i(s) + 4d + 1)\Delta_p - K_5p] \text{ for } t(k) \leq u < t(k+1)\}$$

occur. Then for

$$(3.65) \quad 1 + \Delta_p/(3L^2) \leq n \leq [t(k+1) - t(k)]/(3L^2),$$

it holds

$$(3.66) \quad \tilde{K}_p(\vartheta(t(k) + 3L^2(n-1), \mathbf{j}), t(k) + 3L^2(n-1)) \geq \frac{1}{2}\gamma_0\mu_A[K_5p]^d \geq 2.$$

As observed, this implies that  $\mathcal{E}_n$  also occurs for the  $n$  in (3.65) and then also

$$(3.67) \quad V(\bar{t}, L) \geq \frac{t(k+1) - t(k) - \Delta_p}{3L^2} - 2 > \frac{2}{15L^2}p^q\Delta_p$$

for  $p \geq$  some constant  $p_7$ .

However, we know from (3.42) that on the event (3.39), (3.63) indeed holds outside a set of conditional probability  $\Delta_p^{-d-1}$ . To estimate the probability that (3.64) fails for a relevant value of  $u$ , we introduce the random time

$$(3.68) \quad \tau := \inf\{w \in [t(k), t(k+1)] : \vartheta(w, \mathbf{j}) \notin \prod_{s=1}^d [(i(s) - 4d)\Delta_p, (i(s) + 4d + 1)\Delta_p - K_5p]\}.$$

This definition for  $\tau$  holds if the set in the right hand side of (3.68) is not empty; otherwise we set  $\tau$  equal to  $t(k+1)$ . We shall prove in the remainder of this step that

$$(3.69) \quad \begin{aligned} P\{\vartheta(u, \mathbf{j}) \notin \prod_{s=1}^d [(i(s) - 4d)\Delta_p, (i(s) + 4d + 1)\Delta_p - K_5p] \\ \text{for some } u \in [t(k), t(k+1)) \\ \text{and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\ = P\{\tau < t(k+1) \text{ and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\ \leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] \end{aligned}$$

for  $p \geq$  some constant  $p_8$ . As we observed above, this will imply

$$\begin{aligned}
 (3.70) \quad & P\{V(\bar{t}, L) \leq \frac{2}{15L^2}p^q\Delta_p \text{ and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \leq P\{(3.63) \text{ fails} \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \quad + P\{(3.64) \text{ fails but } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\
 & \leq \Delta_p^{-d-1} + 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right],
 \end{aligned}$$

for  $p \geq p_8$  and on the event (3.39).

To prove (3.69) we note that on the event  $D(x, t(k), \mathbf{j})$  it holds

$$\int_{t(k)}^\tau [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)] du \leq 16DK_{12}p^q$$

(see (3.56)). Consequently, on  $\{\tau < t(k+1)\} \cap D(x, t(k), \mathbf{j})$  we also have

$$\begin{aligned}
 M(\tau) & \geq \|\vartheta(\tau, \mathbf{j}) - m(\mathbf{j})\|_2 - 16DK_{12}p^q \\
 & \geq \|\vartheta(\tau, \mathbf{j}) - m(\mathbf{i})\|_2 - \|m(\mathbf{i}) - m(\mathbf{j})\|_2 - 16DK_{12}p^q \\
 & \geq 4d\Delta_p - K_5p - \sqrt{d}\Delta_p - 16DK_{12}p^q \geq \Delta_p + \sqrt{d}\frac{9}{8}\Delta_p,
 \end{aligned}$$

provided  $p \geq$  some constant  $p_9$ . In particular, on  $\{\tau < t(k+1)\} \cap D(x, t(k), \mathbf{j})$

$$\sup_{t(k) \leq s \leq t(k+1)} |M(s) - M(t(k))| \geq M(\tau) - M(t(k)) \geq \Delta_p \text{ (see (3.57)).}$$

We already proved in (3.59) and (3.60) that

$$P\left\{\sup_{t(k) \leq s \leq t(k+1)} |M(s) - M(t(k))| \geq \Delta_p \mid \mathcal{H}(\mathbf{i}, k)\right\} \leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right].$$

This implies the promised inequality (3.69).

*Step (vi).* We finally prove Lemma 9 in this step, by combining the preceding steps. It follows from the definition of  $\tilde{A}(x, t(k), \mathbf{j})$  that  $[\tilde{A}(x, t(k), \mathbf{j})]^c$  can occur only if  $(x, t(k))$  is occupied, but

$$\begin{aligned}
 (3.71) \quad & \text{the } \mathcal{Z}_p(\mathbf{i}, k)\text{-process started at } (x, t(k)) \text{ does not have} \\
 & \text{a } B\text{-particle in } m(\mathbf{j}) + \mathcal{C}\left(\frac{1}{8}\Delta_p\right) \text{ at time } t(k+1).
 \end{aligned}$$

In turn, this last event can occur only if  $D(x, t(k), \mathbf{j})$  occurs, or if

$$(3.72) \quad \|\vartheta(u, \mathbf{j}) - m(\mathbf{j})\|_2 \leq \frac{1}{16}\Delta_p \text{ for some } u \in [t(k), t(k+1)],$$

as well as (3.71) occur. However, the probability of the intersection of (3.72) and (3.71) is small. Indeed, if

$$\sigma := \inf\{u \geq t(k) : \|\vartheta(u, \mathbf{j}) - m(\mathbf{j})\|_2 \leq \frac{1}{16}\Delta_p\},$$

then on the intersection of (3.72) and (3.71), the particle  $\widehat{\rho}(\sigma)$  is within distance  $\Delta_p/(16)$  of  $m(\mathbf{j})$  at time  $\sigma$ , but outside  $m(\mathbf{j}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  at time  $t(k+1)$ . (Recall that we are working with the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ . This has no recuperation, so the particle  $\widehat{\rho}(\sigma)$  will still be a  $B$ -particle in this process at time  $t(k+1)$ .) In other words, the particle  $\widehat{\rho}(\sigma)$ , which is the distinguished one at time  $\sigma$ , travels a distance at least  $\Delta_p/8 - \Delta_p/16 = \Delta_p/16$  during  $[\sigma, t(k+1)]$ . Consequently,

$$\begin{aligned} &P\{(3.72) \text{ and } (3.71) \text{ occur} \mid \mathcal{H}(\mathbf{i}, k)\} \\ &\leq P\left\{\sup_{s \leq t(k+1) - t(k)} \|S_{t(k+1)} - S_s\|_2 \geq \frac{1}{16}\Delta_p \mid \mathcal{H}(\mathbf{i}, k)\right\} \\ &\leq 8d \exp\left[-K_{17} \frac{\Delta_p}{p^q}\right] \text{ (see (2.42) in [KSa])} \end{aligned}$$

for some constant  $K_{17} = K_{17}(d, D)$ . It follows that on the event (3.39)

$$\begin{aligned} &P\{(3.71) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\ &\leq P\{D(x, t(k), \mathbf{j}) \mid \mathcal{H}(\mathbf{i}, k)\} + 8d \exp\left[-K_{17} \frac{\Delta_p}{p^q}\right] \\ &\leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] \\ &\quad + P\{V(\bar{t}, L) \leq \frac{2}{15L^2}\bar{t} \text{ and } D(x, t(k), \mathbf{j}) \text{ occurs} \mid \mathcal{H}(\mathbf{i}, k)\} \\ &\quad + 2 \exp\left[-\frac{K_3}{3}\varepsilon^2\beta^2(L, d)L^{-2\bar{t}}\right] + 8d \exp\left[-K_{17} \frac{\Delta_p}{p^q}\right] \text{ (by (3.62))} \\ &\leq 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] + 2 \exp\left[-\frac{K_3}{3}\varepsilon^2\beta^2(L, d)L^{-2\bar{t}}\right] \\ &\quad + 8d \exp\left[-K_{17} \frac{\Delta_p}{p^q}\right] + \Delta_p^{-d-1} + 4 \exp\left[-\frac{K_{16}\Delta_p}{4p^q}\right] \end{aligned}$$

(see (3.70)) for  $p \geq$  a suitable constant  $p_2$ . Summation of this estimate over the at most  $[1 + \Delta_p/4]^d$  possible  $x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)$  now proves (3.40).  $\square$

For the remainder of this section we shall only consider  $p \geq p_2$ .

LEMMA 10. *For  $p \geq p_2$  we can choose  $\lambda_0 = \lambda_0(p) > 0$  such that on the event (3.39),*

$$(3.73) \quad \sum_{x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)} P\{[\widetilde{A}(x, t(k), \mathbf{j}) \cap \widetilde{B}(x, t(k), \mathbf{j}, \lambda_0)]^c \mid \mathcal{H}(\mathbf{i}, k)\} \leq 2\Delta_p^{-1},$$

and

$$(3.74) \quad P\{C(\mathbf{i}, k, \mathbf{j}, \lambda_0) \text{ in } \{Y_t(\lambda_0)\} \mid \mathcal{H}(\mathbf{i}, k)\} \leq 2\Delta_p^{-1}.$$

*Proof.* The left hand side of (3.73) equals

$$\begin{aligned} & \sum_{x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)} P\{[\tilde{A}(x, t(k), \mathbf{j})]^c \mid \mathcal{H}(\mathbf{i}, k)\} \\ & + \sum_{x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)} P\{\tilde{A}(x, t(k), \mathbf{j}) \cap [\tilde{B}(x, t(k), \mathbf{j}, \lambda_0)]^c \mid \mathcal{H}(\mathbf{i}, k)\}. \end{aligned}$$

In view of Lemma 9 it therefore suffices for (3.73) to prove that on the event  $\{\mathcal{Z}_p(\mathbf{i}, k) \text{ is good}\}$

$$(3.75) \quad \sum_{x \in m(\mathbf{i}) + \mathcal{C}(\frac{1}{8}\Delta_p)} P\{\tilde{A}(x, t(k), \mathbf{j}) \cap [\tilde{B}(x, t(k), \mathbf{j}, \lambda_0)]^c \text{ in } \{Y_t(\lambda_0)\} \mid \mathcal{H}(\mathbf{i}, k)\} \leq \Delta_p^{-1}.$$

We claim that each summand in (3.75) is at most

$$1 - \exp[-\lambda_0[t(k+1) - t(k)]] \leq \lambda_0[t(k+1) - t(k)].$$

Indeed, by the definitions (3.17), (3.18) of  $\tilde{A}$  and  $\tilde{B}$ , once we know that  $\tilde{A}(x, t(k), \mathbf{j})$  occurs, the event  $[\tilde{B}(x, t(k), \mathbf{j}, \lambda_0)]^c$  can occur only if some particle  $\rho_i$  has a recuperation event in  $\{Y_t(\lambda_0)\}$  during  $[s_i, s_{i+1}]$ , for  $0 \leq i \leq \ell$ . Here  $\rho_i$  are certain particles and the  $s_i$  are increasing and such that  $s_{\ell+1} - s_0 = t(k+1) - t(k)$ . These  $\rho_i$  and  $s_i$  are determined by the  $\mathcal{Z}_p(\mathbf{i}, k)$ -process started at  $(x, t(k))$ , and therefore are independent of the recuperation events during  $[t(k), \infty)$ . This proves our claim. (3.75) now follows for some small  $\lambda_0(p)$ , since there are at most  $[1 + \Delta_p/4]^d$  terms in the sum in (3.75).

The preceding paragraph proves (3.73). (3.74) is now an immediate consequence of (3.19) and the fact that  $C(\mathbf{i}, k, \mathbf{j}, \lambda) = \emptyset$  if  $(\mathbf{i}, k)$  is not active.  $\square$

Lemma 10 will help us to bound the probability that there are many sites  $(\mathbf{i}, k)$  in an open cluster  $\mathcal{C}$  with a good bottom and with a closed edge from  $(\mathbf{i}, k)$  to a site in  $\partial_{ext} \mathcal{C}$ . In order to obtain (3.23) from such a bound we first have to show that there is only a small probability that a  $\mathcal{C}_0$ -barrier  $S$  has of order  $|S|$  sites with a parent (on  $\mathcal{D}$ ) with a bad bottom. (A bottom  $\mathcal{Z}_p(\mathbf{i}, k)$  is called *bad* if it is not good.) This will be the goal of Lemmas 11 and 12. For a  $\mathbb{Z}^{d+1}$ -connected set  $S$  and integer  $p \geq 2$  we define

$$(3.76) \quad S_p^* = \bigcup_{\substack{(\mathbf{i}', k): \|\mathbf{i}' - \mathbf{j}\| \leq 4d-1 \\ \text{for some } (\mathbf{j}, k+1) \in S}} \widehat{\mathcal{B}}_p(\mathbf{i}', k).$$

Recall that we used the vertex  $(\mathbf{i}, k)$  as a kind of renormalized site to replace the block  $\widehat{\mathcal{B}}_p(\mathbf{i}, k)$ . Forming  $S_p^*$  from  $S$  is a construction going in the other direction. From the collection  $S$  of renormalized sites we reconstruct the

blocks which are near the ones represented by sites in  $S$ . We define further for any positive integer  $\nu$  and  $r \geq p$  the blocks

$$\mathcal{L}_{r,\nu}(\mathbf{m}, u) := \prod_{s=1}^d [\nu m(s)\Delta_r, \nu(m(s) + 1)\Delta_r) \times [\nu u\Delta_r, \nu(u + 1)\Delta_r).$$

Note that the blocks  $\mathcal{L}_{r,\nu}(\mathbf{m}, u)$  form a partition of  $\mathbb{Z}^{d+1}$  into disjoint cubes.

LEMMA 11. *Let  $r \geq p \geq 2$  and  $q = 2d + 1$  (as before). There exists a constant  $K_{18}$ , depending on  $d$  only, such that for each  $\mathbb{Z}^{d+1}$ -connected set  $S$  and each integer  $\nu \geq 1$ , there exists a  $\mathbb{Z}^{d+1}$ -connected set  $\Lambda_{p,r}(S, \nu) \subset \mathbb{Z}^{d+1}$  such that*

$$(3.77) \quad |\Lambda_{p,r}(S, \nu)| \leq K_{18} \left[ \frac{|S|\Delta_p}{\nu\Delta_r} + 1 \right] p^q$$

and such that

$$(3.78) \quad \bigcup_{(\mathbf{m}, u) \in \Lambda_{p,r}(S, \nu)} \mathcal{L}_{r,\nu}(\mathbf{m}, u) \supset S_p^*.$$

*Proof.* This lemma is essentially the same as Lemma 1 in [CGGK]. For the convenience of the reader we repeat the main steps of the proof. Let  $|S| = n$ . Since  $S$  is connected it has a spanning tree with  $n - 1$  edges, and then there exists a path  $(v_0, v_1, \dots, v_a)$  on  $\mathbb{Z}^{d+1}$  of length  $a \leq 2n - 2$  whose vertices are exactly the vertices of  $S$  (some vertices are repeated; the path is not self-avoiding, in general).

For  $0 \leq u \leq a$  let  $v_u = (\mathbf{i}_u, k_u)$ . Now set

$$\mu = \nu \frac{\Delta_r}{\Delta_p} = \nu \Delta_{r-p},$$

and consider the vertices  $v_{j\mu}$  for  $0 \leq j \leq a/\mu$  (note that  $\mu$  is an integer, by our choice of  $C_0$  and  $\Delta_r$ ). For  $0 \leq j \leq a/\mu$  let  $(\mathbf{m}_j, u_j)$  be the unique vertex in  $\mathbb{Z}^{d+1}$  such that

$$(3.79) \quad (\mathbf{i}_{j\mu}\Delta_p, k_{j\mu}p^q\Delta_p) \in \mathcal{L}_{r,\nu}(\mathbf{m}_j, u_j).$$

We now take

$$\Lambda_{p,r}(S, \nu) = \bigcup \{ (\mathbf{m}, u) : \text{there exists a } 0 \leq j \leq a/\mu \text{ such that } \|\mathbf{m} - \mathbf{m}_j\| \leq (4d + 2) \text{ and } |u - u_j| \leq 3p^q \}.$$

Then, since  $(\mathbf{m}_j, u_j)$  takes at most  $(a/\mu + 1) \leq (2n\Delta_p/\nu\Delta_r + 1)$  values, it holds

$$|\Lambda_{p,r}(S, \nu)| \leq \left[ \frac{2n\Delta_p}{\nu\Delta_r} + 1 \right] (8d + 5)^d (6p^q + 1),$$

and (3.77) holds.

Next we verify (3.78). Assume that  $y$  is a vertex in  $S_p^*$ . Then there is some  $v_u = (\mathbf{i}_u, k_u)$  and some  $(\mathbf{i}', k')$  with  $|i' - i_u| \leq 4d - 1$ ,  $k' = k_u - 1$

such that  $y \in \widehat{\mathcal{B}}_p(\mathbf{i}', k')$ . In particular,  $|y(s) - i_u(s)\Delta_p| \leq 4d\Delta_p, 1 \leq s \leq d$ , and  $|y(d+1) - k_u p^q \Delta_p| \leq p^q \Delta_p$ . Also, there exists some  $j$  such that  $j\mu \leq u < (j+1)\mu$ , and  $0 \leq j \leq a/\mu$ , and consequently  $\|(\mathbf{i}_u, k_u) - (\mathbf{i}_{j\mu}, k_{j\mu})\| \leq \mu$ . Finally, by virtue of (3.79),

$$\begin{aligned} & \|(\mathbf{i}_{j\mu}\Delta_p, k_{j\mu}p^q\Delta_p) - (\mu\mathbf{m}_j\Delta_p, \mu u_j\Delta_p)\| \\ &= \|(\mathbf{i}_{j\mu}\Delta_p, k_{j\mu}p^q\Delta_p) - (\nu\mathbf{m}_j\Delta_r, \nu u_j\Delta_r)\| \leq \Delta_r \end{aligned}$$

and

$$\begin{aligned} & |y(s) - \nu m_j(s)\Delta_r| \\ & \leq |y(s) - i_u(s)\Delta_p| + |i_u(s)\Delta_p - i_{j\mu}(s)\Delta_p| + |i_{j\mu}(s)\Delta_p - \nu m_j(s)\Delta_r| \\ & \leq 4d\Delta_p + (\nu + 1)\Delta_r \leq (4d + 2)\nu\Delta_r, \quad 1 \leq s \leq d, \end{aligned}$$

and similarly

$$|y(d+1) - \nu u_j\Delta_r| \leq 3p^q\nu\Delta_r.$$

The relation (3.78) now follows easily.

Finally, the connectedness of  $\Lambda_{p,r}(S, \nu)$  follows from the fact that  $\Lambda_{p,r}(S, \nu)$  is the union of the rectangular boxes

$$\prod_{s=1}^d [m_j(s) - 4d - 2, m_j(s) + 4d + 2] \times [u_j - 3p^q, u_j + 3p^q], \quad 1 \leq j \leq a/\mu.$$

Each of these boxes is clearly  $\mathbb{Z}^{d+1}$ -connected and the boxes corresponding to the two successive values  $j$  and  $j+1$  intersect, because they have the point  $(\mathbf{m}_{j+1}, u_{j+1})$  in common. Clearly, this point lies in the  $(j+1)$ -th box. It also lies in the  $j$ -th box, because  $m_j(s) = \lfloor i_{j\mu}(s)/\mu \rfloor$  and  $m_{j+1}(s) = \lfloor i_{(j+1)\mu}(s)/\mu \rfloor, 1 \leq s \leq d$ , (by (3.79)) and  $|i_{(j+1)\mu}(s) - i_{j\mu}(s)| \leq \|v_{(j+1)\mu} - v_{j\mu}\| \leq \mu$ . Similarly,  $u_j = \lfloor k_{j\mu}p^q/\mu \rfloor$  and  $|k_{(j+1)\mu} - k_{j\mu}| \leq \mu$ .  $\square$

In the next lemma  $\mathfrak{C}_0$  always will be such that

$$(3.80) \quad \mathfrak{C}_0 \subset \mathcal{D}, \mathbf{0} \in \mathfrak{C}_0, \text{ and}$$

$$\mathfrak{C}_0 \text{ is } \mathbb{Z}^{d+1}\text{-connected when viewed as a subset of } \mathbb{Z}^{d+1}.$$

LEMMA 12. We can choose  $\lambda_0 > 0, p_0$  and  $\mathfrak{C}_0$  such that (3.80) and (3.23) are satisfied.

*Proof.* The hard part of the work was done in [KSa] and [KSc]. It is too long to repeat and we shall be content with reducing the lemma to some results in those references.

*Step (i).* Basically we are going to show that there is only a small probability that there exists a  $\mathfrak{C}_0$ -barrier with ‘many’ vertices which are  $\mathcal{D}$ -adjacent to a vertex with a bad bottom (see (3.33) for the definition of a good bottom). In this step we reduce the bounding of the number of barriers with a large number

of vertices that have a parent  $(\mathbf{i}, k)$  in  $\mathcal{D}$  with a bad bottom to estimates in [KSc].

Let  $\mathfrak{C}_0$  have the properties in (3.80). It is easy to see that then any  $\mathfrak{C}_0$ -barrier  $S$  must contain some vertices  $(k_+, 0, \dots, 0)$  and  $(k_-, 0, \dots, 0)$  on the positive and negative first coordinate axes, respectively. Moreover, if  $|S| = n$ , then

$$\text{diameter}(S) = \max_{x, y \in S} \|x - y\| \leq n.$$

It follows that we may take  $1 \leq k_{\pm} \leq n$  and that  $S \subset [-n, n]^{d+1}$ . Since  $S$  must be  $\mathbb{Z}^{d+1}$  connected and must contain  $(k_+, 0, \dots, 0)$  for some  $1 \leq k_+ \leq n$ , there are at most  $n[K_{19}]^n$  possibilities for  $S$ , (with  $K_{19}$  depending on  $d$  only; see for instance [Ka], formula (5.22)). We remind the reader that in [KSc] we also defined bad  $r$ -blocks of the form

$$(3.81) \quad \mathcal{B}_r(\mathbf{m}, \ell) := \prod_{s=1}^d [m(s)\Delta_r, (m(s) + 1)\Delta_r) \times [\ell\Delta_r, (\ell + 1)\Delta_r),$$

and their *pedestals*

$$(3.82) \quad \mathcal{V}_r(\mathbf{m}, \ell) := \prod_{s=1}^d [(m(s) - 3)\Delta_r, (m(s) + 4)\Delta_r) \times \{(\ell - 1)\Delta_r\}.$$

Note that  $\mathcal{B}_r(\mathbf{m}, \ell r^q) \subset \widehat{\mathcal{B}}_r(\mathbf{m}, \ell)$ . The block in (3.81) is called bad (in the sense of [KSc]) if (see (3.31) and (3.32) for  $\mathcal{Q}_p$  and  $U_p$ )

$$U_r(x, v) < \gamma_r \mu_A C_0^{dr} \text{ for some } (x, v) \text{ with integer } v \text{ for which}$$

$$\mathcal{Q}_r(x) \times \{v\} \subset \prod_{s=1}^d [(m(s) - 3)\Delta_r, (m(s) + 4)\Delta_r) \times [(\ell - 1)\Delta_r, (\ell + 1)\Delta_r).$$

Similarly, the pedestal in (3.82) is called bad (in the sense of [KSc]) if

$$U_r(x, (\ell - 1)\Delta_r) < \gamma_r \mu_A C_0^{dr} \text{ for some } x \text{ for which}$$

$$\mathcal{Q}_r(x) \subset \prod_{s=1}^d [(m(s) - 3)\Delta_r, (m(s) + 4)\Delta_r).$$

Here the  $\gamma_r$  are increasing in  $r$  and satisfy

$$0 < \gamma_0 \leq \gamma_r \leq \gamma_{\infty} \leq \frac{1}{2}, \quad r \geq 0$$

for some  $\gamma_0, \gamma_{\infty}$ . The precise form of the  $\gamma_r$  used in [KSc] is not important at the moment. If  $\widehat{\mathcal{B}}_p(\mathbf{i}, k)$  has a bad bottom as defined in (3.33), then

$$U_p(x, t_p(k)) < \gamma_0 \mu_A C_0^{dp} \text{ for some } x \text{ for which } \mathcal{Q}_p(x) \subset Z_p(\mathbf{i}).$$

In this case,  $\mathcal{Q}_p(x) \subset \prod_{s=1}^d [(i'(s) - 3)\Delta_p, (i'(s) + 4)\Delta_p)$  for some  $i'$  with

$$(3.83) \quad i(s) - 4d + 2 \leq i'(s) \leq i(s) + 4d - 2, \quad 1 \leq s \leq d.$$



Therefore,  $\mathcal{B}_p(\mathbf{i}', kp^q)$  is bad in the sense of [KSc] for some  $\mathbf{i}'$  satisfying (3.83).

