Illinois Journal of Mathematics
Volume 50, Number 2, Summer 2006, Pages 385–394
S 0019-2082

A NOTE ON X-HARMONIC FUNCTIONS

E. B. DYNKIN

Dedicated to the memory of Joseph Leo Doob whose work was my inspiration and admiration for many years

Abstract. The Martin boundary theory allows one to describe all positive harmonic functions in an arbitrary domain E of a Euclidean space starting from the functions $k^y(x) = g(x,y)/g(a,y)$, where $g(x,y)$ is the Green function of the Laplacian and a is a fixed point of E. In two previous papers a similar theory was developed for a class of positive functions on a space of measures. These functions are associated with a superdiffusion X and we call them X-harmonic. Denote by $\mathcal{M}_c(E)$ the set of all finite measures μ supported by compact subsets of E. X-harmonic functions are functions on $\mathcal{M}_c(E)$ characterized by a mean value property formulated in terms of exit measures of a superdiffusion. Instead of the ratio $g(x,y)/g(a,y)$ we use a Radon-Nikodym derivative of the probability distribution of an exit measure of X with respect to the probability distribution of another such measure. The goal of the present note is to find an expression for this derivative.

1. Introduction

1.1. X-harmonic functions. Suppose that L is a second order uniformly elliptic operator in a domain E of $\mathbb{R}^d$. An L-diffusion is a continuous strong Markov process $\xi = (\xi_t, \Pi_x)$ in E with generator L. Let ψ be a function from $E \times \mathbb{R}_+$ to $\mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. An (L, ψ)-superdiffusion is a model of an evolution of a random cloud. It is described by a family of random measures (X_D, P_μ), where $D \subset E$ and μ is a finite measure on E.[1] If μ is concentrated on D, then X_D is concentrated on ∂D. We call X_D the *exit measure from* D. Heuristically, it describes the mass distribution on an absorbing barrier placed on ∂D.

Received July 15, 2005; received in final form October 15, 2005.
2000 *Mathematics Subject Classification.* 60J50, 31C05.
Partially supported by National Science Foundation Grant DMS-0503977.

[1] Assumptions about these random measures are formulated in Section 1.2.

We put $\mu \in \mathcal{M}_c(D)$ if μ is a finite measure concentrated on a compact subset of D. We write $D \Subset E$ if D is a bounded smooth[2] domain such that the closure $\bar{D}$ of D is contained in E. We say that a function $H : \mathcal{M}_c(E) \to \mathbb{R}_+$ is *X-harmonic* if, for every $D \Subset E$ and every $\mu \in \mathcal{M}_c(D)$,

$$P_\mu H(X_D) = H(\mu). \tag{1.1}$$

For every domain $D \subset E$ we have an inclusion $\mathcal{M}_c(D) \subset \mathcal{M}_c(E)$. We say that H is *X-harmonic in D* if

$$P_\mu H(X_O) = H(\mu) \quad \text{for all } O \Subset D, \mu \in \mathcal{M}_c(O). \tag{1.2}$$

1.2. Superdiffusions. We write $f \in \mathcal{B}$ if f is a positive $\mathcal{B}$-measurable function. We denote by $\mathcal{B}(E)$ the class of all Borel subsets of E and by $\mathcal{M}(E)$ the set of all finite measures on the σ-algebra $\mathcal{B}(E)$.

Suppose that to every open set $D \subset E$ and every $\mu \in \mathcal{M}(E)$ there corresponds a random measure[3] (X_D, P_μ) on $\mathbb{R}^d$ such that, for every $f \in \mathcal{B}(E)$,

$$P_\mu e^{-\langle f, X_D\rangle} = e^{-\langle V_D(f), \mu\rangle}, \tag{1.3}$$

where $u = V_D(f)$ satisfies the equation[4]

$$u + G_D\psi(u) = K_D f. \tag{1.4}$$

Here

$$G_D f(x) = \Pi_x \int_0^{\tau_D} f(\xi_s)\, ds, \quad K_D f(x) = \Pi_x 1_{\tau_D < \infty} f(\xi_{\tau_D}) \tag{1.5}$$

are the *Green operator* and the *Poisson operator* of ξ in D. We call the family $X = (X_D, P_\mu)$ an *(L, ψ)-superdiffusion* if, besides (1.3)–(1.4), it satisfies the following condition.

1.2.A (Markov property). For every $\mu \in \mathcal{M}_c(E)$ and every $D \Subset E$,

$$P_\mu YZ = P_\mu(YP_{X_D}Z)$$

if $Y \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\subset D}$ generated by $X_O, O \subset D$, and $Z \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\supset D}$ generated by $X_{O'}, O' \supset D$.

The existence of a (ξ, ψ)-superprocesses is proved in [Dyn02, Theorem 4.2.1] for

$$\psi(x; u) = b(x)u^2 + \int_0^\infty (e^{-tu} - 1 + tu) N(x; dt) \tag{1.6}$$

[2]We call smooth domains those of the class $C^{2,\lambda}$.