Now suppose  $\widehat{\mathcal{B}}_p(\mathbf{i}, k)$  has a bad bottom and there exists a  $\mathcal{D}$ -edge from  $(\mathbf{i}, k)$  to  $(\mathbf{j}, k + 1) \in S$ . Then there is an  $\mathbf{i}'$  with  $\|\mathbf{i}' - \mathbf{i}\| \leq 4d - 2$ , and hence  $\|\mathbf{i}' - \mathbf{j}\| \leq 4d - 1$ , such that  $\mathcal{B}_p(\mathbf{i}', kp^q)$  is bad in the sense of [KSc]. By the definition (3.76)  $\widehat{\mathcal{B}}_p(\mathbf{i}', k) \subset S_p^*$ , and therefore,  $\mathcal{B}_p(\mathbf{i}', kp^q) \subset \widehat{\mathcal{B}}_p(\mathbf{i}', k) \subset S_p^*$ . Moreover,  $\|\mathbf{i}' - \mathbf{j}\| \leq 4d - 1$ , so that at most  $(8d - 1)^d$  vertices  $(\mathbf{j}, k + 1) \in S$  can give rise to the same  $(\mathbf{i}', k)$ . This shows that

$$(3.84) \quad \begin{aligned} & \text{[the number of } (\mathbf{j}, k + 1) \in S \text{ with a parent } (\mathbf{i}, k) \text{ for which} \\ & \quad \widehat{\mathcal{B}}_p(\mathbf{i}, k) \text{ has a bad bottom]} \\ & \leq (8d - 1)^d \times \text{[the number of bad } p\text{-blocks } \mathcal{B}_p(\mathbf{i}', kp^q) \\ & \quad \text{in the sense of [KSc] contained in } S_p^*]. \end{aligned}$$

In the next step we apply [KSc] to estimate the right hand side of (3.84).

*Step (ii).* In analogy with [KSc] we now make the following definitions for a barrier  $S$ . In these definitions, an  $r$ -block is of the form (3.81) and ‘good’ or ‘bad’ are meant in the sense of [KSc].

$$(3.85) \quad \begin{aligned} \widehat{\phi}_r(S_p^*) &= \text{number of bad } r\text{-blocks which intersect } S_p^*, \\ \widehat{\psi}_r(S_p^*) &= \text{number of } r\text{-blocks which intersect } S_p^* \text{ and which have} \\ & \quad \text{a good pedestal, but contain a bad } (r - 1)\text{-block,} \end{aligned}$$

$$(3.86) \quad \widehat{\Phi}_r(n) = \widehat{\Phi}_r(n, \mathfrak{C}_0) = \sup\{\widehat{\phi}_r(S_p^*) : S \text{ a } \mathfrak{C}_0\text{-barrier of cardinality } n\},$$

and

$$(3.87) \quad \widehat{\Psi}_r(n) = \widehat{\Psi}_r(n, \mathfrak{C}_0) = \sup\{\widehat{\psi}_r(S_p^*) : S \text{ a } \mathfrak{C}_0\text{-barrier of cardinality } n\}.$$

(The quantities in (3.85)–(3.87) also depend on  $p$ , but we do not explicitly indicate this in our notation.) In this step we shall prove that for every choice of  $K$  and  $\varepsilon_0$ , there exist constants  $p_0, n_0$  such that for all  $p \geq p_0, n \geq n_0$ ,

$$(3.88) \quad P\{\widehat{\Phi}_r(n, \mathfrak{C}_0) \geq \varepsilon_0 n \text{ for some } r \geq p\} \leq \frac{2}{nK}.$$

It is crucial that this estimate is uniform in  $\mathfrak{C}_0$  satisfying (3.80). In fact,  $p_0, n_0$  depend only on  $d, \gamma_0, \mu_A, \varepsilon_0, K$ , but not on  $\mathfrak{C}_0$ .

We saw in step (i) that all  $\mathfrak{C}_0$ -barriers  $S$  of cardinality  $n$  have to satisfy

$$(3.89) \quad S \subset [-n, n]^{d+1},$$

so that we may restrict the sup in (3.86) to  $S$  which satisfy the condition (3.89). The quantities  $\widehat{\phi}_r$  and  $\widehat{\Phi}_r$  are analogues of the following quantities

introduced in [KSc]

$\phi_r(\widehat{\pi}) :=$  number of bad (in the sense of [KSc])  $r$ -blocks which intersect the space-time path  $\widehat{\pi}$

and

$$(3.90) \quad \Phi_r(\ell) = \sup_{\widehat{\pi} \in \Xi(\ell, t)} \phi_r(\widehat{\pi}),$$

with

$$(3.91) \quad \Xi(\ell, t) = \{\widehat{\pi} : \widehat{\pi} \text{ is a space-time path over the time interval } [0, t] \text{ and located in } \mathcal{C}(t \log t), \text{ with exactly } \ell \text{ jumps during } [0, t]\}$$

( $\mathcal{C}(\cdot)$  is defined in (1.13)). We showed in [KSc], Proposition 8, that for any choice of  $K$  and  $\varepsilon_0 > 0$ , there exist constants  $r_0, t_1$ , such that for all  $t \geq t_1$

$$(3.92) \quad P\{\Phi_r(\ell) \geq \varepsilon_0 C_0^{-6r}(t + \ell) \text{ for some } r \geq r_0, \ell \geq 0\} \leq \frac{2}{t^K}.$$

One can check that the lengthy proof of (3.92) uses the restriction that  $\widehat{\pi} \in \Xi(\ell, t)$  in the sup in (3.90) only for the bound in (4.32) in [KSc]. This bound says (after a small change to the present notation) that for integers  $\nu \geq 1$  and  $r \geq p$ , the number of blocks  $\mathcal{L}_{r,\nu}(\mathbf{m}, u)$  which intersect any given  $\widehat{\pi} \in \Xi(\ell, t)$  is at most

$$(3.93) \quad \lambda(\ell) := 3^d \left( \frac{t + \ell}{\nu \Delta_r} + 2 \right).$$

In the present case we can replace this estimate by (3.77). This tells us that for  $r \geq p$ , any set  $S_p^*$  defined by (3.76) for  $S$  a  $\mathfrak{C}_0$ -barrier of cardinality  $n$ , intersects at most

$$(3.94) \quad K_{18} \left[ \frac{n \Delta_p}{\nu \Delta_r} + 1 \right] p^q$$

blocks  $\mathcal{L}_{r,\nu}(\mathbf{m}, u)$ . Apart from an insignificant change from the factor  $3^d$  to  $K_{18}$  this takes the place of the bound (3.93), provided we replace  $(t + \ell)$  by  $n \Delta_p p^q$ . We further have to replace  $R(t)$  of (4.16) in [KSc] by  $\widehat{R}(n)$ , which we take to be the unique integer  $R$  for which

$$C_0^R \geq [K_4 \log n]^{1/d} > C_0^{R-1}.$$

If diameter  $(S) = n$ , then by (3.89) and (3.76)

$$S_p^* \subset [-(n + 4d)\Delta_p, (n + 4d)\Delta_p]^d \times [-p^q \Delta_p, np^q \Delta_p].$$

Simple estimates for the Poisson distribution (compare Lemmas 5 and 9 in [KSa]) show then that we can take  $K_4 = K_4(d, \mu_A, K)$  so large that

$$P\{\widehat{\Phi}_r(n) > 0 \text{ for some } r \geq \widehat{R}(n) \vee \log p\} \leq \frac{1}{n^K}, \quad n \geq 1.$$

This estimate takes the place of (4.17) in [KSc]. We can then follow the proof of Lemma 7 in [KSc] with only trivial changes to show that there exist constants  $C_5, \kappa_0, n_0$  which depend on  $d, \gamma_0, \mu_A, K$  (but not on  $p, r, n$  or  $\mathfrak{C}_0$ ), such that for all  $n \geq n_0, \kappa \geq \kappa_0, p \leq r \leq \widehat{R}(n) - 1$ ,

$$P\{\widehat{\Psi}_{r+1}(n) \geq \frac{\kappa n}{\Delta_{r+1}} p^q \Delta_p [\rho_{r+1}]^{1/(d+1)}\} \leq \exp\left[-nC_5 \kappa p^q \Delta_p \exp\left[-\frac{\gamma_0 \mu_A}{2(d+1)} C_0^{(d-3/4)r}\right]\right],$$

where

$$\rho_{r+1} = 3^{d+1} C_0^{6(d+1)(r+1)} \exp\left[-\frac{1}{2} \gamma_r \mu_A C_0^{(d-3/4)r}\right].$$

With this estimate in hand one can copy the proof of Proposition 8 in [KSc] with the simple replacement of  $\kappa_0(t+\ell)/\Delta_{r+1}$  by  $\kappa_0 n p^q \Delta_p / \Delta_{r+1}$ . This yields that

$$\widehat{\Phi}_r(n) \leq n 6 \kappa_0 C_0^{6(d+1)} \Delta_p p^q \exp\left[-\frac{\gamma_0 \mu_A}{2(d+1)} C_0^{(d-3/4)r}\right] < \varepsilon_0 n$$

outside a set of probability  $2n^{-K}$ , for  $n \geq n_0$  and  $r_0(d, \gamma_0, \mu_A, \varepsilon_0) \vee p \leq r \leq \widehat{R}(n) - 1$ . By taking  $p_0 \geq r_0$  and  $r \geq p \geq p_0$  one obtains (3.88).

*Step (iii).* Without loss of generality we take  $p_0 \geq p_2$  (which was determined in Lemma 10). For  $p \geq p_0, K = 2$  and  $r = p$ , (3.84) and (3.88), imply that for any  $n_1 \geq n_0$  and any  $\mathfrak{C}_0$  satisfying (3.80)

$$\begin{aligned} \sum_{n \geq n_1} P\{\text{there exists a } \mathfrak{C}_0\text{-barrier } S \text{ with } |S| = n \text{ and at least} \\ (8d - 1)^d \varepsilon_0 n \text{ vertices which have a parent } (\mathbf{i}, k) \\ \text{such that } \widehat{\mathcal{B}}_p(\mathbf{i}, k) \text{ has a bad bottom}\} \\ \leq \sum_{n \geq n_1} \frac{2}{n^2}. \end{aligned}$$

We shall take  $\varepsilon_0 = \varepsilon_0(d)$  such that  $(8d - 1)^d \varepsilon_0 = 1/(12)$ . We further take  $n_1$  so large that  $\sum_{n \geq n_1} 2n^{-2} \leq 1/3$ . Finally, we fix

$$\mathfrak{C}_0 = \{(k, 0, \dots, 0) : 0 \leq k \leq n_1\}.$$

It is clear that there do not exist any sets  $S$  of fewer than  $n_1$  elements which separate this segment of the first coordinate axis from  $\infty$  in  $\mathbb{Z}^{d+1}$ . Thus

$$\begin{aligned} \sum_{n \geq 1} P\{\text{there exists a } \mathfrak{C}_0\text{-barrier } S \text{ with } |S| = n \text{ and} \\ \text{at least } n/12 \text{ vertices which have a parent} \\ (\mathbf{i}, k) \text{ such that } \widehat{\mathcal{B}}_p(\mathbf{i}, k) \text{ has a bad bottom}\} \\ \leq \frac{1}{3}. \end{aligned}$$

Now, for  $S$  to be a  $\mathfrak{C}_0$ -barrier, it must contain a subset of at least  $n/6$  vertices  $(\mathbf{j}, k + 1)$  which have a parent  $(\mathbf{i}, k)$  such that  $C(\mathbf{i}, k, \mathbf{j}, \lambda_0)$  occurs (see (3.20)). In view of our last estimate, it suffices for (3.23) to prove that

$$(3.95) \quad \sum_{n \geq n_1} P\{\text{there exists a } \mathbb{Z}^{d+1}\text{-connected set } S \text{ with } |S| = n, \\ \text{which separates } \mathfrak{C}_0 \text{ from } \infty \text{ on } \mathbb{Z}^{d+1} \\ \text{and which contains at least } n/12 \text{ vertices } (\mathbf{j}, k + 1) \\ \text{with a parent } (\mathbf{i}, k) \text{ such that } \widehat{\mathcal{B}}_p(\mathbf{i}, k) \\ \text{has a good bottom and } C(\mathbf{i}, k, \mathbf{j}, \lambda) \text{ occurs}\} \\ < \frac{2}{3}.$$

In this step will shall prove (3.95).

Now suppose we are given any set of vertices  $(\mathbf{i}_1, k_1), \dots, (\mathbf{i}_m, k_m)$  and further  $\mathbf{j}_r$ ,  $1 \leq r \leq m$ , such that  $(\mathbf{i}_r, k_r)$  is a parent of  $(\mathbf{j}_r, k_r + 1)$ . Assume that

$$(3.96) \quad \|\mathbf{i}_r - \mathbf{i}_s\| \geq 8d + 7 \text{ for all } r, s \text{ with } k_r = k_s.$$

We claim that then for  $p \geq p_0$  and  $\lambda = \lambda_0(p)$

$$(3.97) \quad P\{\widehat{\mathcal{B}}_p(\mathbf{i}_r, k_r) \text{ has a good bottom, but } C(\mathbf{i}_r, k_r, \mathbf{j}_r, \lambda_0) \text{ occurs} \\ \text{in } \{Y_t(\lambda_0)\} \text{ for all } 1 \leq r \leq m\} \leq [2\Delta_p^{-1}]^m.$$

Recall that  $\lambda$  is the recuperation rate and  $p$  is the parameter determining the block sizes used to define  $\tilde{A}$  and  $\tilde{B}$ ;  $\lambda_0$  was determined in Lemma 10. (3.97) is immediate from (3.74). To see this, assume without loss of generality that  $k_r \leq k_s$  for all  $1 \leq r \leq s \leq m$ . Then for  $r < s$ ,

$$(3.98) \quad C(\mathbf{i}_r, k_r, \mathbf{j}_r, \lambda) \in \mathcal{H}(\mathbf{i}_s, k_s).$$

Indeed, for  $k_r < k_s$  this follows from the fact that  $C(\mathbf{i}_r, k_r, \mathbf{j}, \lambda)$  depends on information during  $[0, t_p(k_r + 1)] \subset [0, t_p(k_s)]$  only. For  $k_r = k_s$  but  $r < s$  (3.98) follows from the fact that  $C(\mathbf{i}_r, k_r, \mathbf{j}_r, \lambda)$  depends only on information during  $[0, t_p(k_r)]$  and on particles in  $Z_p(\mathbf{i}_r)$  at time  $t_p(k_r) = t_p(k_s)$ , and  $Z_p(\mathbf{i}_r) \cap Z_p(\mathbf{i}_s) = \emptyset$  by virtue of (3.96). Thus (3.98) holds. We already remarked that also  $\{\mathcal{Z}_p(\mathbf{i}_r, k_r) \text{ is good}\} \in \mathcal{H}(\mathbf{i}_r, k_r)$ . Therefore,

$$P\{\mathcal{Z}_p(\mathbf{i}_s, k_s) \text{ is good and } C(\mathbf{i}_s, k_s, \mathbf{j}_s, \lambda_0) \text{ in } \{Y_t(\lambda_0)\} \mid \mathcal{Z}_p(\mathbf{i}_r, k_r) \text{ is} \\ \text{good and } C(\mathbf{i}_r, k_r, \mathbf{j}_r, \lambda_0) \text{ for } r < s\} \leq 2[\Delta_p]^{-1},$$

by (3.74). (3.97) follows.

The rest of the proof is routine. We already saw in Step (i) that there are at most  $n[K_{19}]^n$  possible  $\mathfrak{C}_0$ -barriers  $S$  of size  $n$ . For such a set  $S$  to have the property in (3.95), it must have a subset of at least  $n/12$  vertices  $(\mathbf{j}, k + 1)$

with a parent  $(\mathbf{i}, k)$  such that  $C(\mathbf{i}, k, \mathbf{j}, \lambda)$  occurs. There are at most  $2^n$  choices for the set of  $(\mathbf{j}, k + 1)$ , since  $S$  has only  $2^n$  subsets. When these  $(\mathbf{j}, k + 1)$  have been chosen, they have at most  $3^d n$  parents  $(\mathbf{i}, k)$ , because any vertex has at most  $3^d$  parents. A subset of these parents must have good bottoms. Thus the total number of choices for the set  $\{(\mathbf{i}_1, k_1), \dots, (\mathbf{i}_m, k_m)\}$  of pairs  $(\mathbf{i}, k)$  for which  $\widehat{\mathcal{B}}_p(\mathbf{i}, k)$  has a good bottom, while also  $C(\mathbf{i}, k, \mathbf{j}, \lambda_0)$  occurs, is at most

$$n[K_{19}2 \cdot 2^{3^d}]^n.$$

The number of parents  $(\mathbf{i}, k)$  needed so that each of  $n/12$  vertices has at least one parent among these  $(\mathbf{i}, k)$  is at least  $3^{-d}(n/12)$ . And for at least one choice of  $\mathbf{a} = (a(1), \dots, a(d))$ , the residue class  $\{i_r(s) \equiv a(s) \pmod{8d + 7}, 1 \leq s \leq d\}$  has at least  $(8d + 7)^{-d} 3^{-d} n/12$  members. By (3.97), the probability that for all these  $(\mathbf{i}, k)$   $\widehat{\mathcal{B}}_p(\mathbf{i}, k)$  has a good bottom, while  $C(\mathbf{i}, k, \mathbf{j}, \lambda_0)$  occurs, is at most

$$[2\Delta_p^{-1}]^{(8d+7)^{-d} 3^{-d} (n/12)}.$$

Thus, the  $n$ -th summand in (3.95) is at most

$$n[K_{19}2 \cdot 2^{3^d}]^n [2\Delta_p^{-1}]^{(24d+21)^{-d} (n/12)} \leq n \left[ K_{20} [\Delta_p]^{-1/(12 \cdot (24d+21)^d)} \right]^n,$$

for some constant  $K_{20}(d)$ . This shows that (3.95) holds for large enough  $p$  and completes the proof of Lemma 12. □

As pointed out in Lemma 7, (3.23) implies that  $\lambda_c > 0$ .

#### 4. The maximal number of jumps in a path

We need a few definitions to state the purpose of this section. In this section we consider only the system of  $A$ -particles and there is no interaction between any particles. Accordingly, recuperation plays no role in this section. We start as usual with the  $N_A(x, 0-)$  as i.i.d. mean  $\mu_A$  Poisson variables. Sometimes we will add one  $A$ -particle at the origin at time 0. Thus  $N_A(x, 0) = N_A(x, 0-)$  or  $N_A(x, 0) = N_A(x, 0-) + \delta(x, \mathbf{0})$ . A  $J$ -path is a space-time path  $\widehat{\pi} : [0, \infty) \rightarrow \mathbb{Z}^d \times \mathbb{R}_+$  such that at all times  $t \geq 0$ ,  $\widehat{\pi}(t)$  is the space-time position of some  $A$ -particle and such that each jump in  $\widehat{\pi}$  coincides with a jump of some  $A$ -particle. Thus, such a path at all times follows an  $A$ -particle. It may switch from following one particle to following another particle  $\rho$  at any time when it is at the same space-time point as  $\rho$ . (B. Tóth suggested that one should think of the  $A$ -particles as horses; the path always rides some horse, but may change from one horse to another when the two horses are at the same place at the same time.) The ‘ $J$ ’ in the designation of these paths is to indicate the importance of the jumps. In fact, we are interested in the following random variables:

$$(4.1) \quad j(t, \widehat{\pi}) := \text{number of jumps of } \widehat{\pi} \text{ during } [0, t],$$

and

$$(4.2) \quad J(t, x) := \sup\{j(t, \hat{\pi}) : \hat{\pi} \text{ is a } J\text{-path with } \hat{\pi}(0) = (x, 0)\}.$$

If  $x$  is unoccupied at time 0 (i.e.,  $N_A(x, 0) = 0$ ), then we take  $J(t, x) \equiv 0$ . In this section we shall show that  $J(t, x)$  is  $O(t)$  a.s. We note that this is obvious in the discrete time setting. The problem only arises in continuous time and we have only found a quite elaborate proof of this result.

PROPOSITION 13. *There exists a constant  $C_{12} < \infty$  such that, for each fixed  $x$ ,*

$$(4.3) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} J(t, x) \leq C_{12} \text{ a.s.}$$

We present the proof in Lemmas 14–23. First we recall some notation and results of [KSa].  $C_0$  is a large integer chosen as in [KSa] (6.3)–(6.5) and, as before,  $\Delta_r = C_0^{6r}$ . As in (3.81) and (3.82) we define the  $r$ -block

$$\mathcal{B}_r(\mathbf{i}, k) := \prod_{s=1}^d [i(s)\Delta_r, (i(s) + 1)\Delta_r) \times [k\Delta_r, (k + 1)\Delta_r).$$

We further define

$$V_r(\mathbf{i}) := \prod_{s=1}^d [(i(s) - 3)\Delta_r, (i(s) + 4)\Delta_r),$$

and the pedestal of  $\mathcal{B}_r(\mathbf{i}, k)$  is then

$$\mathcal{V}_r(\mathbf{i}, k) = V_r(\mathbf{i}) \times \{(k - 1)\Delta_r\}.$$

$\mathcal{Q}_r(x)$  and  $U_r(x, v)$  are as defined in (3.31) and (3.32). For want of a better term, we shall talk about good blocks and good pedestals. However, the term ‘good’ here does not have the same meaning as in the good bottoms, good blocks and good pedestals used in Section 3. In Section 3 a good object contained ‘many’ particles, whereas here a good object will be one containing ‘few’ particles. Since the definitions of Section 3 will not be used further in this paper we hope that this does not lead to confusion. Also the constants  $C_0, \gamma_i$  will be as in (6.2)–(6.5) and (5.10) of [KSa] (rather than as in [KSc]). The only property of them which is important here is that the  $\gamma_r$  now are decreasing in  $r$  and that

$$(4.4) \quad 0 < \gamma_\infty \leq \gamma_r \leq \gamma_0.$$

The  $r$ -block  $\mathcal{B}_r(\mathbf{i}, k)$  is called *good* if

$$U_r(x, v) \leq \gamma_r \mu_A C_0^{dr} \text{ for all } (x, v) \text{ for which } \mathcal{Q}_r(x) \subset V_r(\mathbf{i}) \text{ and } v \in [(k - 1)\Delta_r, (k + 1)\Delta_r).$$

Similarly, the pedestal  $\mathcal{V}_r(\mathbf{i}, k)$  is called good, if

$$U_r(x, (k - 1)\Delta_r) \leq \gamma_r \mu_A C_0^{dr} \text{ for all } x \text{ for which } \mathcal{Q}_r(x) \subset V_r(\mathbf{i}).$$

A *bad* block or pedestal is one which is not good. Finally,

$$\phi_r(\widehat{\pi}) = \phi_{r,t}(\widehat{\pi}) := \text{number of bad } r\text{-blocks which intersect } \widehat{\pi}|_{[0,t]},$$

the restriction of  $\widehat{\pi}$  to  $[0, t]$ .

For simplicity we shall think of  $t$  as fixed and abbreviate  $\widehat{\pi}|_{[0,t]}$  to  $\widehat{\pi}$  if it is clear that only the restriction of  $\widehat{\pi}$  to  $[0, t]$  plays a role. Also, we shall write  $\phi_r(\widehat{\pi})$  instead of  $\phi_{r,t}(\widehat{\pi})$ .  $\Phi_r(\ell)$  and  $\Xi(\ell, t)$  are defined exactly as in (3.90)–(3.91). These definitions and notations all agree with [KSa].  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$  for  $\mathbf{a} \in \{0, 1, \dots, 11\}^d$  and  $b = 0$  or  $1$  will mean that  $i(s) \equiv a(s) \pmod{12}$  and  $k \equiv b \pmod{2}$ .

We shall bound  $j(t, \widehat{\pi})$  by a number of sums of the form

$$(4.5) \quad \sum_{r \geq 1} \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} M(r, \mathbf{i}, k) I(\widehat{\pi}, r, \mathbf{i}, k),$$

where  $\sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)}$  is a sum over all  $(\mathbf{i}, k)$  for which  $\mathcal{B}_r(\mathbf{i}, k)$  is a good  $r$ -block which intersects  $\widehat{\pi}|_{[0,t]}$ ;  $I(\widehat{\pi}, r, \mathbf{i}, k)$  is the indicator function of some event, and several different choices for  $M$  and  $I$  will be made below. Let  $C_1$  be the constant in Theorem 1 in [KSc] and let  $H_1, H_2$  be the events

$$H_1(t) := \{\text{all } J\text{-paths starting at } (\mathbf{0}, 0) \text{ stay in } \mathcal{C}(C_1 t) \text{ during } [0, t]\}$$

and, for  $\widehat{\pi} \in \Xi(\ell, t)$ ,

$$(4.6) \quad H_2(\widehat{\pi}, r) := \left\{ \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} I(\widehat{\pi}, r, \mathbf{i}, k) \leq \varepsilon_r(t + \ell) \right\}$$

for some small  $\varepsilon_r$ . As we shall see, we shall be able to get a bound on  $P\{[H_1 \cap H_2]^c\}$  in several cases. Finally, it will be the case in our applications that for fixed  $r \geq 1$  we can define nonrandom collections  $\mathcal{S}(\mathbf{a}, b)$  of  $(\mathbf{i}, k)$  with  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$  and a collection of random variables  $\{\widetilde{M}(r, \mathbf{i}, k) : (\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)\}$  with the following properties (with  $\Theta$  as in (4.15) below):

$$(4.7) \quad \begin{aligned} &\text{for } \widehat{\pi} \in \Xi(\ell, t), \text{ on the event } \Theta(t) \cap H_2(\widehat{\pi}, r), \\ &\text{any } \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} |M(r, \mathbf{i}, k)| I(\widehat{\pi}, r, \mathbf{i}, k) \text{ is bounded by} \\ &2 \cdot (12)^d \sum_{(\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)} \widetilde{M}(r, \mathbf{i}, k) \\ &\text{for one of the possible } \mathcal{S}(\mathbf{a}, b), \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} &\text{the } \{\widetilde{M}(r, \mathbf{i}, k) : (\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)\} \text{ are independent, and} \\ &\text{satisfy } \widetilde{M}(r, \mathbf{i}, k) \geq |M(r, \mathbf{i}, k)| \text{ and } E \exp[\theta_r \widetilde{M}(r, \mathbf{i}, k)] \leq \Gamma_r \\ &\text{for some constants } \theta_r > 0 \text{ and } 1 \leq \Gamma_r < \infty. \end{aligned}$$

Lemma 15 shows how to estimate the double sum (4.5) in such a situation, but first we need some information on the location of  $J$ -paths starting at the origin.

Even though  $J(t, x)$  does not involve  $B$ -particles, we shall make use of  $B$ -particles in the proof of Lemma 20. Also in the proof of that lemma, we shall need to consider initial conditions which are not of the form of i.i.d. Poisson variables  $N_A(x, 0-)$  plus some extra particles at time 0. We therefore do not make this assumption in the next lemma. In particular, we only assume that the  $\{Y_t\}$ -process (which has no recuperation) is formed by adding one  $B$ -particle at the origin at time 0, and that this process is coupled with the  $A$ -system by giving the same path to each  $A$ -particle present at time 0- in this process as in the  $A$ -system. (This is exactly as in Section 2.) In addition, the initial  $N_A(x, 0-)$  have to be such that  $Y_0 \in \Sigma_0$  a.s.  $\Sigma_0$  is the state space introduced in [KSc], [KSb]. All particles still perform independent continuous time simple random walks.

LEMMA 14. *Under the conditions just described we have for  $x \in \mathbb{Z}^d$  and  $s \geq 0$*

$$(4.9) \quad \begin{aligned} &\{ \text{there is a } J\text{-path from } (\mathbf{0}, 0) \text{ through } (x, s) \} \\ &\quad \subset \{ \text{there is a } B\text{-particle at } (x, s) \text{ in the } \{Y_t\}\text{-process} \}. \end{aligned}$$

*In particular,*

$$(4.10) \quad \begin{aligned} P\{[H_1(s)]^c\} &= P\{\text{some } J\text{-path starting at } (\mathbf{0}, 0) \\ &\quad \text{leaves } \mathcal{C}(C_1s) \text{ during } [0, s]\} \\ &\leq 2P\{\text{some } J\text{-path which started at } (\mathbf{0}, 0) \\ &\quad \text{is outside } \mathcal{C}(C_1s) \text{ at time } s\} \\ &\leq 2E\{(\text{number of } B\text{-particles outside } \mathcal{C}(C_1s) \\ &\quad \text{at time } s, \text{ in the } \{Y_t\}\text{-process})\}. \end{aligned}$$

*Proof.* Clearly adding an  $A$ -particle to the  $A$ -system can only increase the collection of  $J$ -paths, so that we may assume that we start the  $A$ -system with  $N_A(x, 0) = N_A(x, 0-) + \delta(x, \mathbf{0})$ . (We repeat that the  $N_A(x, 0-)$  do not have to be i.i.d. Poisson variables in this lemma.) We can then couple the  $A$ -system and the  $\{Y_t\}$ -process so that they have the same particles and so that each



particle follows the same path in both processes. The only difference between the processes is that in the  $A$ -system all particles have type  $A$ , while in  $\{Y_t\}$  there are particles of both types.

Now assume that there is a  $J$ -path  $\hat{\pi}$  in the  $A$ -system from  $(\mathbf{0}, 0)$  to  $(x, s)$ . Then there exists some sequence of times  $s_0 = 0 < s_1 < \dots < s_\ell < s$  and particles  $\rho_i$  such that  $\hat{\pi}$  agrees with the path of  $\rho_i$  during  $[s_i, s_{i+1}]$ ,  $0 \leq i \leq \ell$  (with  $s_{\ell+1} = s$ ). In addition  $\rho_{i+1}$  and  $\rho_i$  are at the same position at time  $s_i$ , while  $\rho_0$  starts at  $(\mathbf{0}, 0)$  and  $\rho_\ell$  is at  $x$  at time  $s$ . Now in the  $\{Y_t\}$ -process all particles at  $\mathbf{0}$  are given type  $B$  at time  $s_0 = 0$ . But then  $\rho_0$  has type  $B$  for all  $t \geq 0$ . Then  $\rho_1$  will have type  $B$  at least from time  $s_1$  on. One then sees by induction on  $i$  that  $\rho_{i+1}$  has type  $B$  on  $[s_i, \infty)$ . In particular,  $\rho_\ell$  has type  $B$  at the time  $s > s_\ell$ , at which time it is at  $x$ . This implies (4.9).

Next, we have in the  $A$ -system

$$(4.11) \quad \begin{aligned} &P\{\text{some } J\text{-path starting at } (\mathbf{0}, 0) \text{ leaves } \mathcal{C}(C_1 s) \text{ during } [0, s]\} \\ &\leq 2P\{\text{some } J\text{-path starting at } (\mathbf{0}, 0) \text{ is outside } \mathcal{C}(C_1 s) \text{ at time } s\}. \end{aligned}$$

This follows from a reflection argument, as in the proof of Proposition 3 in [KSc]. The last inequality in (4.10) then follows from (4.9). □

We now return to the usual initial conditions, that is we take the  $\{N_A(x, 0-) : x \in \mathbb{Z}^d\}$  as i.i.d., mean  $\mu_A$  Poisson variables. We also add an extra particle to the system at the origin at time 0. We note that Proposition 5 and Remark 2 after it in [KSc] show that in this case  $Y_0 \in \Sigma_0$  a.s., so that we can apply Lemma 14 in this case. If the  $N_A(x, 0-)$  are i.i.d. Poisson variables, then (4.10), together with (1.3) in [KSc], shows that for all large  $t$

$$(4.12) \quad P\{\text{some } J\text{-path starting at } (\mathbf{0}, 0) \text{ leaves } \mathcal{C}(C_1 t) \text{ during } [0, t]\} \leq 2e^{-t}.$$

This will allow us to restrict our further estimates to  $J$ -paths  $\hat{\pi}$  which stay in  $\mathcal{C}(C_1 t)$  during  $[0, t]$ . If we take  $t$  so large that  $C_1 t \leq t \log t$ , then these paths also stay in  $\mathcal{C}(t \log t)$  and therefore belong to  $\bigcup_{\ell \geq 0} \Xi(\ell, t)$  (see (3.91) for  $\Xi$ ). This explains why the next few lemmas speak about such paths only. In fact, it is useful to make a further reduction. To this end, we define, as in (6.10) of [KSa],  $R = R(t)$  as the integer for which

$$(4.13) \quad C_0^R \geq [\log t]^{1/d} > C_0^{R-1}.$$

We then have just as in Lemma 9 of [KSa] that

$$(4.14) \quad \begin{aligned} &P\{\text{for some } r \geq R \text{ there exists a bad } r\text{-block} \\ &\quad \text{which intersects } \mathcal{C}(t \log t)\} \leq \frac{1}{t^2}. \end{aligned}$$

Accordingly we define the event

$$(4.15) \quad \Theta(t) = \{\text{for all } r \geq R(t) \text{ no bad } r\text{-block intersects } \mathcal{C}(t \log t)\}.$$

We then have

$$(4.16) \quad P\{\Theta(t)^c\} \leq \frac{1}{t^2},$$

so that we can restrict most estimates to subevents of  $\Theta(t)$ . Since we are only concerned with the existence of certain  $J$ -paths it is convenient to define

$$(4.17) \quad \Xi(J, \ell, t) := \text{the collection of } J\text{-paths in } \Xi(\ell, t).$$

Many constants  $K_i, R_i$  and  $t_i$  appear in the remainder of this section. It is crucial that these *do not depend on  $t$  or  $\ell$* , even though we usually do not state this explicitly.

LEMMA 15. *Let  $M$  and  $I$  be such that (4.7) and (4.8) hold. Then, there exist constants  $0 < K_1 - K_3 < \infty$ , all depending on  $d$  only, such that for all  $t \geq 2$  and for each  $\ell \geq 0, r \geq 1$*

$$(4.18) \quad \begin{aligned} &P\{\Theta(t) \text{ and for some } \hat{\pi} \in \Xi(J, \ell, t), \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M(r, \mathbf{i}, k) I(\hat{\pi}, r, \mathbf{i}, k) \geq x\} \\ &\leq P\{\text{for some } \hat{\pi} \in \Xi(J, \ell, t), \Theta(t) \cap [H_2(\hat{\pi}, r)]^c \text{ occurs}\} \\ &\quad + K_1 [t \log t]^d \exp [K_2(t + \ell) / \Delta_r] \exp \left[ -\frac{x\theta_r}{2(12)^d} + \varepsilon_r(t + \ell) \log \Gamma_r \right]. \end{aligned}$$

For

$$(4.19) \quad x \geq \frac{4(12)^d}{\theta_r} \left[ \varepsilon_r \log \Gamma_r + \frac{K_2}{\Delta_r} \right] (t + \ell)$$

this yields

$$(4.20) \quad \begin{aligned} &P\{\Theta(t) \text{ and for some } \hat{\pi} \in \Xi(J, \ell, t), \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M(r, \mathbf{i}, k) I(\hat{\pi}, r, \mathbf{i}, k) \geq x\} \\ &\leq P\{\text{for some } \hat{\pi} \in \Xi(J, \ell, t), \Theta(t) \cap [H_2(\hat{\pi}, r)]^c \text{ occurs}\} \\ &\quad + K_1 [t \log t]^d \exp[-K_3 x \theta_r]. \end{aligned}$$

*Proof.* The first term in the right hand side of (4.18) takes care of the event that  $H_2(\hat{\pi}, r)$  fails for any  $\hat{\pi}$ . It therefore suffices for (4.18) to estimate

$$(4.21) \quad \begin{aligned} &P\{\text{for some } \hat{\pi} \in \Xi(J, \ell, t), H_2(\hat{\pi}, r) \text{ occurs and} \\ &\quad \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M(r, \mathbf{i}, k) I(\hat{\pi}, r, \mathbf{i}, k) \geq x\}. \end{aligned}$$

If the event here occurs, then there is a  $\hat{\pi} \in \Xi(\ell, t)$  and a subset,  $\mathcal{S}$  say, of the points  $(\mathbf{i}, k)$  for which  $\mathcal{B}_r(\mathbf{i}, k)$  intersects  $\hat{\pi}$ , such that  $I(\hat{\pi}, r, \mathbf{i}, k) = 1$  for

$(\mathbf{i}, k) \in \mathcal{S}$  and

$$\sum_{(\mathbf{i}, k) \in \mathcal{S}} |M(r, i, k)| \geq x.$$

Moreover,  $\mathcal{S}$  contains at most  $\varepsilon_r(t + \ell)$  points (because  $H_2(\widehat{\pi}, r)$  occurs). We can split  $\mathcal{S}$  into the  $2(12)^d$  subsets

$$\mathcal{S}(\mathbf{a}, b) = \text{collection of } (\mathbf{i}, k) \text{ in } \mathcal{S} \text{ with } (\mathbf{i}, k) \equiv (\mathbf{a}, b),$$

with  $\mathbf{a} \in \{0, 1, \dots, 11\}^d, b = 0$  or  $1$ . Then (4.21) is bounded by the sum of

$$(4.22) \quad P\left\{ \sum_{(\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)} M(r, i, k) \geq \frac{x}{2(12)^d} \right\}$$

over all possible  $\mathcal{S}(\mathbf{a}, b)$  corresponding to some  $\widehat{\pi} \in \Xi(\ell, t)$ .

We know that any  $\widehat{\pi} \in \Xi(\ell, t)$  intersects at most

$$(4.23) \quad \lambda_r(\ell) := 3^d \left( \frac{t + \ell}{\Delta_r} + 2 \right)$$

$r$ -blocks (see (6.30) in [KSa] for  $\nu = 1$  and with  $r + 1$  replaced by  $r$ ). The set of  $(\mathbf{i}, k)$  for which  $\mathcal{B}_r(\mathbf{i}, k)$  intersects  $\widehat{\pi}$  has to be  $\mathcal{L}$ -connected (see the lines following (3.2) for  $\mathcal{L}$ ). Thus, as  $\widehat{\pi}$  varies over  $\Xi(\ell, t)$ , and the starting point of  $\widehat{\pi}$  varies over  $\mathcal{C}(t \log t)$ , there are at most  $[2t \log t + 1]^d \exp[K_9 \lambda_r(\ell)]$  different possibilities for the collections  $\{(\mathbf{i}, k) : \mathcal{B}_r(\mathbf{i}, k) \text{ intersects } \widehat{\pi}\}$ . Here  $K_9$  is some constant which depends on  $d$  only. Each  $\mathcal{S}$  has to be a subset of this collection, and once  $\mathcal{S}$  is given there are  $2 \cdot 12^d$  possibilities for  $(\mathbf{a}, b)$ . Thus, there are at most

$$(4.24) \quad 2(12)^d 2^{\lambda_r(\ell)} [2t \log t + 1]^d \exp[K_9 \lambda_r(\ell)]$$

possibilities for  $\mathcal{S}(\mathbf{a}, b)$ .

Finally, for a fixed choice of  $\mathcal{S}(\mathbf{a}, b)$  we have by (4.8) that the probability in (4.22) is bounded by

$$(4.25) \quad \begin{aligned} P\left\{ \sum_{(\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)} \widetilde{M}(r, \mathbf{i}, k) \geq \frac{x}{2(12)^d} \right\} \\ \leq \exp\left[-\frac{x\theta_r}{2(12)^d}\right] E\left\{ \exp\left[\theta_r \sum_{(\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)} \widetilde{M}(r, \mathbf{i}, k)\right] \right\} \\ \leq \exp\left[-\frac{x\theta_r}{2(12)^d}\right] \Gamma_r^{\varepsilon_r(t+\ell)}, \end{aligned}$$

where, for the last inequality, we used that  $\mathcal{S}(\mathbf{a}, b)$  has at most  $\varepsilon_r(t + \ell)$  elements on  $H_2(\widehat{\pi}, r)$  and (4.8). This implies (4.18) for suitable  $K_1, K_2$ , because (4.21) is bounded by a sum of at most (4.24) terms, each of which is bounded by (4.25).

The inequality (4.20) now follows from (4.18) and (4.19), because the latter implies that

$$\frac{x\theta_r}{2(12)^d} \geq \frac{x\theta_r}{4(12)^d} + [\varepsilon_r \log \Gamma_r + \frac{K_2}{\Delta_r}](t + \ell). \quad \square$$

Our first task is now to establish a representation for  $j(t, \hat{\pi})$  of the form (4.5), at least outside an event of small probability. Fix some  $R_1 \geq 1$  and consider a sample point for which  $\Theta(t)$  occurs. If  $\hat{\pi}$  is a  $J$ -path which stays in  $\mathcal{C}(t \log t)$  during  $[0, t]$ , then all  $r$ -blocks with  $r \geq R$  which intersect  $\hat{\pi}|_{[0, t]}$  must be good. Here  $R = R(t)$  as defined in (4.13). Now recall that for each  $r$ , each point of space-time belongs to a unique  $r$ -block  $\mathcal{B}_r(\mathbf{i}, k)$ . We shall say that a jump in  $\hat{\pi}$  from  $x$  to  $y$  at time  $s$  is *located at*  $(x, s)$ . For such a jump, either  $(x, s)$  belongs to a good  $r$ -block for all  $r \geq R_1$ , or there is a unique  $r(x, s) \in (R_1, R]$  such that  $(x, s)$  belongs to a good  $r$ -block for  $r \geq r(x, s)$ , but belongs to a bad  $[r(x, s) - 1]$ -block. In the former case we set  $r(x, s) = R_1$ . Note that for any jump  $(x, s)$ ,  $r(x, s)$  is defined and the jump lies in some  $r(x, s)$ -block. Moreover, this is a good block, by the choice of  $r(x, s)$ .

We have for  $t \geq$  some  $t_1$  that outside the event in (4.12) but in  $\Theta(t)$ , it holds

$$(4.26) \quad J(t, \mathbf{0}) = \sup_{\hat{\pi}(0)=\mathbf{0}} \sum_{r=R_1}^{R(t)} [\text{number of jumps } (x, s) \text{ of } \hat{\pi} \text{ with } (x, s) \in \mathcal{C}(t \log t) \times [0, t] \text{ and } r(x, s) = r]$$

(the sup here is over the same set as in (4.2)). The union of the exceptional event in (4.12) and  $[\Theta(t)]^c$  has probability at most  $2/t^2$ . We can ignore these exceptional events here. We now concentrate on estimating the summands appearing in the right hand side in (4.26). Let  $(x, s) \in \mathcal{B}_r(\mathbf{i}, k)$  be a jump of  $\hat{\pi}$ . This jump is the jump of some particle  $\rho$  at time  $s$ . We distinguish two kinds of jumps, according as  $\rho$  was outside or inside the pedestal  $\mathcal{V}_r(\mathbf{i}, k)$  at time  $(k - 1)\Delta_r$ . We define the corresponding quantity

$$M_{\text{out}}(r, \mathbf{i}, k) = [\text{number of jumps located inside } \mathcal{B}_r(\mathbf{i}, k) \text{ by any particle } \rho \text{ that was outside } \mathcal{V}_r(\mathbf{i}, k) \text{ at time } (k - 1)\Delta_r],$$

and its analogue  $M_{\text{in}}(r, \mathbf{i}, k)$  with “outside” replaced by “inside”. We further say that the block  $\mathcal{B}_r(\mathbf{i}, k)$  is *contaminated* if it contains a jump of a particle which was outside  $\mathcal{V}_r(\mathbf{i}, k)$  at time  $(k - 1)\Delta_r$  and take

$$I_1(r, \mathbf{i}, k) := I[\mathcal{B}_r(\mathbf{i}, k) \text{ is contaminated}].$$

We point out that this definition of contaminated is somewhat stricter than the one used in [KSa] (just after (6.9)).

We now start with a bound for

$$(4.27) \quad \sum_{(i,k)}^{(\hat{\pi},r)} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k).$$

LEMMA 16. *There exist constants  $K_i, t_2$  and  $R_2$  such that for  $t \geq t_2, R_2 \leq r \leq R(t)$  and  $\ell \geq 0$ ,*

$$(4.28) \quad P\{\Theta(t) \text{ and there exists a } \hat{\pi} \in \Xi(J, \ell, t) \text{ such that}$$

$$\begin{aligned} & \sum_{(i,k)}^{(\hat{\pi},r)} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k) \geq \frac{K_4(t + \ell)}{\Delta_r} \} \\ & \leq K_5 \exp [ - K_6(t + \ell) / [\log t]^6 ]. \end{aligned}$$

*Proof.* We break the proof up into two steps.

*Step (i).* In this step we reduce the calculations to some calculations for discrete time random walks. This first step is standard weak convergence theory and we leave many details to the reader. We approximate the paths of the various particles by some random walk paths which can jump only at times  $j/n$  for some integer  $n \geq 1$  and  $j = 0, 1, 2, \dots$ . Specifically, we let  $\{S_u^{(n)}\}_{u \geq 0}$  be a random walk starting at  $\mathbf{0}$  which can jump only at times  $j/n$ , with the jump distribution

$$q^{(n)}(y) = P\{S_{(j+1)/n}^{(n)} - S_{j/n}^{(n)} = y\} = \begin{cases} 1 - \frac{D}{n} & \text{if } y = \mathbf{0} \\ \frac{D}{2dn} & \text{if } y = \pm e_i, 1 \leq i \leq d, \end{cases}$$

( $e_i$  is the  $i$ -th coordinate vector). For each particle  $\rho$  we take  $\{S_u^{(n)}(\rho)\}_{u \geq 0}$  as a copy of  $\{S_u^{(n)}\}_{u \geq 0}$ , and we take the walks for the different  $\rho$  as completely independent. We then form what we shall call the  $(n)$ -system by letting  $\rho$  move along the path  $t \mapsto \pi^{(n)}(t, \rho) := \pi(0, \rho) + S_{[tn]/n}^{(n)}(\rho)$  for each of the particles  $\rho$ . Now it easy to see that for any finite collection of particles  $(\rho_{i_1}, \dots, \rho_{i_K})$ , the  $K$  dimensional process  $t \mapsto (\pi^{(n)}(t, \rho_{i_1}), \dots, \pi^{(n)}(t, \rho_{i_K}))$  converges weakly (in the Skorokhod topology on the space  $D([0, \infty), (\mathbb{Z}^d)^K)$  to the process  $t \mapsto (\pi(t, \rho_{i_1}), \dots, \pi(t, \rho_{i_K}))$ . This last process is the process of the true paths of  $(\rho_{i_1}, \dots, \rho_{i_K})$ . A simple way to prove this weak convergence is to apply Theorem 15.6 in [B] (or rather the line following it before the proof of Theorem 15.6). We then define the obvious analogue of  $N^*$  (see (3.29) and preceding lines for  $N^*$ ), namely

$$\begin{aligned} N^{(n)}(x, t) &= (\text{number of particles at } (x, t) \text{ in the } (n)\text{-system}) \\ &= (\text{number of } \rho \text{ with } \pi^{(n)}(t, \rho) = x). \end{aligned}$$

Here we do not include the extra particle added at the origin at time 0; we only include the particles which were among the  $N_A(x, 0-)$  at some  $x$ , just before the start of our system. We also need an approximation to  $N^{(n)}$  which

only counts particles which started in some finite cube. For this we fix some numbering of the particles  $\rho_1, \rho_2, \dots$ . Again, this excludes the extra particle added at  $(\mathbf{0}, 0)$  if there is such a particle. We then set

$$N^{(n,L)}(x, t) := (\text{number of } i \leq L \text{ with } \pi^{(n)}(t, \rho_i) = x).$$

It is convenient to set  $N^{(\infty)}(x, t) = N^*(x, t)$  and similarly

$$N^{(\infty,L)}(x, t) = (\text{number of } \rho \text{ among the first } L \text{ particles with } \pi(t, \rho) = x).$$

Particles which start far out only have a small probability of reaching  $\mathcal{C}(2t \log t)$  during  $[0, 2t]$ . In fact, estimates like the ones for (2.29)–(2.32) in [KSc] prove that for all  $t > 0$  and  $\eta > 0$  there exists an  $L_0 = L_0(t, \eta)$  such that

$$(4.29) \quad P\{N^{(n,L)}(x, s) \neq N^{(n)}(x, s) \text{ for some } (x, s) \in \mathcal{C}(2t \log t) \times [0, 2t]\} \leq \eta, \quad \text{for all } L \geq L_0, 1 \leq n \leq \infty.$$

Note the uniformity in  $n$  here as well as the fact that  $n = \infty$  is permitted in (4.29). We can now replace  $N^*$  by  $N^{(n)}$  or  $N^{(n,L)}$  in many of the definitions. We indicate such a replacement by decorating the appropriate quantity with a superscript  $(n)$  or  $(n, L)$  in a self explanatory fashion, or by adding the qualification “in the  $(n)$  system or  $(n, L)$ -system.” For instance,

$$U_r^{(n,L)}(x, v) := \sum_{y \in \mathcal{Q}_r(x)} N^{(n,L)}(y, v)$$

and the block  $\mathcal{B}_r(\mathbf{i}, k)$  is good in the  $(n, L)$ -system if

$$U_r^{(n,L)}(x, v) \leq \gamma_r \mu_A C_0^{dr} \text{ for all } (x, v) \text{ for which } \mathcal{Q}_r(x) \subset V_r(\mathbf{i}) \text{ and } v \in [(k-1)\Delta_r, (k+1)\Delta_r].$$

(4.29) immediately implies that uniformly in  $1 \leq n \leq \infty$

$$(4.30) \quad P\left\{ \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}^{(n,L)}(r, \mathbf{i}, k) I_1^{(n,L)}(r, \mathbf{i}, k) \neq \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}^{(n)}(r, \mathbf{i}, k) I_1^{(n)}(r, \mathbf{i}, k) \right\} \leq \eta$$

for  $L \geq L_0(t, \eta)$ , provided  $t$  is so large that any  $r$ -block which intersects  $\mathcal{C}(t \log t) \times [0, t]$  is contained in  $\mathcal{C}(2t \log t) \times [0, 2t]$ . (It suffices for this last proviso that  $3\Delta_r = 3C_0^{6r} \leq t$ .) Note that the sum over  $(\mathbf{i}, k)$  runs over those  $(\mathbf{i}, k)$  for which  $\mathcal{B}_r(\mathbf{i}, k)$  is a good  $r$ -block in the full system (and not in the  $(\infty, L)$ -system) which intersect  $\hat{\pi}|_{[0, t]}$ ; cf. (4.5).