[3]A random measure on a measurable space $(S, \mathcal{B}_S)$ is a pair (X, P), where $X(\omega, B)$ is a kernel from an auxiliary measurable space $(\Omega, \mathcal{F})$ to $(S, \mathcal{B}_S)$ and P is a probability measure on $\mathcal{F}$. (We say that $p(x, B), x \in E, B \in \mathcal{B}'$, is a kernel from a measurable space $(E, \mathcal{B})$ to a measurable space $(E', \mathcal{B}')$ if it is a $\mathcal{B}$-measurable function in x and a finite measure in B.)

[4]$\psi(u)$ is a short notation for $\psi(x, u(x))$.

under broad conditions on a positive Borel function $b(x)$ and a kernel N from E to $\mathbb{R}_+$. It is sufficient to assume that

$$b(x), \int_1^\infty tN(x;dt) \quad \text{and} \quad \int_0^1 t^2 N(x;dt) \quad \text{are bounded.} \tag{1.7}$$

An important special case is the function

$$\psi(x,u) = \ell(x)u^\alpha, 1 < \alpha \le 2, \tag{1.8}$$

corresponding to $b = 0$ and

$$N(x,dt) = \tilde{\ell}(x)t^{-1-\alpha}dt,$$

where

$$\tilde{\ell}(x) = \ell(x)\left(\int_0^\infty (e^{-\lambda} - 1 + \lambda)\lambda^{-1-\alpha}d\lambda\right)^{-1}.$$

Condition (1.7) holds if $\ell(x)$ is bounded.

It follows from (1.3)–(1.5) that

$$P_\mu\{X_D(D) = 0\} = 1 \tag{1.9}$$

and

$$P_\mu\{X_D = \mu\} = 1 \quad \text{if } \mu(D) = 0. \tag{1.10}$$

Let $\mathcal{F}$ stand for the σ-algebra in Ω generated by $X_D(B)$, where $D \Subset E$ and $B \in \mathcal{B}(E)$. Denote by $\mathfrak{M}$ the σ-algebra in $\mathcal{M}_c(E)$ generated by the functions $F(\mu) = \mu(B)$ with $B \in \mathcal{B}(E)$. If $\mu \in \mathcal{M}_c(E)$ and $D \Subset E$, then, P_μ-a.s., $X_D \in \mathcal{M}_c(E)$ and X_D is a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathcal{M}_c(E), \mathfrak{M})$. Moreover, if $\mu \in \mathcal{M}_c(D)$, then, P_μ-a.s., $X_D \in \mathcal{M}(\partial D)$. It follows from (1.3) that $H(\mu) = P_\mu Y$ is $\mathfrak{M}$-measurable for every $\mathcal{F}$-measurable $Y \ge 0$.

We have:

1.2.B (Absolute continuity property). For every set $C \in \mathcal{F}_{\supset D}$ either $P_\mu(C) = 0$ for all $\mu \in \mathcal{M}_c(D)$ or $P_\mu(C) > 0$ for all $\mu \in \mathcal{M}_c(D)$.

A proof of this property can be found in [Dyn04b, Theorem 5.3.2].

We denote by $\mathcal{P}_D(\mu, \cdot)$ the probability distribution of X_D under P_μ, that is,

$$\mathcal{P}_D(\mu, A) = P_\mu\{X_D \in A\} \quad \text{for } A \in \mathfrak{M}. \tag{1.11}$$

Fix a reference point $a \in E$ and put $\mathcal{P}_D(A) = \mathcal{P}_D(\delta_a, A)$ (δ_a is the unit mass concentrated at a). By 1.2.B, there exists a Radon-Nikodym derivative

$$H_D^\nu(\mu) = \frac{\mathcal{P}_D(\mu, d\nu)}{\mathcal{P}_D(d\nu)}. \tag{1.12}$$

For every $\mu \in \mathcal{M}_c(D)$, this is a function of $\nu \in \mathcal{M}(\partial D)$ defined up to $\mathcal{P}_D$-equivalence. We continue it to $\mathcal{M}(E) \times \mathcal{M}(E)$ by setting $H_D^\nu(\mu) = 0$ off $\mathcal{M}_c(D) \times \mathcal{M}(\partial D)$. It is proved in [Dyn05, Theorem 1.1] that there exists a version of $H^{\nu}_D(\mu)$ which is $\mathfrak{M}\times\mathfrak{M}$-measurable and X-harmonic in μ in the domain D for every $\nu\in\mathcal{M}(\partial D)$. We will use the notation $H^{\nu}_D(\mu)$ for this version.