Next we claim that for each fixed finite  $L, r$  and fixed finite set  $\mathcal{S}$  of pairs  $(\mathbf{i}, k)$ , as  $n \rightarrow \infty$ ,

$$(4.31) \quad M_{\text{out}}^{(n,L)}(r, \mathbf{i}, k) \text{ converges weakly to } M_{\text{out}}^{(\infty,L)}(r, \mathbf{i}, k)$$

and

$$(4.32) \quad \sum_{(\mathbf{i},k) \in \mathcal{S}} M_{\text{out}}^{(n,L)}(r, \mathbf{i}, k) I_1^{(n,L)}(r, \mathbf{i}, k) \text{ converges weakly} \\ \text{to } \sum_{(\mathbf{i},k) \in \mathcal{S}} M_{\text{out}}^{(\infty,L)}(r, \mathbf{i}, k) I_1^{(\infty,L)}(r, \mathbf{i}, k).$$

This is an immediate application of the continuous mapping theorem (Theorem 5.1 or 5.2) in [B]). Indeed, in any system of  $L$  moving particles with joint paths  $s \mapsto (\pi_1(s), \dots, \pi_L(s)) \in (\mathbb{Z}^d)^L$  we can define  $\bar{U}_r(x, v)$  for  $v \leq t$  as a functional of these paths by

$$\bar{U}_r(x, v) = (\text{number of } i \in [1, L] \text{ with } \pi_i(v) \in \mathcal{Q}_r(x)).$$

We restrict ourselves to paths  $\pi_i$  which are right continuous with left limits, so that we view  $\bar{U}_r(x, v)$  as a functional on the Skorokhod space  $D([0, t], (\mathbb{Z}^d)^L)$ , and we put the Skorokhod topology on this space. Then  $U_r^{(n,L)}(x, v)$  is just the value of  $\bar{U}_r(x, v)$  at the point with  $\pi_i(\cdot) = \pi^{(n,L)}(\cdot, \rho_i)$ . In a similar way we can view  $I[\mathcal{B}_r(\mathbf{i}, k) \text{ is good}]$ ,  $M_{\text{out}}(r, \mathbf{i}, k)$  and  $I_1(r, \mathbf{i}, k)$  as the value at  $\pi_i(\cdot) = \pi^{(n,L)}(\cdot, \rho_i)$  of suitable functionals on  $D([0, t], (\mathbb{Z}^d)^L)$ . We indicate these functionals on  $D([0, t], (\mathbb{Z}^d)^L)$  by a bar over the appropriate symbol. Now it is not hard to see that

$$\bar{I}[\mathcal{B}_r(\mathbf{i}, k) \text{ is good}] \\ = I[\sup\{\bar{U}_r(x, v) : \mathcal{Q}_r(x) \subset V_r(\mathbf{i}), (k-1)\Delta_r \leq v < (k+1)\Delta_r\} \leq \gamma_r \mu_A C_0^{dr}]$$

are continuous functionals on  $D([0, t], (\mathbb{Z}^d)^L)$  at all points  $(\pi_1(\cdot), \dots, \pi_L(\cdot))$  for which each  $\pi_i$  is continuous at each  $\{j\Delta_r : j \in \mathbb{Z}\}$ . In particular, this holds almost surely at the points with  $\pi_i(\cdot) = \pi(\cdot, \rho_i)$ . Similarly  $\bar{M}_{\text{out}}(r, \mathbf{i}, k)$  is continuous at these same points. Therefore, (4.31) and (4.32) indeed follow from the continuous mapping theorem.

Finally we note that the event in the left hand side of (4.28) occurs if and only if

$$(4.33) \quad \sum_{(\mathbf{i},k) \in \mathcal{S}} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k) \geq \frac{K_4(t + \ell)}{\Delta_r},$$

for one of a number of possible collections  $\mathcal{S}$  of pairs  $(\mathbf{i}, k)$ . The possible collections  $\mathcal{S}$  are the collections of the form  $\{(\mathbf{i}, k) : \mathcal{B}_r(\mathbf{i}, k) \text{ is good and intersects } \hat{\pi}\}$ , for some  $\hat{\pi} \in \Xi(J, \ell, t)$ . The number of possibilities for  $\mathcal{S}$  is finite, and whether  $\mathcal{S}$  is a possible collection depends on the class  $\Xi(J, \ell, t)$  and on which blocks  $\mathcal{B}_r(\mathbf{i}, k)$  are good. The indicator function of  $\{\mathcal{S} \text{ is possible collection}\}$  for a fixed collection  $\mathcal{S}$  is also a.s. a continuous functional on  $D([0, t], (\mathbb{Z}^d)^L)$ . We can now combine this observation with (4.30) and (4.32) to obtain the conclusion of this step that the left hand side of (4.28) is bounded

by

$$\limsup_{L \rightarrow \infty} \lim_{n \rightarrow \infty} P\{\text{there exists a } \hat{\pi} \in \Xi(J, \ell, t) \text{ such that}$$

$$\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}^{(n, L)}(r, \mathbf{i}, k) I_1^{(n, L)}(r, \mathbf{i}, k) \geq \frac{K_4(t + \ell)}{\Delta_r} - 1\}.$$

In fact, since the collection of particles present in the  $(n, L)$ -system increases to the collection of particles in the  $(n)$ -system as  $L \rightarrow \infty$ , this expression is bounded by

$$(4.34) \quad \lim_{n \rightarrow \infty} P\{\text{there exists a } \hat{\pi} \in \Xi(J, \ell, t) \text{ such that}$$

$$\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}^{(n)}(r, \mathbf{i}, k) I_1^{(n)}(r, \mathbf{i}, k) \geq \frac{K_4(t + \ell)}{\Delta_r} - 1\}.$$

Note that by the weak convergence arguments just after (4.32), the sum  $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)}$  here may be taken over the  $(\mathbf{i}, k)$  for which  $\mathcal{B}_r(\mathbf{i}, k)$  is a good  $r$ -block in the  $(n)$ -system which intersects  $\hat{\pi}|_{[0, t]}$ . At a few places we shall write  $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r, n)}$  to indicate that we are summing over the good blocks in the  $(n)$ -system.

*Step (ii).* In this step we derive a bound for (4.34) in terms of a large number of independent copies of the random walk  $\{S_u^{(n)}\}_{u \geq 0}$ . We follow the proof of Lemmas 10 and 11 in [KSa] closely.

We take for  $\{S_u^{(n)}(x, s, q)\}_{u \geq 0}$  a copy of  $\{S_u^{(n)}\}_{u \geq 0}$  and take all these copies for different  $x \in \mathbb{Z}^d, s$  of the form  $k/n$  and  $q \geq 1$ , completely independent. We further associate to each particle  $\rho$  a uniform random variable on  $[0, 1]$ ,  $U(\rho)$  say, and all  $U(\rho)$  and  $\{S_u^{(n)}(x, s, q)\}$  are independent. Finally

$$(4.35) \quad \mathcal{W}_r(\mathbf{i}) := \partial \prod_{s=1}^d [(i(s) - 3)\Delta_r, (i(s) + 4)\Delta_r - 1] = \partial V_r(\mathbf{i}),$$

where  $\partial$  denotes the topological boundary. We now fix some  $\mathbf{a} \in \{1, 2, \dots, 11\}^d$  and  $b = 0$  or  $1$ , and we want to look at the contribution to the sum (4.27) from the  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$ . For the sake of argument let  $b = 0$ . Assume that the paths of all  $A$ -particles till time  $(k - 1)\Delta_r$  with  $k$  even have already been constructed in some way. In the case  $k = 0$  this simply means that we begin with a mean  $\mu_A$  Poisson system of  $A$ -particles at time  $-\Delta_r$ . (The only change which is needed for the case  $b = 1$  is that we work with odd  $k$ 's and start with a Poisson system at time  $-2\Delta_r$  in that case.) At each point  $(x, (k - 1)\Delta_r)$  (in space-time) order all particles  $\rho$  present so that their associated uniform variables  $U(\rho)$  are increasing. To the  $q$ -th particle in this order associate the path  $\{x + S_u^{(n)}(x, (k - 1)\Delta_r, q)\}_{u \geq 0}$ . This particle then moves to  $x + S_{1/n}^{(n)}(x, (k - 1)\Delta_r, q)$  at time  $(k - 1)\Delta_r + 1/n$ . We also associate to each



particle at each time an *index*  $(y', v', q', g')$ . A particle has index  $(y', v', q', g')$  at a certain time if its last associated random walk is  $S^{(n)}(y', v', q')$  and if the particle has moved  $g'$  steps (or  $g'/n$  time units) according to  $S^{(n)}(y', v', q')$  since this random walk was associated to the particle. Accordingly, the index associated to the  $q'$ -th particle at  $(x, (k - 1)\Delta_r)$  at time  $(k - 1)\Delta_r + 1/n$  is  $(x, (k - 1)\Delta_r, q', 1)$ . Assume we have constructed the paths of all particles up to and including time  $v \in [(k - 1)\Delta_r, (k + 1)\Delta_r)$  (with  $v$  a multiple of  $1/n$ ) and that each particle has an index. To construct the paths  $1/n$  time units further, we look for each  $y \in \mathbb{Z}^d$  at all particles at  $(y, v)$ . If  $y$  does not belong to

$$(4.36) \quad \bigcup_{\mathbf{j} \equiv \mathbf{a}} \mathcal{W}_r(\mathbf{j}),$$

and a particle at  $(y, v)$  has index  $(z, v', q, g)$ , then this particle moves to  $y + S_{(g+1)/n}^{(n)}(z, v', q) - S_{g/n}^{(n)}(z, v', q) = z + S_{(g+1)/n}^{(n)}(z, v', q)$  and its new index is  $(z, v', q, g + 1)$ . In other words it moves one step further in the random walk it is presently associated with, and the last component of its index increases by 1. If, on the other hand,  $y$  lies in the union (4.36), then all particles at  $y$  are again ranked according to increasing values of their uniform random variables and a new random walk is associated to these particles. The particle with rank  $q'$  will move to  $y + S_{1/n}^{(n)}(y, v, q')$  at time  $v + 1/n$ . Its index will then be  $(y, v, q', 1)$ . We continue this procedure till all positions at time  $(k + 1)\Delta_r$  have been determined. We then start anew with  $k$  replaced by  $k + 1$ . That is, we order all particles at one site  $(x, (k + 1)\Delta_r)$  and move the  $q$ -th particle at that site to  $x + S_{1/n}^{(n)}(x, (k + 1)\Delta_r, q)$  and give it the index  $(x, (k + 1)\Delta_r, q, 1)$ , and so on.

Basically, the above procedure switches each particle to a new random walk every time the particle visits the set (4.36). It is clear that in the above construction all the  $A$ -particles perform independent random walks with transition probability  $q_A^n$ . Now, a particle  $\rho$  whose jumps contribute to one of the sums (4.33) has to lie outside  $\mathcal{V}_r(\mathbf{i}, k)$  at time  $(k - 1)\Delta_r$ , but has some jump in  $\mathcal{B}_r(\mathbf{i}, k)$  during  $[k\Delta_r, (k + 1)\Delta_r)$ . Its space-time path in the discrete time system must contain a piece  $(x_\xi, v), (x_{\xi+1}, v + 1/n), \dots, (x_\zeta, v + (\zeta - \xi)/n)$  with  $v$  a multiple of  $1/n$ , which satisfies

$$(4.37) \quad x_\xi \in \mathcal{W}_r(\mathbf{i}), x_\zeta \in \partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$$

and  $x_\kappa$  lies strictly between  $\mathcal{W}_r(\mathbf{i})$  and

$$\partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1] \text{ for } \xi < \kappa < \zeta,$$

and which is traversed during  $[(k-1)\Delta_r, (k+1)\Delta_r]$  (compare (6.17) in [KSa]). At the times  $v + j/n, 1 \leq j \leq (\zeta - \xi)$ ,  $\rho$  is at a position in the open cube  $\prod_{s=1}^d ((i(s) - 3)\Delta_r, (i(s) + 4)\Delta_r)$  and hence does not visit (4.36). Therefore, the random walk associated to  $\rho$  remains the same at the times  $v + j/n, 0 \leq j \leq (\zeta - \xi)$ . It follows that for  $(x_\xi, v)$  to be the first point of such an excursion from  $\mathcal{W}_r(\mathbf{i})$  to  $\partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$  it is necessary that for an appropriate  $q$ ,  $S_{1/n}^{(n)}(x_\xi, v, q) \neq \mathbf{0}$  and  $\sup_{u \leq 2\Delta_r n} \|S_{u/n}^{(n)}(x_\xi, v, q) - S_{1/n}^{(n)}(x_\xi, v, q)\| \geq 2\Delta_r - 1$ . The last condition has to be satisfied because the minimal distance between  $\mathcal{W}_r(\mathbf{i})$  and  $\partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$  is  $2\Delta_r$ , and we are counting jumps in  $\mathcal{B}_r(\mathbf{i}, k)$  after time  $(k - 1)\Delta_r$ . These jumps must occur in the time interval  $[(k - 1)\Delta_r, (k + 1)\Delta_r]$ , i.e., in at most  $2\Delta_r n$  steps of  $S^{(n)}(x_\xi, v, q)$ . (Note our terminology here:  $S_{u/n}^{(n)}$  takes a step each time  $u$  increases by 1, but it has a jump only if  $S_{(u+1)/n}^{(n)} \neq S_{u/n}^{(n)}$ .) Suppose  $S^{(n)}(x_\xi, v, q)$  indeed leaves  $\mathcal{W}_r(\mathbf{i})$  and reaches  $\partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$  before it returns to  $\mathcal{W}_r(\mathbf{i})$ . In this case, let  $m = m(x_\xi, v, q)$  be the smallest integer for which  $S_{m/n}^{(n)}(x_\xi, v, q) \in \partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$ . In the notation of (4.37), this is the number of steps it takes  $S^{(n)}(x_\xi, v, q)$  to reach  $x_\zeta$ . The number of jumps of  $\rho$  in  $\mathcal{B}_r(\mathbf{i}, k)$  between time  $v$  and the next return to  $\mathcal{W}_r(\mathbf{i})$  is then bounded by the number of jumps of  $\{S_{u/n}^{(n)}(x, v, q)\}$  for  $m \leq u \leq m + 2\Delta_r n$ . This number is independent of all random walks  $S^{(n)}(y, w, s)$  with  $(y, w, s) \neq (x_\xi, v, q)$  and of the  $S_{u/n}^{(n)}(x_\xi, v, q)$  for  $u \leq m$ . Moreover, if  $\mathcal{B}_r(\mathbf{i}, k)$  is good (in the  $(n)$ -system), then there are at most  $\gamma_r \mu_A C_0^{dr} + 1$  particles at the space-time point  $(x_\xi, v)$ . Indeed  $N^{(n)}(x_\xi, v) \leq U_r^{(n)}(x_\xi, v) \leq \gamma_0 \mu_A C_0^{dr}$  by the definition of a good block, and the only possible particle at  $(x_\xi, v)$  possibly not counted by  $N^{(n)}(x_\xi, v)$  is an extra particle which was added at time 0 at the origin (see (3.29)). Therefore, in  $\sum_{(\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, \mathbf{b})}^{(\tilde{\pi}, r, n)} M_{\text{out}}^{(n)}(r, \mathbf{i}, k) I_1^{(n)}(r, \mathbf{i}, k)$  we only need to count jumps of some  $\{S_u^{(n)}(x, v, q)\}$  with  $q \leq \gamma_0 \mu_A C_0^{dr} + 1$ . It follows that the total number of jumps in  $[(k - 1)\Delta_r, (k + 1)\Delta_r]$  in the good block  $\mathcal{B}_r(\mathbf{i}, k)$  of particles outside  $\mathcal{V}_r(\mathbf{i})$  at time  $(k - 1)\Delta_r$  is stochastically bounded by

$$(4.38) \quad \sum_{x \in \mathcal{W}_r(\mathbf{i})} \sum_{v \in [(k-1)\Delta_r, (k+1)\Delta_r]} \sum_{q \leq \gamma_0 \mu_A C_0^{dr} + 1} I[S_{1/n}^{(n)}(x, v, q) \neq \mathbf{0}, \\ \sup_{u \leq 2\Delta_r n} \|S_{u/n}^{(n)}(x, v, q) - S_{1/n}^{(n)}(x, v, q)\| \geq 2\Delta_r - 1] \\ \times [\text{number of jumps of } S_{u/n}^{(n)}(x, v, q), m \leq u \leq m + 2\Delta_r n]$$

( $v$  is restricted to the multiples of  $1/n$  in the second sum; the bound here is valid in each  $(n)$ -system with  $n < \infty$ ). Each of the random variables

[number of jumps of  $S_{u/n}^{(n)}(x, v, q), m \leq u \leq m + 2\Delta_r n$ ] converges (as  $n \rightarrow \infty$ ) in distribution to a Poisson variable,  $X(x, v, q)$  say, with mean  $2\Delta_r D$ . Furthermore

$$(4.39) \quad \sum_{x \in \mathcal{W}_r(\mathbf{i})} \sum_{v \in [(k-1)\Delta_r, (k+1)\Delta_r)} \sum_{q \leq \gamma_0 \mu_A C_0^{dr+1}} I[S_{1/n}^{(n)}(x, v, q) \neq \mathbf{0}, \sup_{u \leq 2\Delta_r n} \|S_{u/n}^{(n)}(x, v, q) - S_{1/n}^{(n)}(x, v, q)\| \geq 2\Delta_r - 1]$$

converges (as  $n \rightarrow \infty$ ) in distribution to a Poisson random variable,  $T = T(\mathbf{i}, k)$  say, of mean

$$(4.40) \quad K_7 \Delta_r^d C_0^{dr} D \lim_{n \rightarrow \infty} P\{ \sup_{u \leq 2\Delta_r n} \|S_u^{(n)}(x, v, q) - S_1^{(n)}(x, v, q)\| \geq 2\Delta_r - 1\} \leq K_8 C_0^{7dr} \exp[-K_{10}\Delta_r]$$

for some constants  $K_7 - K_{10}$  which depend on  $d, D$  and  $\gamma_0 \mu_A$  only. Moreover,  $T$  and all  $X(x, v, q)$  are independent. Thus (see (4.31))  $P\{M_{\text{out}}(r, \mathbf{i}, k) > x\} \leq P\{\widetilde{M}(r, \mathbf{i}, k) \geq x\}$  for  $\widetilde{M}(r, \mathbf{i}, k) = \sum_{j=1}^T X_j$  with Poisson variables  $X_j$  with mean  $2\Delta_r D$ , independent of each other and of  $T$ .

We finally show that (4.7) and (4.8) hold for the  $M_{\text{out}}(r, \mathbf{i}, k), (\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)$ , for fixed  $(\mathbf{a}, b)$ , and with the  $\widetilde{M}(r, \mathbf{i}, k)$  as above and  $\mathcal{S}(\mathbf{a}, b)$  any collection of pairs  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$ . Firstly, the sums in (4.38) for different  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$  use different random walks  $\{S_u^{(n)}\}$  and therefore are independent. From the argument in the last paragraph it then follows that the  $M_{\text{out}}(r, \mathbf{i}, k), (\mathbf{i}, k) \in \mathcal{S}(\mathbf{a}, b)$ , are dominated by an independent family of random variables  $\widetilde{M}(r, \mathbf{i}, k)$ , each of which has the distribution of  $\sum_{j=1}^T X_j$ . A straightforward calculation gives

$$(4.41) \quad E\{e^{\theta \widetilde{M}(r, \mathbf{i}, k)}\} \leq \exp\left[K_8 C_0^{7dr} \exp[-K_{10}\Delta_r + 2\Delta_r D(e^\theta - 1)]\right].$$

Thus (4.8) holds for any  $r \geq 1$  with  $\theta_r = \theta$  and  $\log \Gamma_r = K_{11}$  for any  $\theta > 0$  for which  $2D(e^\theta - 1) < K_{10}/2$  and for a constant  $K_{11} = K_{11}(d, D, \gamma_0 \mu_A) \geq \sup_{r \geq 1} \{K_8 C_0^{7dr} \exp[-K_{10}\Delta_r + 2\Delta_r D(e^\theta - 1)]\}$ .

In order to apply Lemma 15, we have to have an estimate for

$$(4.42) \quad P\{\text{for some } \widehat{\pi} \in \Xi(\ell, t), [H_2(\widehat{\pi}, r)]^c \text{ occurs}\},$$

when  $r \leq R(t)$ . But this is trivial for  $r \leq R(t)$ , if we take  $\varepsilon_r = 3^{d+1}/\Delta_r$ . Indeed, with this  $\varepsilon_r$  and  $r \leq R(t)$ ,  $H_2(\widehat{\pi}, r)$  never fails, because the total number of  $r$ -blocks intersecting a given  $\widehat{\pi} \in \Xi(\ell, t)$  is at most  $\lambda_r(\ell) \leq \varepsilon_r(t + \ell)$  (see (4.23) and recall that  $\Delta_r = C_0^{6r} \leq C_0^6 [\log t]^{6/d}$  by (4.13), and finally that we can take  $t_2$  so that  $C_0^6 [\log t]^{6/d} \leq t$  for  $t \geq t_2$ ).

Lemma 16 now follows from (4.20) with  $x$  equal to the right hand side of (4.19) with  $\theta_r, \varepsilon_r$  and  $\Gamma_r$  as above. □

Lemma 16 takes care of all contributions to (4.26) from jumps at some  $(x, s)$  in some good  $\mathcal{B}_r(\mathbf{i}, k)$  with  $r(x, s) = r \in [R_1, R(t)]$ , and such that the particle which jumps at  $(x, s)$  was outside  $V_r(\mathbf{i}, k)$  at time  $(k - 1)\Delta_r$ . Next we consider the jumps at some  $(x, s)$  in some good  $\mathcal{B}_r(\mathbf{i}, k)$  with  $r(x, s) = r \in [R_1 + 1, R(t)]$ , and such that the particle which jumps at  $(x, s)$  was inside  $V_r(\mathbf{i}, k)$  at time  $(k - 1)\Delta_r$ . Note that  $r(x, s) = r > R_1$  implies that these jumps lie in addition in a bad  $(r - 1)$ -block. We shall therefore estimate  $\sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} M_{\text{in}}(r, \mathbf{i}, k) I_2(\widehat{\pi}, r, \mathbf{i}, k)$  where

$$I_2(\widehat{\pi}, r, \mathbf{i}, k) := I[\mathcal{B}_r(\mathbf{i}, k) \text{ contains a bad } (r - 1)\text{-block which intersects } \widehat{\pi}].$$

LEMMA 17. *There exist constants  $R_3$  and  $t_3$  such that for  $t \geq t_3, R_3 \leq r \leq R(t)$  and  $\ell \geq 0$ ,*

$$(4.43) \quad P\left\{\Theta(t) \text{ and there exists a } \widehat{\pi} \in \Xi(J, \ell, t) \text{ such that} \right. \\ \left. \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} M_{\text{in}}(r, \mathbf{i}, k) I_2(\widehat{\pi}, r, \mathbf{i}, k) \geq \frac{8(12)^d K_2(t + \ell)}{\Delta_r} \right\} \\ \leq 2 \exp \left[ -\sqrt{(t + \ell)} \right].$$

*Proof.* Again this proof relies on [KSa]. First we modify the  $M_{\text{in}}$  somewhat, so that we can verify (4.7) and (4.8). If  $\mathcal{B}_r(\mathbf{i}, k)$  is good, then  $\mathcal{V}_r(\mathbf{i}, k)$  contains at most  $[7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1$  particles, so that  $M_{\text{in}}(r, \mathbf{i}, k)$  counts the number of jumps in  $\mathcal{B}_r(\mathbf{i}, k)$  during  $[k\Delta_r, (k + 1)\Delta_r)$  of at most  $[7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1$  particles. (Again the one is added to the number of particles to take into account the extra particle added at time 0.)  $M_{\text{in}}(r, \mathbf{i}, k)$  is therefore bounded by the total number of jumps during  $[k\Delta_r, (k + 1)\Delta_r)$  of the first  $[7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1$  particles in  $\mathcal{V}_r(\mathbf{i}, k)$  in some arbitrary ordering of particles; if there are fewer than  $[7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1$  particles in  $\mathcal{V}_r(\mathbf{i}, k)$  we add artificial particles to raise the number to  $[7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1$  and count the jumps of the artificial particles as well. Denote the resulting number of jumps by  $\widetilde{M}_{\text{in}}(r, \mathbf{i}, k)$ . By construction, each of the  $\widetilde{M}_{\text{in}}(r, \mathbf{i}, k)$  is a Poisson variable with mean  $D\Delta_r \{ [7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1 \}$ . Moreover, if  $\mathcal{B}_r(\mathbf{i}, k)$  and  $\mathcal{B}_r(\mathbf{i}', k')$  have disjoint pedestals, then their corresponding  $\widetilde{M}_{\text{in}}$  are independent since they count jumps of disjoint sets of particles, and the cardinalities of the sets are non random. Thus (4.7) and (4.8) hold for  $\widetilde{M}(r, \mathbf{i}, k)$  Poisson variables with mean  $D\Delta_r \{ [7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1 \}$ , and correspondingly

$$(4.44) \quad \theta_r = 1, \log \Gamma_r = D\Delta_r \{ [7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1 \} (e - 1).$$

Next we check (4.6). By definition of  $I_2$ ,  $\sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} I_2(\widehat{\pi}, r, \mathbf{i}, k)$  is bounded by  $\phi_{r-1}(\widehat{\pi})$ , the number of bad  $(r - 1)$ -blocks which intersect  $\widehat{\pi}$ . However,  $\phi_{r-1}$  is already estimated in Lemma 15 of [KSa]. The proof of Lemma 15 there (especially the one but last member of (6.47)) tells us that for suitable

constants  $K_i, C_i, t \geq$  some  $t_3$  and  $r - 1 \geq d$

$P\{\Theta(t)$  and there exists a  $\hat{\pi} \in \Xi(\ell, t)$  such that

$$\begin{aligned} & \sum_{(i,k)}^{(\hat{\pi},r)} I_2(\hat{\pi}, r, \mathbf{i}, k) \geq K_{13}\kappa_0(t + \ell) \exp [ - K_{12}C_0^{(r-1)/4} ] \} \\ & \leq \sum_{q=r-1}^{R(t)-1} \exp [ - C_7\kappa_0(t + \ell) \exp[-C_0^{q/2}] ] \\ & \quad + \sum_{q=r-1}^{R(t)-1} \exp \left[ - C_{10}\kappa_0(t + \ell) \exp \left[ - \frac{1}{2(d+1)} \gamma_0\mu_A C_0^{(d-\frac{3}{4})q} \right] \right] \\ & \leq \exp [ - \sqrt{(t + \ell)} ]. \end{aligned}$$

Thus, if we take

$$\varepsilon_r = K_{13}\kappa_0 \exp [ - K_{12}C_0^{(r-1)/4} ]$$

then

$P\{\Theta(t)$  and there exists a  $\hat{\pi} \in \Xi(\ell, t)$  such that

$$\begin{aligned} & \sum_{(i,k)}^{(\hat{\pi},r)} I_2(\hat{\pi}, r, \mathbf{i}, k) \geq \varepsilon_r(t + \ell) \} \\ & \leq \exp [ - \sqrt{(t + \ell)} ]. \end{aligned}$$

Finally, we apply (4.20) with  $x = 8(12)^d K_2(t + \ell)/\Delta_r$  to obtain (4.43) for  $R_3 \leq r \leq R(t), t \geq t_3$ , with suitable  $R_3, t_3$ .  $\square$

We go back to (4.26). Each jump at  $(x, s)$  on some  $J$ -path is counted in some  $M_{\text{out}}(r, \mathbf{i}, k)$  or some  $M_{\text{in}}(r, \mathbf{i}, k)$ . Lemma 16 takes care of all jumps of the former kind with  $R_2 \leq r(x, s) \leq R(t)$ , whereas Lemma 17 takes care of the jumps of the latter kind, but only if  $(R_1 + 1) \vee R_3 \leq r(x, s) \leq R(t)$ . On  $\Theta(t)$  there are no jumps with  $r > R(t)$  to consider. Without loss of generality we can take  $R_1 \geq R_2 \vee R_3$ . The only contributions to  $J(t, \mathbf{0})$  which we still must estimate are then counted in

$$(4.45) \quad \sum_{(i,k)}^{(\hat{\pi},R_1)} M_{\text{in}}(R_1, \mathbf{i}, k).$$

This sum will be broken up into several subsums. But we must first introduce a certain constant  $\tilde{C}_1$ . Let

$$(4.46) \quad \tilde{\mu} = 2\gamma_0\mu_A.$$

In Theorem 1 of [KSc] we defined a constant  $C_1$ . This constant depends only on  $\mu_A, d$  and  $D$ , since these are the only parameters appearing in the model (now that the  $A$  and  $B$ -particles have the same jumprate  $D$ ). Therefore, if  $\mu_A$  is replaced by  $\tilde{\mu}$ , then Theorem 1 of [KSc] again holds, but now with  $C_1$

replaced by some constant  $\tilde{C}_1$ . Without loss of generality we take  $\tilde{C}_1$  to be an integer greater than or equal to 1.

We now break the sum (4.45) up into the two sums:

$$\sum_{(\mathbf{i}, k)}^{(\tilde{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_3(R_1, \mathbf{i}, k) \text{ and } \sum_{(\mathbf{i}, k)}^{(\tilde{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_4(R_1, \mathbf{i}, k),$$

where

$$\begin{aligned} I_3(r, \mathbf{i}, k) &:= I \left[ \mathcal{B}_r(\mathbf{i}, k) \text{ is good and there exists a } J\text{-path from} \right. \\ &\quad \left. \text{some point } (x', s') \in \mathcal{B}_r(\mathbf{i}, k), \text{ to a point } (x'', s'') \in \right. \\ &\quad \left. \left( \partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1] \right) \right. \\ &\quad \left. \times \left[ s', \left( s' + \frac{\Delta_r}{2\tilde{C}_1} \right) \wedge (k + 1)\Delta_r \right] \right] \end{aligned}$$

and

$$I_4(r, \mathbf{i}, k) = 1 - I_3(r, \mathbf{i}, k).$$

It will turn out that the sum with  $I_4$  can easily be reduced to the sum with  $I_3$  (see Lemma 23). However, the latter sum will have to be split up further. We define

$$\begin{aligned} I_{3,1}(r, \mathbf{i}, k) &= I \left[ \mathcal{B}_r(\mathbf{i}, k) \text{ is a good } r\text{-block, but some particle} \right. \\ &\quad \left. \text{which is outside } V_r(\mathbf{i}) \text{ at time } (k - 1)\Delta_r \text{ enters} \right. \\ &\quad \left. \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1] \right. \\ &\quad \left. \text{during } [(k - 1)\Delta_r, (k + 1)\Delta_r] \right]; \end{aligned}$$

$$\begin{aligned} I_{3,2}(r, \mathbf{i}, k) &= I \left[ \text{there exists a } J\text{-path using only particles} \right. \\ &\quad \left. \text{in } V_r(\mathbf{i}) \text{ at time } (k - 1)\Delta_r \text{ and running from} \right. \\ &\quad \left. \text{some point } (x', s') \in \mathcal{B}_r(\mathbf{i}, k) \text{ to a point } (x'', s'') \in \right. \\ &\quad \left. \left( \partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1] \right) \right. \\ &\quad \left. \times \left[ s', \left( s' + \frac{\Delta_r}{2\tilde{C}_1} \right) \wedge (k + 1)\Delta_r \right] \right]. \end{aligned}$$

If  $I_{3,1}(r, \mathbf{i}, k) = 0$ , i.e., if no particles from the outside of  $V_r(\mathbf{i})$  enter  $\prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$  during  $[(k - 1)\Delta_r, (k + 1)\Delta_r]$ , but there is a  $J$ -path from  $\mathcal{B}_r(\mathbf{i}, k)$  to  $(\partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]) \times [(k - 1)\Delta_r, (k + 1)\Delta_r]$ ,

then this  $J$ -path cannot use any particles which come from outside  $V_r(\mathbf{i})$ . Consequently

$$(4.47) \quad I_3(r, \mathbf{i}, k) \leq I_{3,1}(r, \mathbf{i}, k) + I_{3,2}(r, \mathbf{i}, k).$$

LEMMA 18. *There exist constants  $K_4, R_4$  and  $t_4$  such that for  $t \geq t_4, R_4 \leq r \leq R(t)$  and  $\ell \geq 0$ ,*

$$(4.48) \quad P\{\Theta(t) \text{ and there exists a } \hat{\pi} \in \Xi(J, \ell, t) \text{ such that}$$

$$\begin{aligned} & \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{in}}(r, \mathbf{i}, k) I_{3,1}(r, \mathbf{i}, k) \geq \frac{K_4(t + \ell)}{\Delta_r} \} \\ & \leq 2 \exp[-\sqrt{t + \ell}]. \end{aligned}$$

*Proof.* The sum  $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r, n)} I_{3,1}(r, \mathbf{i}, k)$  has already been estimated (in part) on the event  $\Theta(t)$  in the proof of Lemma 16 (or alternatively in Lemmas 10 and 11 of [KSa]). However, an extra percolation argument is needed to get a bound which is sharp enough for our purposes. It is useful to summarize a few of the steps of the proof of Lemma 16. Define

$$\begin{aligned} I_5(r, \mathbf{i}, k) := I & \left[ \mathcal{B}_r(\mathbf{i}, k) \text{ is good, but there is a particle which is in } \mathcal{W}_r(\mathbf{i}) \right. \\ & \text{at some time } u \in [(k - 1)\Delta_r, (k + 1)\Delta_r] \\ & \text{and which visits } \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1] \\ & \left. \text{at some later time in } [u, (k + 1)\Delta_r] \right]. \end{aligned}$$

Clearly  $I_{3,1}(r, \mathbf{i}, k) \leq I_5(r, \mathbf{i}, k)$ . One now uses the construction by means of the discrete time random walks  $\{S_u^{(n)}(x, s, q)\}$  as in the proof of Lemma 16. The discrete time analogue of  $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_5(r, \mathbf{i}, k)$  is then the number of good  $r$ -blocks  $\mathcal{B}_r(\mathbf{i}, k)$  (in the  $(n)$ -system) which intersect  $\hat{\pi}$  and for which there exists a particle whose path contains a piece with the properties in (4.37). The discrete time analogue of  $I_5(r, \mathbf{i}, k)$  itself is stochastically bounded by the triple sum in (4.39). As we saw, the weak limit of (4.39) is a Poisson variable  $T(\mathbf{i}, k)$  with mean bounded by (4.40). Since  $I_5$  can take only the values 0 and 1, this means that

$$P\{I_5(r, \mathbf{i}, k) = 1\} \leq K_8 C_0^{7dr} \exp[-K_{10}\Delta_r].$$

We define

$$p(r) = \exp[-\Delta_r^\kappa] \text{ with some fixed } 0 < \kappa < (d/6) \wedge 1,$$

and without loss of generality take  $R_4$  so large that  $p(r) \geq K_8 C_0^{7dr} \exp[-K_{10}\Delta_r]$  for  $r \geq R_4$ , so that

$$P\{I_5(r, \mathbf{i}, k) = 1\} \leq p(r)$$

for the values of  $r$  of interest in this lemma.

Next, the triple sums in (4.39) for different  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$  are independent, as observed just before (4.41). Thus, for any  $(\mathbf{a}, b)$ , the family  $\{I_5(r, \mathbf{i}, k) : (\mathbf{i}, k) \equiv (\mathbf{a}, b)\}$  lies stochastically below a family of independent binomial random variables,  $\{Z(r, \mathbf{i}, k) : (\mathbf{i}, k) \equiv (\mathbf{a}, b)\}$  say, which satisfy

$$P\{Z(r, \mathbf{i}, k) = 1\} = 1 - P\{Z(r, \mathbf{i}, k) = 0\} = p(r).$$

Therefore,

$$\begin{aligned} (4.49) \quad P\{\Theta(t) \text{ and for some } \hat{\pi} \in \Xi(J, \ell, t), \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_5(r, \mathbf{i}, k) \geq x\} \\ \leq \sum_{(\mathbf{a}, b)} P\{\text{there exists a path } \hat{\pi} \in \Xi(J, \ell, t) \text{ such that} \\ \sum_{\substack{(\mathbf{i}, k) \\ (\mathbf{i}, k) \equiv (\mathbf{a}, b)}}^{(\hat{\pi}, r)} Z(r, \mathbf{i}, k) \geq \frac{x}{2(12)^d}\}. \end{aligned}$$

The following argument appears already in the proof of Lemma 8, as well as Lemmas 10, 11, in [KSa]; see also the proof of Theorem 9 in [L]. For convenience we repeat it here, because it will also be used in the proofs of Lemmas 21, 27 and at the end of the next section. We choose an integer  $\nu = \nu_r$  such that

$$[p(r)]^{-1/(d+1)} \leq \nu \leq 2[p(r)]^{-1/(d+1)},$$

and form the blocks

$$\mathcal{D}(\mathbf{m}, q) := \left( \prod_{s=1}^d [\nu m(s) \Delta_r, \nu(m(s) + 1) \Delta_r] \times [q\nu \Delta_r, (q + 1)\Delta_r] \right).$$

Each of these blocks is a disjoint union of  $\nu^{d+1}$   $r$ -blocks. Any space-time path  $\hat{\pi} \in \Xi(\ell, t)$  intersects at most

$$\tilde{\lambda}_r(\ell) := 3^d \left( \frac{t + \ell}{\nu \Delta_r} + 2 \right)$$

blocks  $\mathcal{D}(\mathbf{m}, q)$  (see (6.30) in [KSa]). It follows that any space-time path  $\hat{\pi} \in \Xi(\ell, t)$  is contained in a union of at most  $\tilde{\lambda}_r(\ell)$  blocks  $\mathcal{D}(\mathbf{m}, q)$ . Moreover, if

$$(4.50) \quad \mathcal{S}(\mathbf{m}_0, \dots, \mathbf{m}_{\lambda-1}) := \bigcup_{q=0}^{\lambda-1} \mathcal{D}(\mathbf{m}_q, q)$$

for some  $\lambda \leq \tilde{\lambda}_r(\ell)$  is the union of all such blocks which intersect some path  $\hat{\pi} \in \Xi(\ell, t)$ , then  $(\mathbf{m}_0, 0), (\mathbf{m}_1, 1), \dots, (\mathbf{m}_{\lambda-1}, \lambda - 1)$ , viewed as a subset of  $\mathbb{Z}^{d+1}$  has to be  $\mathcal{L}$ -connected and has to be contained in  $\mathcal{C}(2t \log t)$  (see (3.2) for  $\mathcal{L}$ ). There are therefore at most  $\exp[K_{14} \tilde{\lambda}_r(\ell)]$  possible choices for



$(\mathbf{m}_0, 0), (\mathbf{m}_1, 1), \dots, (\mathbf{m}_{\lambda-1}, \lambda - 1)$ , where  $K_{14}$  is some constant which depends on  $d$  only. For each such choice the union (4.50) contains  $\lambda\nu^{d+1} \leq \tilde{\lambda}_r(\ell)\nu^{d+1}$   $r$ -blocks  $\mathcal{B}_r(\mathbf{i}, k)$ . Therefore

$$(4.51) \quad P\{\text{the sum of the } Z(r, \mathbf{i}, k) \text{ for which } \mathcal{B}_r(\mathbf{i}, k) \text{ is contained in (4.50) is } \geq \frac{x}{2(12)^d}\} \leq P\{\tilde{T} \geq \frac{x}{2(12)^d}\},$$

where  $\tilde{T}$  has a binomial distribution corresponding to  $\tilde{\lambda}_r(\ell)\nu^{d+1}$  trials with a success probability  $p(r)$ . Thus  $E\{\tilde{T}\} = \tilde{\lambda}_r(\ell)\nu^{d+1}p(r) \leq 2^{d+1}\tilde{\lambda}_r(\ell)$  and simple large deviation estimates for the binomial distribution show that there exists a universal constant  $K_{15} > 0$  such that

$$P\{\tilde{T} \geq y\} \leq \exp[-K_{15}y] \text{ for all } y \geq 2^{d+2}\tilde{\lambda}_r(\ell) \geq 2E\{\tilde{T}\}.$$

We now choose

$$(4.52) \quad x = \left(\frac{2K_{14}}{K_{15}} + 2^{d+2}\right)2(12)^d\tilde{\lambda}_r(\ell).$$

Then the probability in (4.51) is at most

$$\exp[-K_{15}\frac{x}{2(12)^d}] \leq \exp[-2K_{14}\tilde{\lambda}_r(\ell)].$$

Now

$$(4.53) \quad P\left\{\Theta(t) \text{ and } \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_{3,1}(r, \mathbf{i}, k) \geq x \text{ for some } \hat{\pi} \in \Xi(J, \ell, t)\right\} \leq P\left\{\Theta(t) \text{ and } \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_5(r, \mathbf{i}, k) \geq x \text{ for some } \hat{\pi} \in \Xi(J, \ell, t)\right\} \leq \sum_{\mathbf{m}_0, \dots, \mathbf{m}_{\lambda-1}} \sum_{(\mathbf{a}, b)} P\left\{\sum_{\substack{(\mathbf{i}, k) \equiv (\mathbf{a}, b) \\ \mathcal{B}_r(\mathbf{i}, k) \subset \mathcal{S}(\mathbf{m}_0, \dots, \mathbf{m}_{\lambda-1})}} Z(r, \mathbf{i}, k) \geq \frac{x}{2(12)^d}\right\} \leq \sum_{\mathbf{m}_0, \dots, \mathbf{m}_{\lambda-1}} 2(12)^d \exp[-2K_{14}\tilde{\lambda}_r(\ell)].$$

Since there are only  $\exp[K_{14}\tilde{\lambda}_r(\ell)]$  possible choices for the union (4.50), it follows that the right, and hence also the left hand side of (4.53) is, for  $t \geq$  a suitable  $t_4$  and  $R_4 \leq r \leq R(t)$ , at most

$$(4.54) \quad 2(12)^d \exp[-K_{14}\tilde{\lambda}_r(\ell)] \leq 2(12)^d \exp\left[-K_{14}3^d \frac{(t + \ell) \exp[-\Delta_r^c/(d + 1)]}{2\Delta_r}\right] \leq \exp[-\sqrt{t + \ell}]$$

where we used the value of  $\kappa$  and (4.13) for the last inequality. Thus for some constant  $K_{16}$  which depends on  $d$  only, and

$$\varepsilon_r \geq \frac{K_{16}}{\Delta_r} \exp[-\Delta_r^\kappa/(d+1)] \geq \frac{x}{t+\ell}$$

and  $t_4$  sufficiently large, we have for  $\widehat{t} \geq t_4, R_4 \leq r \leq R(\widehat{t})$ ,

$$(4.55) \quad P\left\{\Theta(t) \text{ and for some } \widehat{\pi} \in \Xi(J, \ell, t), \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} I_{3,1}(r, \mathbf{i}, k) \geq \varepsilon_r(t+\ell)\right\} \\ \leq \exp[-\sqrt{t+\ell}]$$

For the sake of definiteness we shall take  $\varepsilon_r = 2[\Delta_r]^{-2d-3}$ . This bounds the first term in the right hand side of (4.18)

In addition we have already seen in the proof of Lemma 17 that (4.7) and (4.8) for the  $M_{\text{in}}(r, \mathbf{i}, k)$  hold with  $\widehat{M}(r, \mathbf{i}, k)$  a Poisson variable and  $\theta_r, \log \Gamma_r$  given in (4.44). Finally we apply Lemma 15 once more, this time with  $x = 8(12)^d K_2(t+\ell)/\Delta_r$ , to obtain Lemma 18, but possibly with different values for  $K_4$  than in (4.28).  $\square$

We now start on some technical preparations for estimating  $\sum I_{3,2}$ . For  $q = 1, 2, \dots$  and  $a \in \mathbb{R}$  we define

$$[a]_q = a(a-1)\dots(a-q+1).$$

We also need the following  $\sigma$ -fields and random variables.

$$(4.56) \quad \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r) := \sigma\text{-field generated by the } N_A(x, 0-), x \in \mathbb{Z}^d, \\ \text{and all paths during } [0, (k-1)\Delta_r], \text{ as well as} \\ \text{the paths on } [(k-1)\Delta_r, \infty) \text{ of all particles} \\ \text{outside } V_r(\mathbf{i}) \text{ at time } (k-1)\Delta_r,$$

$$(4.57) \quad \widetilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r) := \sigma\text{-field generated by the locations and} \\ \text{numbers of particles in } V_r(\mathbf{i}) \\ \text{at time } (k-1)\Delta_r,$$

$$L(x, u) = L_r(x, \mathbf{i}, k-1, u) \\ = [\text{number of particles at } x \text{ at time } (k-1)\Delta_r + u \\ \text{which were in } V_r(\mathbf{i}) \text{ at time } (k-1)\Delta_r].$$

LEMMA 19. *There exists an  $R_5$ , such that if  $r \geq R_5$  and  $\Delta_r/2 \leq u \leq 3\Delta_r$ , then for distinct  $a_1, \dots, a_\ell \in \mathbb{Z}^d$  and  $q_1, \dots, q_\ell \in \{1, 2, \dots\}$  it holds on the*

event  $\{\mathcal{V}_r(\mathbf{i}, k) \text{ is good}\}$ ,

$$\begin{aligned}
 (4.58) \quad E\left\{ \prod_{i=1}^{\ell} [L(a_i, u)]_{q_i} \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r) \right\} \\
 = E\left\{ \prod_{i=1}^{\ell} [L(a_i, u)]_{q_i} \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r) \right\} \\
 \leq [2\gamma_0\mu_A]^{\sum_{i=1}^{\ell} q_i}.
 \end{aligned}$$

*Proof.* Once we know the numbers and locations of the particles in  $\mathcal{V}_r(\mathbf{i}, k)$ , the  $L(x, u)$  are determined by the increments after  $(k-1)\Delta_r$  in the paths of the particles in  $\mathcal{V}_r(\mathbf{i}, k)$ . These increments are independent of  $\mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)$ . The conditioning on  $\mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)$  only effects the  $L(x, u)$  through the determination of which particles are in  $\mathcal{V}_r(\mathbf{i}, k)$ , because these are the only particles to be counted in  $L(x, u)$ . The equality in (4.58) is immediate from this.

We now first prove (4.58) in the special case  $\ell = 1, q_1 = 1$ . For brevity we write  $a$  instead of  $a_1$ . The conditional expectations in (4.58) are now at most

$$(4.59) \quad \sum_{y \in V_r(\mathbf{i})} N^*(y, (k-1)\Delta_r) P\{y + S_u = a\} + \sup_y P\{y + S_u = a\}.$$

The last term has to be included because the extra particle added at time 0 is not counted in the  $N^*$ , even though it may be in  $V_r(\mathbf{i})$  at time  $(k-1)\Delta_r$ . We have to show that (4.59) is at most  $2\gamma_0\mu_A$ . This part of the proof is very similar to the proof of Lemma 8, with  $p$  replaced by  $r$ . In fact it is somewhat easier. We take  $\mathcal{M}(\ell)$  as in Lemma 8 (with  $p$  replaced by  $r$ ) but this time define  $\Lambda$  by

$$\Lambda = \Lambda(\mathbf{i}, r) = \{\ell \in \mathbb{Z}^d : \mathcal{M}(\ell) \subset V_r(\mathbf{i})\},$$

and for each  $\ell \in \Lambda$  we take  $y_\ell \in \mathcal{M}(\ell)$  such that

$$P\{y_\ell + S_u = a\} = \max_{y \in \mathcal{M}(\ell)} P\{y + S_u = a\}.$$

From here on one can follow the proof of Lemma 8, merely reversing some inequality signs and making use of

$$\sum_{y \in \mathcal{M}(\ell)} N^*(y, (k-1)\Delta_r) \leq \gamma_0\mu_A C_0^{dr}$$

for all  $\ell \in \Lambda(\mathbf{i}, p)$ , which holds because  $\mathcal{V}_r(\mathbf{i}, k)$  is good. Note that the analogue of (3.36) now becomes

$$\begin{aligned}
 (4.60) \quad & \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A P\{y_\ell + S_u = a\} \\
 & \leq \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A P\{y + S_u = a\} \\
 & \quad + \sum_{\ell \in \Lambda} \sum_{y \in \mathcal{M}(\ell)} \gamma_0 \mu_A |P\{y_\ell + S_u = a\} - P\{y + S_u = a\}|.
 \end{aligned}$$

Clearly the first double sum in the right hand side here is at most  $\gamma_0 \mu_A$  for all  $a$ . Moreover, the last double sum is at most  $K_2 \gamma_0 \mu_A C_0^r [\log u]^d u^{-1/2}$  by (5.26) and (6.37) in [KSa]. Also,  $\sup_y P\{y + S_u = a\} = O(u^{-1/2})$  by the local central limit theorem. The desired bound  $2\gamma_0 \mu_A$  for (4.59) for  $r \geq$  some  $R_5$  now follows.

We now turn to the general case of (4.58). Write  $Q$  for  $\sum_{i=1}^\ell q_i$  and let

$$I_i(\rho) = I[\rho \text{ moves from } \mathcal{V}_r(\mathbf{i}, k) \text{ to } a_i \text{ at time } u].$$

Note that  $\prod_{i=1}^\ell [L(a_i, u)]_{q_i}$  equals the number of distinct ordered  $Q$ -tuples of particles, with  $q_i$  of these particles located at  $a_i$  at time  $u$ , and which were in  $V_r(\mathbf{i})$  at time  $(k-1)\Delta_r$ . Set  $Q_0 = 0$  and for  $j \geq 1$  set  $Q_j = \sum_{i=1}^j q_i$ . Then this number of  $Q$ -tuples can be written as

$$(4.61) \quad \sum^{(Q)} \prod_{j=1}^\ell \prod_{i=Q_{j-1}+1}^{Q_j} I_j(\rho_i),$$

where  $\sum^{(Q)}$  denotes the sum over all ordered  $Q$ -tuples of *distinct* particles  $\rho_1, \dots, \rho_Q$  which are in  $V_r(\mathbf{i})$  at time  $(k-1)\Delta_r$ . Let us write  $y_i$  for the position of  $\rho_i$  at time  $(k-1)\Delta_r$ . If we take the conditional expectation of (4.61), given  $\tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r)$ , we find that the middle member of (4.58) equals

$$\begin{aligned}
 & E\left\{ \sum^{(Q)} \prod_{j=1}^\ell \prod_{i=Q_{j-1}+1}^{Q_j} I_j(\rho_i) \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r) \right\} \\
 & = \sum^{(Q)} \prod_{j=1}^\ell \prod_{i=Q_{j-1}+1}^{Q_j} P\{S_u = a_j - y_i\} \\
 & \leq \prod_{j=1}^\ell \prod_{i=Q_{j-1}+1}^{Q_j} \left[ \sum_{\rho_i \in V_r(\mathbf{i})} P\{S_u = a_j - y_i\} \right] \\
 & \leq \prod_{j=1}^\ell \prod_{i=Q_{j-1}+1}^{Q_j} [2\gamma_0 \mu_A] = [2\gamma_0 \mu_a]^Q.
 \end{aligned}$$

The first equality here holds because the  $\rho_i$  are distinct, and their paths are therefore independent. The first inequality holds, because all products which appear in the left hand side also appear in the right hand side after expanding the right hand side. The second inequality is true by virtue of the bound  $2\gamma_0\mu_A$  for (4.59).  $\square$

LEMMA 20. *Without loss of generality we can take  $R_5$  so large that for  $r \geq R_5$*

$$(4.62) \quad P\{I_{3,2}(r, \mathbf{i}, k) = 1 \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \leq \Delta_r^{-2(d+1)^2}$$

on the event

$$(4.63) \quad \{\mathcal{V}_r(\mathbf{i}, k) \text{ is good}\}.$$

*Proof.* The event  $\{I_{3,2}(r, \mathbf{i}, k) = 1\}$  is determined by the location of the particles in  $\mathcal{V}_r(\mathbf{i}, k)$  and by the paths of these particles during  $[(k-1)\Delta_r, (k+1)\Delta_r)$ . From this one easily sees that the conditional probability in the left hand side of (4.62) equals

$$(4.64) \quad P\{I_{3,2}(r, \mathbf{i}, k) = 1 \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r)\}.$$

To estimate this probability on the event (4.63), we note that on this event there are at most  $\gamma_0\mu_A 7^d \Delta_r^d + 1$  particles in  $V_r(\mathbf{i})$  at time  $(k-1)\Delta_r$ . The probability that any given one of these particles,  $\rho$  say, has two jumps within  $1/n$  time units from each other during  $[(k-1)\Delta_r, (k+1)\Delta_r)$  is at most

$$\begin{aligned} & \sum_{j \geq 1} P\{j\text{-th jump of } \rho \text{ after } (k-1)\Delta_r \text{ occurs before } (k+1)\Delta_r \text{ and the} \\ & \quad \text{next jump occurs } \leq 1/n \text{ time units later} \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ & \leq \frac{D}{n} \sum_{j \geq 1} P\{j\text{-th jump of } \rho \text{ after } (k-1)\Delta_r \text{ occurs before} \\ & \quad (k+1)\Delta_r \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ & \leq \frac{2\Delta_r D^2}{n}. \end{aligned}$$

Therefore, on the event (4.63),

$$(4.65)$$

$$\begin{aligned} & P\{\text{some particle in } \mathcal{V}_r(\mathbf{i}, k) \text{ has two jumps within } 1/n \text{ time units} \\ & \quad \text{of each other during } [(k-1)\Delta_r, (k+1)\Delta_r) \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ & \leq (\gamma_0\mu_A 7^d \Delta_r^d + 1) \frac{2\Delta_r D^2}{n}. \end{aligned}$$

For the remainder of this proof we take  $n = \Delta_r^{3(d+1)^2}$ . Assume that the event in the left hand side of (4.65) does not occur. Now if  $\{I_{3,2}(r, \mathbf{i}, k) = 1\}$  occurs, and the  $J$ -path in this event starts at  $(x', s')$  and  $j/n \leq s' < (j+1)/n$ , then each particle at  $x'$  at time  $s'$  is also at  $x'$  at one or both of the times  $j/n, (j+1)/n$ . We can therefore let the  $J$ -path begin at  $(x', j/n)$  or  $(x', (j+1)/n)$ . Consequently, after raising  $R_5$ , if necessary, to make  $\Delta_r/(2\tilde{C}_1) + 1/n \leq (\Delta_r - 1)/\tilde{C}_1$ , the left hand side of (4.62) is at most

(4.66)

$$\begin{aligned} & \frac{(\gamma_0 \mu_A 7^d \Delta_r^d + 1) 2 \Delta_r D^2}{\Delta_r^{3(d+2)^2}} \\ & + \sum_{\Delta_r - 1/n \leq j/n < 2\Delta_r + 1/n} \sum_{\substack{i(s)\Delta_r \leq x(s) < (i(s)+1)\Delta_r \\ 1 \leq s \leq d}} P\{\text{there exists a } J\text{-path} \\ & \text{from } (x, (k-1)\Delta_r + j/n) \text{ to } \partial \prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1] \\ & \text{of time duration } \leq \Delta_r/(2\tilde{C}_1) + 1/n \leq (\Delta_r - 1)/\tilde{C}_1 \text{ and which uses} \\ & \text{only particles which are in } V_r(\mathbf{i}) \text{ at time } (k-1)\Delta_r \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\}. \end{aligned}$$

We next prepare for the estimation of the probability in the right hand side here. Fix  $j/n$  and  $x$  for the time being. We shall condition on the numbers and locations of the particles at time  $(k-1)\Delta_r + j/n$  which were in  $V_r(\mathbf{i})$  at time  $(k-1)\Delta_r$ . Recall that the number of such particles at  $x$  is denoted by  $L(x, j/n) = L_r(x, \mathbf{i}, k-1, j/n)$ . We are going to apply Proposition 4 and Remark 2 after it and (the proof of) Theorem 1 in [KSc]. To this end we bring in the following process. First we start the  $A$ -system by choosing the  $N_A(x, 0-)$  as i.i.d. mean  $\mu_A$  Poisson variables and add an extra  $A$ -particle at  $(\mathbf{0}, 0)$ . We let this  $A$ -system run till time  $(k-1)\Delta_r$ . We then continue from time  $(k-1)\Delta_r$  with only the  $A$ -particles in  $\mathcal{V}_r(\mathbf{i}, k)$ . At time  $(k-1)\Delta_r + j/n$  we add one further  $B$ -particle at  $x$ . We then let this process with the extra  $B$ -particle continue from time  $(k-1)\Delta_r + j/n$ , using the same rules as for the  $\{Y_t\}$  process, that is,  $A$ -particles turn into  $B$ -particles when they coincide with a  $B$ -particle, but particles cannot recuperate from type  $B$  to type  $A$ . Let  $\nu(x, j/n)$  denote the number of  $B$ -particles outside  $x + \mathcal{C}(\Delta_r - 1)$  at time  $(k-1)\Delta_r + j/n + (\Delta_r - 1)/\tilde{C}_1$  in the resulting process. Then

$$\begin{aligned} (4.67) \quad E\{\nu(x, j/n) \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} &= E\{\nu(x, j/n) \mid \tilde{\mathcal{J}}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ &= E\left\{E\{\nu(x, j/n) \mid L(z, j/n), z \in \mathbb{Z}^d\} \mid \tilde{\mathcal{J}}(\mathbf{i}, (k-1)\Delta_r)\right\}. \end{aligned}$$

(4.58) says that the conditional distribution of the  $L(z, j/n)$ ,  $z \in \mathbb{Z}^d$ , given  $\tilde{\mathcal{J}}(\mathbf{i}, (k-1)\Delta_r)$  satisfies condition (2.51) of [KSc] with  $\mu_A$  replaced by

$$\tilde{\mu} := 2\gamma_0\mu_A.$$

We think of the  $L(\cdot, j/n)$  as the random initial condition for the process from time  $(k-1)\Delta_r + j/n$  on of the particles in  $\mathcal{V}_r(\mathbf{i}, k)$  plus the one extra  $B$ -particle inserted at  $x$  at time  $(k-1)\Delta_r + j/n$ . Proposition 4 and Remark 2 after it and Theorem 1 in [KSc] then show that on the event (4.63) the right hand side of (4.67) is at most equal to the

$$\begin{aligned} (4.68) \quad E\{ & \text{number of } B\text{-particles outside } \mathcal{C}(\Delta_r - 1) \text{ at time } (\Delta_r - 1)/\tilde{C}_1 \\ & \text{in the } \{Y_t\}\text{-process with the initial number of} \\ & \text{ } A\text{-particles distributed as i.i.d., mean } \tilde{\mu} \text{ Poisson} \\ & \text{variables plus one } B\text{-particle at } \mathbf{0} \} \\ & \leq 2e^{-(\Delta_r - 1)/\tilde{C}_1} \text{ (see (1.3) in [KSc]) } \leq 2e^{-\Delta_r/(2\tilde{C}_1)}. \end{aligned}$$

We now return to the estimation of (4.66). By virtue of Lemma 14, the probability in (4.66) is at most

$$\begin{aligned} & P\{\text{some } J\text{-path starting at } (x, (k-1)\Delta_r + j/n) \text{ and using only} \\ & \quad \text{particles from } \mathcal{V}_r(\mathbf{i}, k) \text{ leaves } x + \mathcal{C}(\Delta_r - 1) \\ & \quad \text{during } [0, (\Delta_r - 1)/\tilde{C}_1] \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ & \leq 2E\{\text{number of } B\text{-particles outside } \mathcal{C}(\Delta_r - 1) \text{ at time } (\Delta_r - 1)/\tilde{C}_1, \\ & \quad \text{in the process using only particles from } \mathcal{V}_r(\mathbf{i}, k) \\ & \quad \text{plus one } B\text{-particle at } (x, (k-1)\Delta_r + j/n), \\ & \quad \text{as described above} \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ & \leq 4e^{-\Delta_r/(2\tilde{C}_1)} \text{ (by (4.67) and (4.68)).} \end{aligned}$$

To conclude we substitute the last estimate into (4.66) to obtain

$$\begin{aligned} & P\{I_{3,2}(\tau, \mathbf{i}, k) = 1 \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \\ & \leq \frac{(\gamma_0\mu_A 7^d \Delta_r^d + 1) 2\Delta_r D^2}{\Delta_r^{3(d+1)^2}} + \sum_{\substack{\Delta_r - 1/n \leq j/n \\ < 2\Delta_r + 1/n}} \sum_{\substack{i(s)\Delta_r \leq x(s) < (i(s)+1)\Delta_r \\ 1 \leq s \leq d}} 4e^{-\Delta_r/(2\tilde{C}_1)} \\ & \leq \Delta_r^{-2(d+1)^2}, \end{aligned}$$

on the event (4.63), provided  $R_5$  is taken large enough. □

LEMMA 21. Take  $R_1 = \max\{R_2, R_3, R_4, R_5\}$ . Then there exist constants  $K_{17}$ - $K_{19}$ , depending on  $d$  only, and  $t_5$  such that for  $t \geq t_5$  and  $R_1 \leq r \leq R(t)$

$$(4.69) \quad P\left\{ \sup_{\hat{\pi} \in \Xi(J, \ell, t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_{3,2}(r, \mathbf{i}, k) \geq K_{17} \Delta_r^{-2d-3}(t + \ell) \right\} \\ \leq K_{18} \exp \left[ -K_{19} \Delta_r^{-2d-3}(t + \ell) \right].$$

*Proof.* The argument for this proof has already been used in the proof of Lemma 18. Define

$$Y(\mathbf{i}, k) = I[\mathcal{V}_r(\mathbf{i}, k) \text{ is good, but } I_{3,2}(r, \mathbf{i}, k) = 1].$$

Also, let  $\{Z(\mathbf{i}, k)\}$  be a system of independent random variables with

$$P\{Z(\mathbf{i}, k) = 1\} = 1 - P\{Z(\mathbf{i}, k) = 0\} = \Delta_r^{-2(d+1)^2}.$$

We claim that for fixed  $\mathbf{a} \in \{0, \dots, 11\}^d$ ,  $b = 0$  or  $1$

$$\{Y(\mathbf{i}, k) : (\mathbf{i}, k) \text{ such that } (\mathbf{i}, k) \equiv (\mathbf{a}, b) \text{ and } \mathcal{B}_r(\mathbf{i}, k) \\ \text{intersects } \mathcal{C}(t \log t)\}$$

lies stochastically below the collection

$$\{Z(\mathbf{i}, k) : (\mathbf{i}, k) \text{ such that } (\mathbf{i}, k) \equiv (\mathbf{a}, b) \text{ and } \mathcal{B}_r(\mathbf{i}, k) \\ \text{intersects } \mathcal{C}(t \log t)\}.$$

This claim follows immediately from (4.62). Indeed, the event  $\{\mathcal{V}_r(\mathbf{i}, k) \text{ is good}\}$  lies in  $\mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)$ . Also the events  $\{Y(\mathbf{i}', k') = 1\}$  for  $k' \leq k$ ,  $(\mathbf{i}', k') \neq (\mathbf{i}, k)$ ,  $(\mathbf{i}', k') \equiv (\mathbf{a}, b)$ , belong to  $\mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)$ . Finally,  $P\{Y(\mathbf{i}, k) = 1 \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \leq \Delta_r^{-2(d+1)^2}$ , by Lemma 20. (Note that  $Y(\mathbf{i}, k) = 0$  on the complement of the event (4.63).) With our claim established, it follows that the left hand side of (4.69) is at most

$$(4.70) \quad \sum_{(\mathbf{a}, b)} P\left\{ \sup_{\hat{\pi} \in \Xi(\ell, t)} \sum_{(\mathbf{i}, k) \equiv (\mathbf{a}, b)}^{(\hat{\pi}, r)} Z(\mathbf{i}, k) \geq [2 \cdot 12^d]^{-1} K_{17} \Delta_r^{-2d-3}(t + \ell) \right\}.$$

We shall not give further steps in the proof of (4.69), because from (4.70) on it is the same as for Lemma 11 in [KSa], with  $\chi_{r+1}$  and  $r+1$  there replaced by  $\Delta_r^{-2(d+1)^2}$  and  $r$ , respectively, or the proof following (4.49) in Lemma 18.  $\square$

LEMMA 22. There exist a constant  $t_6$  such that for  $t \geq t_6$  and  $R(t) \geq r \geq \max\{R_2, R_3, R_4, R_5\}$ ,

$$(4.71) \quad P\{\text{for some } \hat{\pi} \in \Xi(J, \ell, t), \\ \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{in}}(r, \mathbf{i}, k) I_{3,2}(r, \mathbf{i}, k) \geq \frac{K_4(t + \ell)}{\Delta_r}\} \\ \leq K_5 \exp \left[ -K_6(t + \ell) / [\log t]^{12d+18} \right].$$



*Proof.* This is now a familiar application of Lemma 15. We use that (4.7) and (4.8) for the  $M_{\text{in}}(r, \mathbf{i}, k)$  hold with  $\widetilde{M}(r, \mathbf{i}, k)$  a Poisson variable and  $\theta_r, \log \Gamma_r$  as in (4.44). Further, (4.69) gives us an estimate for the first term in the right hand side of (4.20), with  $I$  replaced by  $I_{3,2}$  and

$$\varepsilon_r = K_{17} \Delta_r^{-2d-3}$$

in the definition of  $H_2(\widehat{\pi}, r)$ . The lemma follows from Lemma 15 with  $x = 8(12)^d K_2 (t + \ell) / \Delta_r$ . □

The next lemma will deal with  $\sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} M_{\text{in}} I_4$ , but only for  $r = R_1$ .

LEMMA 23. *There exist some constants  $C_{13}, R_6$  and  $t_7$  (all independent of  $\ell$ ) such that for  $t \geq t_7, R_1 \geq \max\{R_j : 2 \leq j \leq 6\}$  and  $\ell \geq C_{13}t$  it holds*

$$(4.72) \quad P\{\Theta(t) \text{ and for some } \widehat{\pi} \in \Xi(J, \ell, t), \\ \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_4(R_1, \mathbf{i}, k) \geq \ell/4\} \\ \leq 2 \exp[-\sqrt{t + \ell}] + K_1 [t \log t]^d \exp[-K_3 \ell/4].$$

*Proof.* We begin with proving the deterministic inequality

$$(4.73) \quad \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} I_4(r, \mathbf{i}, k) \leq 2 \cdot 3^d \widetilde{C}_1 \sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} I_3(r, \mathbf{i}, k) + 4 \cdot 3^d \widetilde{C}_1 t / \Delta_r.$$

This inequality holds for each  $r$  and each  $J$ -path  $\widehat{\pi}$ . To see this, fix  $\widehat{\pi}$  and consider the time intervals  $\chi_j := [j\Delta_r / (2\widetilde{C}_1), (j + 1)\Delta_r / (2\widetilde{C}_1))$ . An  $r$ -block  $\mathcal{B}_r(\mathbf{i}, k)$  can intersect  $\widehat{\pi}|_{[0, t]}$  only during a  $\chi_j$  with  $0 \leq j < 2\widetilde{C}_1 t / \Delta_r$ . Fix such a  $j$  and assume that for this  $j$ ,  $\widehat{\pi}|_{\chi_j}$  intersects exactly  $\sigma_j$  distinct good  $r$ -blocks. There is then a subcollection of at least  $\alpha_j := \lceil 3^{-d} \sigma_j \rceil$  of these blocks such that no two of them have spatial parts which are adjacent on  $\mathcal{L}$ . Denote this subcollection of good  $r$ -blocks by  $\mathcal{B}_r(\mathbf{i}_1, k), \dots, \mathcal{B}_r(\mathbf{i}_{\alpha_j}, k)$ , where  $k$  is such that  $\chi_j \subset [k\Delta_r, (k + 1)\Delta_r)$  (only  $r$ -blocks with this value of  $k$  can intersect  $\widehat{\pi}$  during  $\chi_j$ ). Without loss of generality assume these blocks are ordered in the order in which  $\widehat{\pi}|_{\chi_j}$  first visits them. Then, for each  $u < \alpha_j$  let  $(x', s')$  be the earliest point in  $\mathcal{B}_r(\mathbf{i}_u, k) \cap \widehat{\pi}|_{\chi_j}$ , and  $(x'', s'')$  the earliest point in  $\mathcal{B}_r(\mathbf{i}_{u+1}, k) \cap \widehat{\pi}|_{\chi_j}$ . By our choice of the blocks  $\mathcal{B}_r(\mathbf{i}_\ell, k)$  we then have that  $\|\mathbf{i}_{u+1} - \mathbf{i}_u\| > 1$  and  $(x'', s'') \in \mathcal{B}_r(\mathbf{i}_{u+1}, k)$ . Hence  $x''$  lies outside  $\prod_{s=1}^d [(i_u(s) - 1)\Delta_r, (i_u(s) + 2)\Delta_r)$ , so that the piece of  $\widehat{\pi}$  from  $(x', s')$  to  $(x'', s'')$  is a  $J$ -path from  $(x', s') \in \mathcal{B}_r(\mathbf{i}_u, k)$  to  $(x'', s'')$  with  $s', s'' \in \chi_j$  and  $x''$  outside of  $\prod_{s=1}^d [(i_u(s) - 1)\Delta_r, (i_u(s) + 2)\Delta_r)$  and  $s'' \in [s', (s' + \Delta_r / (2\widetilde{C}_1))$ . Thus there have to be at least  $\alpha_j - 1 \geq 3^{-d} \sigma_j - 1$  good  $r$ -blocks  $\mathcal{B}_r(\mathbf{i}, k)$  which are counted in  $\sum_{(\mathbf{i}, k)}^{(\widehat{\pi}, r)} I_3(r, \mathbf{i}, k)$ . The blocks so obtained for one given value of  $j$  are distinct by construction. However, we may obtain the same block  $\mathcal{B}_r(\mathbf{i}, k)$  a number of times for different values of  $j$ . We already saw that this can

happen only for  $\chi_j \subset [k\Delta_r, (k+1)\Delta_r)$ , and hence only for  $2\tilde{C}_1$  values of  $j$ . Consequently,

$$\begin{aligned} & \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} [I_3(r, \mathbf{i}, k) + I_4(r, \mathbf{i}, k)] \\ &= [\text{number of good } r\text{-blocks which intersect } \hat{\pi}] \\ &\leq \sum_{0 \leq j < 2\tilde{C}_1 t / \Delta_r} \sigma_j, \end{aligned}$$

and

$$\begin{aligned} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_3(r, \mathbf{i}, k) &\geq \frac{1}{2\tilde{C}_1} \sum_{0 \leq j < 2\tilde{C}_1 t / \Delta_r} [\alpha_j - 1] \\ &\geq \frac{1}{2\tilde{C}_1} \sum_{0 \leq j < 2\tilde{C}_1 t / \Delta_r} [3^{-d}\sigma_j - 1] \\ &\geq \frac{1}{2 \cdot 3^d \tilde{C}_1} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} [I_3(r, \mathbf{i}, k) + I_4(r, \mathbf{i}, k)] - 2t / \Delta_r. \end{aligned}$$

(4.73) is now immediate.

Now, by virtue of (4.47), (4.55) and (4.69) it holds for  $t \geq$  some  $t_7$  and for  $r \leq R(t)$ , but large enough,

$$\begin{aligned} P\left\{\Theta(t) \text{ and } \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_3(r, \mathbf{i}, k) \geq \frac{2 + K_{17}}{[\Delta_r]^{2d+3}}(t + \ell) \text{ for some } \hat{\pi} \in \Xi(J, \ell, t)\right\} \\ \leq 2 \exp[-\sqrt{t + \ell}]. \end{aligned}$$

Combined with (4.73) this shows that also

$$\begin{aligned} (4.74) \quad P\left\{\Theta(t) \text{ and } \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_4(r, \mathbf{i}, k) \geq 2 \cdot 3^d \tilde{C}_1 \frac{2 + K_{17}}{[\Delta_r]^{2d+3}}(t + \ell) + \frac{4 \cdot 3^d \tilde{C}_1}{\Delta_r} t \right. \\ \left. \text{for some } \hat{\pi} \in \Xi(J, \ell, t)\right\} \leq 2 \exp[-\sqrt{t + \ell}]. \end{aligned}$$

We now take  $R_6$  so large that for  $R_6 \leq r \leq R(t)$ , (4.74) holds as well as

$$(4.75) \quad 2 \cdot 3^d \tilde{C}_1 \frac{2 + K_{17}}{[\Delta_r]^{2d+3}} \log \Gamma_r \leq \frac{K_2}{4\Delta_r}$$

(with  $\log \Gamma_r$  given by (4.44)) and

$$(4.76) \quad 4(12)^d \frac{4K_2}{\Delta_r} \leq \frac{1}{8}.$$

Finally we take for given  $R_1, C_{13} = C_{13}(R_1) \geq 1$  so large that

$$(4.77) \quad \frac{4 \cdot 3^d \tilde{C}_1}{1 + C_{13}} \log \Gamma_{R_1} \leq \frac{K_2}{4}.$$

With  $R_1 \geq \max\{R_2, \dots, R_6\}$  we then set

$$\varepsilon_{R_1} = 2 \cdot 3^d \tilde{C}_1 \frac{2 + K_{17}}{[\Delta_{R_1}]^{2d+3}} + \frac{4 \cdot 3^d \tilde{C}_1}{(1 + C_{13}) \Delta_{R_1}}.$$

For  $\ell \geq C_{13}t$ , (4.74) (with  $r = R_1$ ) then gives the following bound for the first term in the right hand side of (4.20):

$$(4.78) \quad P \left\{ \Theta(t) \text{ and for some } \hat{\pi} \in \Xi(J, \ell, t), \right. \\ \left. \sum_{(i,k)}^{(\hat{\pi}, R_1)} I_4(\hat{\pi}, R_1, k) \geq \varepsilon_{R_1}(t + \ell) \right\} \\ \leq 2 \exp[-\sqrt{t + \ell}].$$

To prove (4.72) we shall apply Lemma 15 once more. As in Lemma 17 we have (4.7) and (4.8) for  $M_{\text{in}}(R_1, \mathbf{i}, k) \leq \tilde{M}(R_1, \mathbf{i}, k)$  with the  $\tilde{M}(R_1, \mathbf{i}, k)$  Poisson variables with  $\theta_{R_1}, \log \Gamma_{R_1}$  as in (4.44) with  $r$  replaced by  $R_1$ . Moreover, for  $\ell \geq C_{13}t \geq t$ ,

$$x = \frac{\ell}{4} \geq 4(12)^d \frac{4K_2}{\Delta_{R_1}}(t + \ell)$$

satisfies condition (4.19) by virtue of our choices (4.75)–(4.77). Thus (4.72) follows from (4.20) and (4.78).  $\square$

*Proof of Proposition 13.* The definitions of  $M_{\text{out}}$  and  $M_{\text{in}}$ , and the lines just before (4.45) show that for  $\hat{\pi} \in \Xi(J, \ell, t)$ , on the event  $\Theta(t)$ ,

$$(4.79) \quad j(t, \hat{\pi}) \leq \sum_{r=R_1}^{R(t)} \sum_{(i,k)}^{(\hat{\pi}, r)} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k) \\ + \sum_{r=R_1+1}^{R(t)} \sum_{(i,k)}^{(\hat{\pi}, r)} M_{\text{in}}(r, \mathbf{i}, k) I_2(r, \mathbf{i}, k) \\ + \sum_{(i,k)}^{(\hat{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_3(R_1, \mathbf{i}, k) \\ + \sum_{(i,k)}^{(\hat{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_4(R_1, \mathbf{i}, k).$$

Now any  $\hat{\pi} \in \Xi(\ell, t)$  has  $\ell$  jumps during  $[0, t]$  and therefore, if  $\Xi(J, \ell, t)$  is nonempty, then for some  $\hat{\pi}$  one of the four sums in the right hand side here

must be at least  $\ell/4$ . Consequently,

(4.80)

$$P\{\Theta(t) \text{ and } \Xi(J, \ell, t) \neq \emptyset\}$$

$$\begin{aligned} &\leq P\{\Theta(t) \text{ and } \sup_{\hat{\pi} \in \Xi(J, \ell, t)} \sum_{r=R_1}^{R(t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k) \geq \ell/4\} \\ &\quad + P\{\Theta(t) \text{ and } \sup_{\hat{\pi} \in \Xi(J, \ell, t)} \sum_{r=R_1+1}^{R(t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{in}}(r, \mathbf{i}, k) I_2(r, \mathbf{i}, k) \geq \ell/4\} \\ &\quad + P\{\Theta(t) \text{ and } \sup_{\hat{\pi} \in \Xi(J, \ell, t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_3(R_1, \mathbf{i}, k) \geq \ell/4\} \\ &\quad + P\{\Theta(t) \text{ and } \sup_{\hat{\pi} \in \Xi(J, \ell, t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, R_1)} M_{\text{in}}(R_1, \mathbf{i}, k) I_4(R_1, \mathbf{i}, k) \geq \ell/4\}. \end{aligned}$$

We now restrict ourselves to  $\ell \geq C_{13}t$  and take  $R_1 \geq \max\{R_j : 2 \leq j \leq 6\}$  such that also

$$(4.81) \quad \Delta_{R_1} > \frac{C_0^6}{C_0^6 - 1} [16K_4 \vee 64(12)^d K_2].$$

Finally we take  $t \geq \max\{t_j : 1 \leq j \leq 7\}$  and large enough for some further inequalities below. We stress that all these requirements do not depend on the value of  $\ell$ .

Now, to estimate the first term in the right hand side of (4.80), assume that

$$\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k) \leq \frac{K_4(t + \ell)}{\Delta_r}$$

for all  $\hat{\pi} \in \Xi(J, \ell, t)$  and all  $R_1 \leq r \leq R(t)$ . Then also for all such  $\hat{\pi}$  and  $\ell \geq C_{13}t \geq t$ ,

$$\begin{aligned} &\sum_{r=R_1}^{R(t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} M_{\text{out}}(r, \mathbf{i}, k) I_1(r, \mathbf{i}, k) \\ &\quad < \sum_{r=R_1}^{\infty} \frac{K_4(t + \ell)}{\Delta_r} = \frac{K_4(t + \ell)}{\Delta_{R_1}} \frac{C_0^6}{C_0^6 - 1} \quad (\text{since } \Delta_r = C_0^{6r}) \\ &\quad \leq \frac{\ell}{4} \quad (\text{by (4.81)}). \end{aligned}$$

It therefore follows from (4.28), and the fact that  $R(t) \sim [d \log C_0]^{-1} \log \log t$  (see (4.13)), that the first term in the right hand side of (4.80) is at most

$$\sum_{R_1 \leq r \leq R(t)} K_5 \exp[-K_6(t + \ell)/[\log t]^6] \leq K_{20}(\log \log t) \exp[-K_6(t + \ell)/[\log t]^6]$$

for some constant  $K_{20}$ .

In the same way, but using (4.43) instead of (4.28), we obtain that the second term in the right hand side of (4.80) is at most

$$K_{20}(\log \log t) \exp [-\sqrt{(t+\ell)}].$$

The third term in the right hand side also contributes at most

$$K_{20}(\log \log t) \exp[-\sqrt{t+\ell}],$$

by virtue of (4.48) and (4.71). Finally, if  $\ell \geq C_{13}t$ , then by (4.72) the fourth term in the right hand side is at most

$$2 \exp[-\sqrt{t+\ell}] + K_1[t \log t]^d \exp[-K_3\ell/4].$$

We now substitute these estimates in the right hand side of (4.80) and sum over  $\ell \geq C_{13}t$ . This yields

$$(4.82) \quad P\{\Theta(t) \text{ and } \Xi(J, \ell, t) \neq \emptyset \text{ for some } \ell \geq C_{13}t\} \leq K_{21} \exp[-K_{22}\sqrt{t}]$$

for suitable constants  $K_{21}, K_{22}$  and all large  $t$ . We add  $P\{[\Theta(t)]^c\}$  (see (4.16)) to obtain that  $P\{\Xi(J, \ell, t) \neq \emptyset \text{ for some } \ell \geq C_{13}t\} \leq 2t^{-2}$  for large  $t$ . Hence, by Borel-Cantelli, a.s.  $\bigcup_{\ell \geq C_{13}t} \Xi(J, \ell, t) = \emptyset$  for all large integers  $t$ . In view of (4.12) and the lines following it, this implies that a.s.  $J(t, \mathbf{0}) \leq C_{13}t$  for all large integers  $t$ . Since  $J(t, \mathbf{0})$  is nondecreasing in  $t$  this implies Proposition 13 with  $C_{12} = 2C_{13}$ .  $\square$

REMARK 6. Note that (4.82) proves the explicit estimate

$$(4.83) \quad P\{\Theta(t) \text{ and } J(t, x) \geq C_{13}t\} \leq K_{21} \exp[-K_{22}\sqrt{t}]$$

for each fixed  $x \in \mathbb{Z}^d$ , for all large  $t$ .

### 5. Extinction for large $\lambda$

In this section we show that  $\lambda_c < \infty$ . We shall use the  $r$ -blocks  $\mathcal{B}_r(\mathbf{i}, k)$  and their pedestals  $\mathcal{V}_r(\mathbf{i}, k) = V(\mathbf{i}, k) \times \{(k-1)\Delta_r\}$  as defined in Section 4.  $\tilde{C}_1$  is defined just after (4.45). We shall work with the  $\{Y_t(\lambda)\}$ -process in this section. This process starts with independent mean  $\mu_A$  Poisson variables  $N_A(x, 0-)$  for the number of  $A$ -particles “just before time 0” and one additional  $B$ -particle at  $\mathbf{0}$  at time 0, as explained in the abstract. The  $B$ -particles turn back into  $A$ -particles at rate  $\lambda$ , independently of everything else;  $\lambda$  is called the recuperation rate. A particle  $\rho'$  which recuperated at time  $s'$  turns into a  $B$ -particle again at time  $s'' := \inf\{s > s' : \rho' \text{ jumps onto another } B\text{-particle } \rho'' \text{ or vice versa at time } s\}$ .

If there is a  $B$ -particle at the space-time point  $(x, t)$ , then there is a genealogical path which starts at  $(\mathbf{0}, 0)$  and reaches  $(x, t)$ . In particular, this means that, for some  $\ell$ , there exist times  $s_0 = 0 < s_1 < \dots < s_\ell < s_{\ell+1} = t$  and particles  $\rho_0, \rho_1, \dots, \rho_\ell$  such that  $\rho_0$  is a  $B$ -particle at  $(\mathbf{0}, 0)$ ,  $\rho_\ell$  is the given  $B$ -particle at  $(x, t)$ , and  $\rho_i$  jumps onto the position of  $\rho_{i-1}$  or vice versa at

time  $s_i$ ,  $1 \leq i \leq \ell$ ; moreover,  $\rho_i$  is of type  $B$  during  $[s_i, s_{i+1}]$ ,  $0 \leq i \leq \ell$ . (See the construction in the proof of Proposition 5 in [KSa] as well as the comments in the paragraph following (2.1) above.) Note that it is not necessary that all particles  $\rho_i$ ,  $0 \leq i \leq \ell$ , are distinct; it is possible that  $\rho_i = \rho_j$  if  $|i - j| > 1$ . This is due to the possibility of recuperation, and cannot be ruled out, as was done in the case without recuperation studied in [KSc]. We shall extend our definition of  $J$ -path somewhat, so that a genealogical path such as just discussed is also a  $J$ -path. In Section 4 we considered only  $A$ -particles. But the paths of the particles are not influenced at all by the types under our basic assumption that the  $A$  and  $B$ -particles perform the same random walk. We can therefore define a  $J$ -path to be any path which coincides at all times with some particle, irrespective of type. Otherwise these paths are exactly as discussed in the beginning of Section 4. All arguments of the preceding section, and in particular, its principal result, Proposition 13, remain valid. To see this, one simply has to ignore the types of all particles. A genealogical path for a particle at time  $t$  coincides at each time in  $[0, t]$  with some  $B$ -particle. Moreover, each jump of a genealogical path coincides with the jump of some particle, and a genealogical path is therefore a  $J$ -path on  $[0, t]$ . Moreover, in our model, it has to start at  $\mathbf{0}$ , because that is the only site with  $B$ -particles at time 0.

In this section we want to prove the following result:

**PROPOSITION 24.** *For sufficiently large  $\lambda$  there a.s. exists a (random) time  $\tau < \infty$  such that there are no  $B$ -particles in  $\{Y_t(\lambda)\}$  after  $\tau$ .*

The idea of the proof is as follows. Assume that there is a  $B$ -particle at  $(x, t)$  and let  $\hat{\pi} : [0, t] \rightarrow \mathbb{Z}^d \times [0, t]$  be its genealogical path. For a fixed large  $r$  we consider all  $r$ -blocks which intersect  $\hat{\pi}$ . Of course there are at least  $\lfloor t/\Delta_r \rfloor$  such blocks, since each  $r$ -block only extends over an interval of length  $\Delta_r$  in the time direction. The next lemma is the principal one. It states that for each of these  $r$ -blocks at least one of four events  $G(j)$  has to occur. We shall then show in a series of lemmas that there are (with high probability for large  $t$ ) for each  $j$  at most  $t/(10\Delta_r)$   $r$ -blocks which intersect  $\hat{\pi}$  and have  $G(j)$  occurring. Actually the next lemma leaves one exceptional case. At the end of the section we show that with high probability this exceptional case contributes at most a bounded number of  $r$ -blocks which intersect  $\hat{\pi}$ . In total that gives at most  $5t/(10\Delta_r) < \lfloor t/\Delta_r \rfloor$   $r$ -blocks which intersect  $\hat{\pi}$ . This contradiction shows that with high probability there are for large  $t$  no points  $(x, t)$  with a  $B$ -particle.

For integral  $z \geq -1$  we shall use the abbreviation

$$v_r(k, z) := [k + z/(4\tilde{C}_1)]\Delta_r.$$

LEMMA 25. *Let there be a B-particle at  $(x, t)$  in the  $\{Y_t(\lambda)\}$ -process and let  $\widehat{\pi} : [0, t] \rightarrow \mathbb{Z}^d \times [0, t]$  be a genealogical path from  $(\mathbf{0}, 0)$  to  $(x, t)$  and let  $\mathcal{B}_r(\mathbf{i}, k)$  be an  $r$ -block which intersects  $\widehat{\pi}$  in a point  $(y'', s'')$  with  $k \geq 1$  and  $v_r(k, z) \leq s'' < v_r(k, z + 1) \leq t$  for some integer  $z \in [0, 4\tilde{C}_1)$ . Then one of the following four events must occur:*

- $G(1) = G_r(1, \mathbf{i}, k) := \{\mathcal{B}_r(\mathbf{i}, k) \text{ is bad}\};$
- $G(2, z) = G_r(2, z, \mathbf{i}, k)$   
 $:= \{\mathcal{B}_r(\mathbf{i}, k) \text{ is good, and in the } \{Y_t(\lambda)\}\text{-process}$   
*which continues from time  $v_r(k, z - 1)$  with the particles*  
*in  $V_r(\mathbf{i})$  only, there are still some B-particles at time  $v_r(k, z)$ \};*
- $G(3, z) = G_r(3, z, \mathbf{i}, k)$   
 $:= \{\mathcal{B}_r(\mathbf{i}, k) \text{ is good, but there is a particle which is}$   
*outside  $V_r(\mathbf{i})$  at some time  $u \in [v_r(k, z - 1), (k + 1)\Delta_r)$*   
*and which visits  $\prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1]$*   
*at some later time in  $[u, (k + 1)\Delta_r)$ \};*
- $G(4, z) = G_r(4, z, \mathbf{i}, k)$   
 $:= \{\mathcal{B}_r(\mathbf{i}, k) \text{ is good, but there is a J-path from}$   
*some  $(y', s')$  to  $(y'', s'')$  with*  
 $y' \in \partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1],$   
 $y'' \in \prod_{s=1}^d [i(s)\Delta_r, (i(s) + 1)\Delta_r)$  *and*  
 $v_r(k, z - 1) \leq s' \leq s'' \leq v_r(k, z + 1),$   
*and this J-path uses only particles*  
*which were in  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$ \}.*

*Proof.* If  $\mathcal{B}_r(\mathbf{i}, k)$  intersects  $\widehat{\pi}$ , then they must have some point  $(y'', s'')$  with  $y'' \in \prod_{s=1}^d [(i(s)\Delta_r, (i(s) + 1)\Delta_r)$  and  $s'' \in [k\Delta_r, (k + 1)\Delta_r)$  in common (by the definition of  $\mathcal{B}_r(\mathbf{i}, k)$ ). Clearly there must then exist an integer  $z \in [0, 4\tilde{C}_1)$  such that  $v_r(k, z) \leq s'' < v_r(k, z + 1)$ . We fix such a  $(y'', s'')$  for the remainder of this proof, and remind the reader that we assume  $k \geq 1$  in this lemma. Since  $(y'', s'')$  is on the genealogical path for  $(x, t)$  there must be a B-particle present at  $(y'', s'')$ , as we already pointed out. Let  $\rho^*$  be such a B-particle.

If  $G(1)$  fails, then  $\mathcal{B}_r(\mathbf{i}, k)$  is good, so that this may be assumed to be the case in  $G(2, z)$ - $G(4, z)$ . Now assume that none of  $G(1)$ ,  $G(2, z)$  or  $G(3, z)$  occur. Since  $G(3, z)$  fails, any particle at  $(y'', s'')$  is one of the particles which was in  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$ . In particular, this must be true for  $\rho^*$ . Consider the genealogical path for  $\rho^*$ . Let  $\widehat{\pi}_0$  be the piece of this last genealogical path over the time interval  $[v_r(k, z - 1), s'']$ . This is a genealogical path for  $\rho^*$  in a system which starts with all the particles at time  $v_r(k, z - 1)$ . Assume  $\widehat{\pi}_0$  arises from particles  $\rho_i$ , such that  $\rho_i$  jumps to the position of  $\rho_{i-1}$  or vice versa at time  $s_i$ ,  $1 \leq i \leq \ell$ , and that  $\rho_i$  has type  $B$  during  $[s_i, s_{i+1}]$ ,  $0 \leq i \leq \ell$ , with  $s_0 = v_r(k, z - 1)$ ,  $s_{\ell+1} = s''$ , and  $\rho_\ell = \rho^*$ , as explained in the second paragraph of this section. Let  $s_{i_0} \leq v_r(k, z) \leq s_{i_0+1}$ . We claim that one of the  $\rho_i$  with  $i \leq i_0$  must have been outside  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$ . Indeed, if this is not the case, then the system starting with only the particles in  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$  has at least one  $B$ -particle at the time  $v_r(k, z)$ . To see this, observe that if  $\rho_0 \dots, \rho_{i_0}$  all came from  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$ , then, by induction on  $i$ , each of these  $\rho_i$  would be a particle of type  $B$  during  $[s_i, s_{i+1}]$  in the system of particles which were in  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$ . In particular  $\rho_{i_0}$  would be of type  $B$  at time  $v_r(k, z)$  in this system. This would contradict the assumption that  $G(2, z)$  does not occur. Our claim follows.

In particular, there is a maximal index  $m \leq \ell$  for which  $\rho_m$  was outside  $V_r(\mathbf{i})$  at some time in  $[v_r(k, z - 1), s_{m+1}] \subset [v_r(k, z - 1), s'']$ . In fact this maximal  $m$  is less than  $\ell$ , since  $\rho_\ell = \rho^*$  is a particle in  $\prod_{s=1}^d [(i(s)\Delta_r, (i(s)+1)\Delta_r)$  at time  $s''$ , and  $G(3, z)$  fails. Since  $\rho_m$  is outside  $V_r(\mathbf{i})$  at some time  $u \in [v_r(k, z - 1), s'']$ , it does not enter  $\prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1]$  during  $[u, (k+1)\Delta_r)$  (because  $G(3, z)$  fails). This means that at time  $s_{m+1}$ ,  $\rho_m$  and hence also  $\rho_{m+1}$ , are outside  $\prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1]$ . The path  $\widehat{\pi}_0$  therefore must intersect  $\partial \prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1]$  sometime during  $[s_{m+1}, s''] \subset [v_r(k, z - 1), v_r(k, z + 1)]$ , because its endpoint at time  $s''$  lies in  $\mathcal{B}_r(\mathbf{i}, k)$ . Let the latest intersection of  $\widehat{\pi}_0$  with  $\partial \prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1]$  occur at time  $s' \in [s_{m+1}, s'')$  and position  $y'$ . Then the piece of  $\widehat{\pi}_0$  over the time interval  $[s', s'']$  is a  $J$ -path which uses at most the particles  $\rho_{m+1}, \dots, \rho_\ell$ , all of which were in  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$  (by our choice of  $m$ ). Thus  $G(4, z)$  occurs with this  $J$ -path, while  $(y', s')$  and  $(y'', s'')$  have all the required properties.  $\square$

We now start on showing that each  $G(j)$  occurs on relatively few blocks. Here  $G(j)$  is short for  $\bigcup_{0 \leq z < 4\tilde{C}_1} G(j, z)$  in case  $j = 2, 3, 4$ . We remind the reader of the definitions of  $\Phi_r(\ell)$  and  $\Xi(J, \ell, t)$  in (3.90), (3.91), (4.17) (see also (6.1), (6.8) and (6.9) in [KSa]; note that “good” is now defined as in [KSa] and not as in [KSc]). In (4.12) and the lines following it we showed that

$$P\{\text{there exists a } J\text{-path starting at } \mathbf{0} \text{ which is not in } \bigcup_{\ell \geq 0} \Xi(J, \ell, t)\} \leq 2e^{-t}$$



for large  $t$ . In addition, Proposition 13 (or rather (4.82)) says that for suitable constants  $C_{13}, K_{21}, K_{22}$

$$P\{\Theta(t) \text{ and } \bigcup_{\ell \geq C_{13}t} \Xi(J, \ell, t) \neq \emptyset\} \leq K_{21} \exp[-K_{22}\sqrt{t}].$$

Finally, (4.16) says that  $P\{[\Theta(t)]^c\} \leq t^{-2}$  for large  $t$ . It follows from these that for all  $r$

$$\begin{aligned} &P\{\text{there is a } B\text{-particle at time } t \text{ in the } \{Y(\lambda)\}\text{-process}\} \\ &\leq P\{\text{there exists some genealogical path } \hat{\pi} \\ &\quad \text{leading to a } B\text{-particle at time } t\} \\ &\leq 2e^{-t} + t^{-2} + K_{21} \exp[-K_{22}\sqrt{t}] \\ &\quad + P\{\Theta(t) \text{ and there exists a path in } \bigcup_{\ell < C_{13}t} \Xi(J, \ell, t) \\ &\quad \text{which starts at } \mathbf{0}\} \\ &\leq 2e^{-t} + t^{-2} + K_{21} \exp[-K_{22}\sqrt{t}] \\ &\quad + \sum_{j=1}^4 P\{\Theta(t) \text{ and there exists a path in } \bigcup_{\ell < C_{13}t} \Xi(J, \ell, t) \\ &\quad \text{which starts at } \mathbf{0} \text{ and intersects} \\ &\quad \text{more than } t/(10\Delta_r) \text{ } r\text{-blocks } \mathcal{B}_r(\mathbf{i}, k) \\ &\quad \text{with } k \geq 1 \text{ for which } G(j) \text{ occurs}\} \\ &\quad + P\{\Theta(t) \text{ and there exists a path in } \bigcup_{\ell < C_{13}t} \Xi(J, \ell, t) \\ &\quad \text{which starts at } \mathbf{0} \text{ and which intersects} \\ &\quad \text{more than } t/(10\Delta_r) \text{ } r\text{-blocks } \mathcal{B}_r(\mathbf{i}, 0)\}. \end{aligned}$$

It therefore suffices for Proposition 24 to prove for some fixed  $r, \lambda$  and  $1 \leq j \leq 4$

$$(5.1) \quad P\{\Theta(t) \text{ and there exists a path in } \bigcup_{\ell < C_{13}t} \Xi(J, \ell, t) \text{ which starts at } \mathbf{0} \text{ and intersects more than } t/(10\Delta_r) \text{ } r\text{-blocks } \mathcal{B}_r(\mathbf{i}, k) \text{ with } k \geq 1 \text{ for which } G(j) \text{ occurs}\} \rightarrow 0,$$

as well as

$$(5.2) \quad P\{\Theta(t) \text{ and there exists a path in } \bigcup_{\ell < C_{13}t} \Xi(J, \ell, t) \text{ which starts at } \mathbf{0} \\ \text{and which intersects more than } t/(10\Delta_r) \text{ } r\text{-blocks } \mathcal{B}_r(\mathbf{i}, 0)\} \rightarrow 0,$$

as  $t \rightarrow \infty$ .

For  $G(1)$  (5.1) is contained in Lemma 15 of [KSa]. Indeed, Lemma 15 in [KSa] proves that for suitable constants  $K_{12}, K_{13}, \kappa_0$

$$(5.3) \quad P\{\text{there exists some path in } \Xi(\ell, t) \text{ which intersects more than} \\ K_{13}\kappa_0(t + \ell) \exp[-K_{12}C_0^{r/4}] \text{ bad } r\text{-blocks for some } r \geq d, \ell \geq 0\} \\ \leq \frac{2}{t^2}$$

for all large  $t$ . We merely have to take  $r_1 \geq d$  so large that

$$K_{13}\kappa_0(1 + C_{13}) \exp[-K_{12}C_0^{r_1/4}] \leq \frac{1}{10\Delta_{r_1}}$$

to obtain for any  $r \geq r_1$

$$(5.4) \quad P\{\text{there exists some path in } \bigcup_{\ell < C_{13}t} \Xi(\ell, t) \text{ which intersects} \\ \text{more than } t/(10\Delta_r) \text{ bad blocks}\} \\ \leq \frac{2}{t^2},$$

which gives (5.1) for  $j = 1$ .

The next two lemmas will imply (5.1) for  $j = 2$ . It is the only place where the recuperation rate  $\lambda$  plays a role. For simplicity we formulate this lemma only in the form in which we use it, even though there is a more general version. We generalize the definition (4.56) to

$$(5.5) \quad \mathcal{J}_r(\mathbf{i}, u) := \sigma\text{-field generated by the } N_A(x, 0-), x \in \mathbb{Z}^d, \text{ and} \\ \text{all paths during } [0, u], \text{ as well as the paths} \\ \text{on } [u, \infty) \text{ of all particles outside } V_r(\mathbf{i}) \text{ at time } u,$$

Similarly we generalize the definition of a good pedestal. Specifically, we say that  $V_r(\mathbf{i}) \times \{u\}$  is good, if

$$U_r(x, u) \leq \gamma_r \mu_A C_0^{dr} \text{ for all } x \text{ for which } \mathcal{Q}_r(x) \subset V_r(\mathbf{i}).$$

The  $\gamma_r$  are introduced just before (4.4); their only property important to us here is (4.4). The definition of a good  $r$ -block then shows that if  $\mathcal{B}_r(\mathbf{i}, k)$  is good, then so is  $V_r(\mathbf{i}) \times \{u\}$  for any  $u \in [(k - 1)\Delta_r, (k + 1)\Delta_r)$ .

LEMMA 26. For each  $r \geq 1, T \geq 0$  and  $0 < \varepsilon \leq 1$  there exists a  $\lambda_1(r, T, \varepsilon)$  such that for all  $\lambda \geq \lambda_1$ , all  $(\mathbf{i}, k)$  with  $k \geq 1$  and all  $u \in [(k-1)\Delta_r, (k+1)\Delta_r)$

$$(5.6) \quad P\{V_r(\mathbf{i}) \times \{u\} \text{ is good and the system which consists at time } u \in [(k-1)\Delta_r, (k+1)\Delta_r) \text{ of the particles in } V_r(\mathbf{i}) \text{ only, and which develops from time } u \text{ on according to the rules for } \{Y_t(\lambda)\}, \text{ has some } B\text{-particles at time } u + T \mid \mathcal{J}_r(\mathbf{i}, u)\} \leq \varepsilon.$$

*Proof.* Let  $\rho_1, \rho_2, \dots, \rho_f$  be all the particles in  $V_r(\mathbf{i})$  at time  $u$ . If  $V_r(\mathbf{i}) \times \{u\}$  is good, then there are at most  $\nu = \nu_r := [7\Delta_r]^d \gamma_0 \mu_A C_0^{dr} + 1$  particles in  $V_r(\mathbf{i})$  at time  $u$ , so that  $f \leq \nu$  (see the beginning of the proof of Lemma 17 for an explanation of the extra term 1 here). Let  $N$  be a large integer and set  $u_m = u + mT/N, 0 \leq m \leq N$ . It suffices to show that with probability at least  $1 - \varepsilon$  all particles  $\rho_1, \dots, \rho_f$  have type  $A$  at some  $u_m, 0 \leq m \leq N$ . Indeed if this happens at time  $u_m$ , then the particles  $\rho_1, \dots, \rho_f$  will all have type  $A$  at all times after  $u_m$  (since we are ignoring interactions with all other particles in the system of this lemma). But whatever types and locations  $\rho_1, \dots, \rho_f$  have at time  $u_m$ , there is a conditional probability of at least  $\exp[-fDT/N][1 - e^{-\lambda T/N}]^f$  that none of the particles  $\rho_i, 1 \leq i \leq f$ , has a jump during  $[u_m, u_{m+1}]$ , but that all of them have a recuperation event during  $[u_m, u_{m+1}]$ . If this happens, then all  $\rho_i$  will be of type  $A$  at time  $u_{m+1}$ . (Note that here we use our rule that a jump is needed before a recuperated particle can become reinfected.) It follows from this that the left hand side of (5.6) is at most

$$\left[1 - \exp[-fDT/N][1 - e^{-\lambda T/N}]^f\right]^N.$$

Now take  $N_0(\nu_r, T, \varepsilon)$  such that

$$\left[1 - \frac{1}{2} \exp[-fDT/N_0]\right]^{N_0} \leq \varepsilon \text{ for all } f \leq \nu_r,$$

and then  $\lambda_1 = \lambda_1(r, T, \varepsilon)$  such that  $[1 - e^{-\lambda_1 T/N_0}]^{\nu_r} \geq 1/2$ . (5.6) holds for this value of  $\lambda_1$ .  $\square$

LEMMA 27. For each  $r \geq 1$  there exists a  $\lambda_0(r)$  such that for  $\lambda \geq \lambda_0$ , (5.1) with  $G(j)$  replaced by  $\bigcup_{0 \leq z < 4\tilde{c}_1} G_r(2, z, \mathbf{i}, k)$  holds.

*Proof.* Fix  $r \geq 1, z \in \{0, 1, \dots, 4\tilde{C}_1 - 1\}$  and  $\varepsilon > 0$ . Define  $Y(z, \mathbf{i}, k) = I[G_r(2, z, \mathbf{i}, k)]$  and let  $\tilde{Y}(z, \mathbf{i}, k)$  be the indicator function of

$\{V_r(\mathbf{i}) \times \{v_r(k, z - 1)\}$  is good and the system which consists  
 only of the particles in  $V_r(\mathbf{i})$  at time  $v_r(k, z - 1)$  and which  
 develops from time  $v_r(k, z - 1)$  on according to the rules  
 for  $\{Y_t(\lambda)\}$  has some  $B$ -particles at time  $v_r(k, z)\}$ .

It is immediate from the definitions that  $Y(z, \mathbf{i}, k) \leq \tilde{Y}(z, \mathbf{i}, k)$ . Moreover, by applying the preceding lemma with  $u = v_r(k - 1, z)$  and  $T = \Delta_r/(4\tilde{C}_1)$ , we see that we can find a  $\lambda_0 = \lambda_0(\varepsilon)$  such that for all  $\lambda \geq \lambda_0, 0 \leq z < 4\tilde{C}_1$ ,

$$(5.7) \quad P\{\tilde{Y}(z, \mathbf{i}, k) = 1 \mid \mathcal{J}_r(\mathbf{i}, v_r(k, z - 1))\} \leq \varepsilon.$$

A fortiori, the same inequality holds if  $\tilde{Y}$  is replaced by  $Y$ . In fact we have more. Let  $\mathbf{a} \in \{0, 1, \dots, 11\}^d$  and  $b = 0$  or  $1$ . Let further  $Z(z, \mathbf{i}, k)$  be a family of independent random variables with

$$P\{Z(z, \mathbf{i}, k) = 1\} = 1 - P\{Z(z, \mathbf{i}, k) = 0\} = \varepsilon.$$

(5.7) shows that the conditional probability of  $\{\tilde{Y}(z, \mathbf{i}, k) = 1\}$ , given all the  $\tilde{Y}(z, \mathbf{j}, \ell)$  with  $\mathbf{j} \equiv \mathbf{i} \pmod{\mathbf{a}}, \ell \equiv k \pmod{b}$  and  $(\mathbf{j}, \ell)$  preceding  $(\mathbf{i}, k)$  in the lexicographic order, is at most  $\varepsilon$ . Just as in the proof of Lemma 21, this shows that for fixed  $z$ , the family  $\{\tilde{Y}(z, \mathbf{i}, k) : (\mathbf{i}, k) \equiv (\mathbf{a}, b)\}$  lies stochastically below the family  $\{Z(\mathbf{i}, k) : (\mathbf{i}, k) \equiv (\mathbf{a}, b)\}$ . Again this statement remains true if  $\tilde{Y}$  is replaced by the smaller  $Y$ .

We can now continue exactly as in Lemma 11 of [KSa] or the proof following (4.49) in Section 4. Note that if there exists a  $\hat{\pi} \in \Xi(J, \ell, t)$  which intersects more than  $K_1 \varepsilon^{1/(d+1)}(t + \ell)/\Delta_r$  blocks  $\mathcal{B}_r(\mathbf{i}, k)$  for which  $\bigcup_{0 \leq z < 4\tilde{C}_1} G_r(2, z, \mathbf{i}, k)$  occurs, then there exists a  $0 \leq z < 4\tilde{C}_1$  such that  $\hat{\pi}$  intersects more than  $K_1 \varepsilon^{1/(d+1)}(t + \ell)/[4\tilde{C}_1 \Delta_r]$  blocks  $\mathcal{B}_r(\mathbf{i}, k)$  for which  $G_r(2, z, \mathbf{i}, k)$  occurs. We therefore have for  $t \geq 1$  and for some constants  $K_1$ - $K_3$ , which do not depend on  $\varepsilon, \ell$  or  $r$ ,

$$\begin{aligned}
 &P\{\text{there exists an } \ell < C_{13}t \text{ and a path } \hat{\pi} \in \Xi(J, \ell, t) \\
 &\quad \text{such that } \hat{\pi} \text{ intersects more than} \\
 &\quad K_1 \varepsilon^{1/(d+1)}(t + \ell)/\Delta_r \text{ blocks } \mathcal{B}_r(\mathbf{i}, k) \\
 &\quad \text{with } k \geq 1 \text{ for which } \bigcup_{0 \leq z < 4\tilde{C}_1} G_r(2, z, \mathbf{i}, k) \text{ occurs}\}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{0 \leq z < 4\tilde{C}_1} P\{\text{there exists an } \ell < C_{13}t \text{ and a path } \hat{\pi} \in \Xi(J, \ell, t) \\
 &\quad \text{such that } \hat{\pi} \text{ intersects more than} \\
 &\quad K_1 \varepsilon^{1/(d+1)}(t + \ell)/[4\tilde{C}_1 \Delta_r] \text{ blocks} \\
 &\quad \mathcal{B}_r(\mathbf{i}, k) \text{ with } k \geq 1 \text{ for which } G_r(2, z, \mathbf{i}, k) \text{ occurs}\} \\
 &\leq \sum_{0 \leq z < 4\tilde{C}_1} \sum_{(\mathbf{a}, b)} \sum_{0 \leq \ell < C_{13}t} P\{\text{there exists a path } \hat{\pi} \in \Xi(J, \ell, t) \\
 &\quad \text{such that } \hat{\pi} \text{ intersects more than} \\
 &\quad K_1 \varepsilon^{1/(d+1)}(t + \ell)/[8(12)^d \tilde{C}_1 \Delta_r] \text{ blocks } \mathcal{B}_r(\mathbf{i}, k) \\
 &\quad \text{with } (\mathbf{i}, k) \equiv (\mathbf{a}, b), k \geq 1 \text{ for which } G_r(2, z, \mathbf{i}, k) \text{ occurs}\} \\
 &\leq \sum_{(\mathbf{a}, b)} \sum_{0 \leq \ell < C_{13}t} K_2 \exp\left[-K_3 \frac{(t + \ell)}{\Delta_r} \varepsilon^{1/(d+1)}\right].
 \end{aligned}$$

For  $\varepsilon$  so small that  $K_1(1 + C_{13})\varepsilon^{1/(d+1)} < 1/10$ , this gives (5.1) for  $G(2) = \bigcup_{0 \leq z < 4\tilde{C}_1} G(2, z)$ . □

The case  $j = 3$  of (5.1) has already been handled in the proof of Lemma 18, where we introduced  $I_5$ . Indeed, since simple random walk cannot jump across  $\mathcal{W}_r(\mathbf{i})$ ,  $I[\bigcup_{0 \leq z < 4\tilde{C}_1} G(3, z, \mathbf{i}, k)] \leq I_5(r, \mathbf{i}, k)$ . Thus, the number of good blocks  $\mathcal{B}_r(\mathbf{i}, k)$  which intersect any given  $J$ -path  $\hat{\pi}|_{[0, t]} \in \Xi(\ell, t)$  and for which the event  $\bigcup_{0 \leq z < 4\tilde{C}_1} G(3, z, \mathbf{i}, k)$  occurs is bounded by  $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_5(r, \mathbf{i}, k)$ . The inequalities (4.53) and (4.54) therefore apply. Moreover, the  $x$  in (4.52) is less than  $(t + \ell)/(10\Delta_r)$  for  $\ell < C_{13}t$  and  $r$  large enough, say for  $r \geq r_2$ . Thus (5.1) with  $\bigcup_{0 \leq z < 4\tilde{C}_1} G(3, z)$  in the place of  $G(j)$  holds for  $r \geq r_2$ .

Finally we turn to (5.1) with  $j = 4$ . As we shall show now, all the steps for this estimate are already given in the estimates for  $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} I_{3,2}(r, \mathbf{i}, k)$  in the preceding section. Define

$$\begin{aligned}
 \tilde{G}(4, z) &= \tilde{G}_r(4, z, \mathbf{i}, k) \\
 &= \{\mathcal{V}_r(\mathbf{i}, k) \text{ is good, but there is a } J\text{-path from some} \\
 &\quad (y', s') \text{ to } (y'', s'') \text{ with } y' \in \partial \prod_{s=1}^d [(i(s) - 1)\Delta_r, (i(s) + 2)\Delta_r - 1], \\
 &\quad y'' \in \prod_{s=1}^d [i(s)\Delta_r, (i(s) + 1)\Delta_r) \\
 &\quad \text{and } v_r(k, z - 1) \leq s' \leq s'' \leq v_r(k, z + 1), \\
 &\quad \text{and this } J\text{-path uses only particles which were in } \mathcal{V}_r(\mathbf{i}) \\
 &\quad \text{at time } v_r(k, z - 1)\}.
 \end{aligned}$$

Then, by the definition of a good block  $\mathcal{B}_r(\mathbf{i}, k)$  and its pedestal  $\mathcal{V}_r(\mathbf{i}, k) = V_r(\mathbf{i}) \times \{(k-1)\Delta_r\}$ , we have  $G(4, z) \subset \tilde{G}(4, z)$ . Moreover, we have the following analogue of (4.66) for  $n = \Delta_r^{3(d+1)^2}$ :

$$\begin{aligned}
 (5.8) \quad & P\{\tilde{G}_r(4, z, \mathbf{i}, k) \text{ occurs} \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \\
 & \leq \frac{(\gamma_0 \mu_A 7^d \Delta_r^d + 1) 2\Delta_r D^2}{\Delta_r^{3(d+2)^2}} \\
 & \quad + \sum_{\substack{v_r(k, z-1) - 1/n \leq j/n \\ \leq v_r(k, z+1) + 1/n}} \sum_{x \in \partial \prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1]} \\
 & \quad \quad P\{\mathcal{V}_r(\mathbf{i}, k) \text{ is good and there exists} \\
 & \quad \quad \text{a } J\text{-path from } (x, j/n) \text{ to } \prod_{s=1}^d [i(s)\Delta_r, (i(s)+1)\Delta_r) \\
 & \quad \quad \text{of time duration } \leq \Delta_r / (2\tilde{C}_1) + 1/n \leq (\Delta_r - 1) / \tilde{C}_1 \\
 & \quad \quad \text{and which uses only particles which are in } V_r(\mathbf{i}) \\
 & \quad \quad \text{at time } v_r(k, z-1) \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\}.
 \end{aligned}$$

We can then follow the proof of Lemma 20 from (4.66) on to obtain that the left hand side of (5.8) is at most

$$\begin{aligned}
 (5.9) \quad & \frac{(\gamma_0 \mu_A 7^d \Delta_r^d + 1) 2\Delta_r D^2}{\Delta_r^{3(d+1)^2}} \\
 & \quad + \sum_{\substack{v_r(k, z-1) - 1/n \leq j/n \\ \leq v_r(k, z+1) + 1/n}} \sum_{x \in \partial \prod_{s=1}^d [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r - 1]} 4e^{-\Delta_r / \tilde{C}_1}.
 \end{aligned}$$

In turn, it is easy to see that there exists an  $r_3$  such that for  $r \geq r_3$  the expression (5.9) is for each  $k \geq 1, 0 \leq z < 4\tilde{C}_1$  at most  $\Delta_r^{-2(d+1)^2}$  (as in Lemma 20). We then also obtain for  $r \geq r_3$

$$\begin{aligned}
 & P\{\mathcal{B}_r(\mathbf{i}, k) \text{ is good and } G_r(4, z, \mathbf{i}, k) \text{ occurs} \\
 & \quad \text{for some } 0 \leq z < 4\tilde{C}_1 \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \\
 & \leq \sum_{0 \leq z < 4\tilde{C}_1} P\{\tilde{G}_r(4, z, \mathbf{i}, k) \mid \mathcal{J}_r(\mathbf{i}, (k-1)\Delta_r)\} \leq 4\tilde{C}_1 \Delta_r^{-2(d+1)^2}.
 \end{aligned}$$

As in the last lemma the collection of random variables

$$\tilde{Y}_r(\mathbf{i}, k) := I[\tilde{G}_r(4, z, \mathbf{i}, k) \text{ occurs for some } 0 \leq z < 4\tilde{C}_1]$$

with  $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$  lies stochastically below a family of independent random variables  $Z_r(\mathbf{i}, k)$  satisfying

$$P\{Z_r(\mathbf{i}, k) = 1\} = 1 - P\{Z_r(\mathbf{i}, k) = 0\} = 4\tilde{C}_1\Delta_r^{-2(d+1)^2}.$$

Again we can now follow the proof of Lemma 11 or (5.43) in [KSa] (or the proof of (4.55) above) to conclude that (for  $r \geq r_3$ )

$$P\left\{ \sup_{\hat{\pi} \in \Xi(\ell, t)} \sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)} \tilde{Y}_r(\mathbf{i}, k) \geq K_{23} \frac{(t + \ell)}{\Delta_r^{2d+3}} \right\} \leq K_{24} \exp \left[ -K_{25} \frac{(t + \ell)}{\Delta_r^{2d+3}} \right]$$

( $\sum_{(\mathbf{i}, k)}^{(\hat{\pi}, r)}$  is as in (4.5)). If we take  $r_4 \geq r_3$  such that  $K_{23}(1 + C_{13})[\Delta_{r_4}]^{-2d+3} \leq [10\Delta_{r_4}]^{-1}$ , then (5.1) for any  $r \geq r_4$  and with  $G(j)$  replaced by

$$\bigcup_{0 \leq z < 4\tilde{C}_1} G_r(4, z, \mathbf{i}, k)$$

is an immediate consequence.

Because the case  $k = 0$  was excluded in Lemma 25 we still need an estimate for the sup over  $\{\hat{\pi} \in \Xi(J, \ell, t) : \hat{\pi}(0) = \mathbf{0}\}$  of the number of blocks  $\mathcal{B}_r(\mathbf{i}, 0)$  which intersect  $\hat{\pi}$ . There are at most  $t/(10\Delta_r)$  blocks  $\mathcal{B}_r(\mathbf{i}, 0)$  with  $\|\mathbf{i}\| \leq K_{26}t^{1/d}$ , with  $K_{26}$  some constant which depends on  $d$  and  $\Delta_r$  only. If there is a block  $\mathcal{B}_r(\mathbf{i}, 0)$  with  $\|\mathbf{i}\| > K_{26}t^{1/d}$  which intersects  $\hat{\pi}$ , then some initial piece of  $\hat{\pi}$  forms a  $J$ -path from  $\mathbf{0}$  to the outside of  $\mathcal{C}(K_{26}t^{1/d})$ . Since all points in blocks  $\mathcal{B}_r(\mathbf{i}, 0)$  have time coordinate less than  $\Delta_r \leq [2C_1]^{-1}K_{26}t^{1/d}$  (for large  $t$ ), we obtain by means of (4.11),(4.12)

$$\begin{aligned} (5.10) \quad & P\{\text{there exists some } \hat{\pi} \in \Xi(J, \ell, t) \text{ with } \hat{\pi}(0) = \mathbf{0} \\ & \quad \text{such that } \hat{\pi} \text{ intersects more than } t/(10\Delta_r) \text{ } r\text{-blocks } \mathcal{B}_r(\mathbf{i}, 0)\} \\ & \leq P\{\text{there exists a } J\text{-path from } \mathbf{0} \text{ to the outside of} \\ & \quad \mathcal{C}(K_{26}t^{1/d}) \text{ of time duration less than } [2C_1]^{-1}K_{26}t^{1/d}\} \\ & \leq 4 \exp \left[ -[2C_1]^{-1}K_{26}t^{1/d} \right]. \end{aligned}$$

Thus, also (5.2) holds.

We now take  $r = \max\{r_i : 1 \leq i \leq 4\}$  and  $\lambda \geq \lambda_0(r)$ . Then (5.1) holds for  $1 \leq j \leq 4$  and also (5.2) holds. As discussed right after the statement of Proposition 24, these properties imply Proposition 24.

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