1.3. Main results. To every Polish space S there corresponds a Polish space

$$\mathcal{Z}_S=\bigcup_{n=0}^{\infty} Z^n_S.$$

For $n>0$, $Z^n_S=S^n$ is the product on n replicas of S (Z^0_S consists of a single element $\emptyset$). We call $\mathcal{Z}_S$ the configuration space over S.

We consider the configuration space over $S=D\times\mathcal{M}$, where $\mathcal{M}=\mathcal{M}(\partial D)$. We also use the configuration spaces $\mathcal{Z}_D$ and $Z_{\mathcal{M}}$. We denote by $z^n=(z_1,\dots,z_n)$ and $\nu^n=(\nu_1,\dots\nu_n)$ generic elements of $\mathcal{Z}^n_D$ and $\mathcal{Z}^n_{\mathcal{M}}$. A pair (z^n,ν^n) represents a generic point of Z^n_S. Every function f on $\mathcal{Z}_{\mathcal{M}}$ can be continued to $\mathcal{Z}_S$ by setting $f(z^n,\nu^n)=f(\nu^n)$. A similar continuation can be defined for functions on $\mathcal{Z}_D$.

We will first establish an expression for the transition function (1.11) and then deduce from this expression a formula for the X-harmonic function (1.12).

A special role is played by a mapping $N:\mathcal{Z}_{\mathcal{M}}\to\mathcal{M}$ defined by the formula

$$N(\nu^n)=\nu_1+\dots+\nu_n\quad\text{for } \nu^n=(\nu_1,\dots,\nu_n).$$

Fix domains $\tilde{D}\Subset D\Subset E$. We will introduce in Section 2 a positive Borel function ρ_μ on $\mathcal{Z}_D$ depending on a parameter $\mu\in\mathcal{M}_c(\tilde{D})$ and in Section 3 a probability measure $\mathbb{P}$ on $\mathcal{Z}_S$. (Both ρ_μ and $\mathbb{P}$ depend on $\tilde{D}$ and D.)[5]

There exists an $\mathfrak{M}\times\mathfrak{M}$-measurable function $\varphi_\mu(\nu)$ such that

$$\mathbb{P}\{\rho_\mu|N\}=\varphi_\mu(N). \tag{1.13}$$

THEOREM 1.1. *Let $D\Subset E$ and let u be the minimal solution of the problem*

$$\begin{aligned} Lu&=\psi(u)\quad\textit{in } D,\\ u&=\infty\quad\textit{on } \partial D.\end{aligned} \tag{1.14}$$

If $a\in\tilde{D}\Subset D$, then for every $f\in\mathcal{B}(\mathcal{M})$ and every $\mu\in\mathcal{M}_c(\tilde{D})$,

$$\int_{\mathcal{M}}\mathcal{P}_D(\mu,d\nu)f(\nu)=ce^{-\langle u,\mu\rangle}\mathbb{P}f(N)\varphi_\mu(N), \tag{1.15}$$

where c is a constant depending on $\tilde{D}$ and D.

[5] Instead of configurations over S we could consider configurations over $\tilde{S}=\partial\tilde{D}\times\mathcal{M}(\partial D)\subset S$ (the functions ρ_μ vanish off $\tilde{S}$), but we prefer to deal with a configuration space independent of $\tilde{D}$.

THEOREM 1.2. *In the notation of Theorem 1.1,*

$$H_D^\nu(\mu) = e^{u(a)-\langle u,\mu\rangle} \frac{\varphi_\mu(\nu)}{\varphi_a(\nu)} \quad \text{for all } \nu \in \mathcal{M}, \mu \in \mathcal{M}_c(\tilde{D}), \tag{1.16}$$

where $a \in \tilde{D}$ *and* $\varphi_a = \varphi_{\delta_a}$.

2. The function ρ_μ

2.1. We give an expression for the function ρ_μ in terms of a class of directed graphs which we call diagrams. A diagram is the union of a finite set of disjoint rooted trees with marked leaves. Each rooted tree has a single root. There exists only one tree with one leaf and only one tree with two leaves. All distinguishable trees with three leaves are presented in Figure 1.

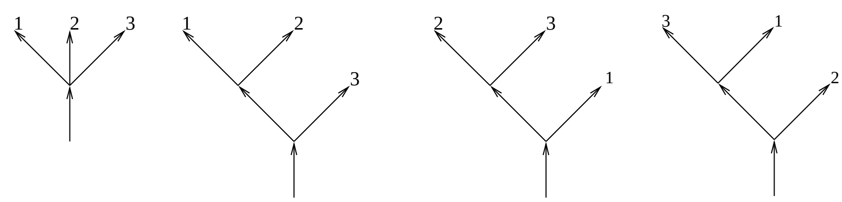

FIGURE 1

Fix domains $\tilde{D} \Subset D \Subset E$. To define ρ_μ on $\mathcal{Z}_D^n = D^n$ we consider all diagrams with n leaves. We set $\rho_\mu(z^n) = 0$ for $z^n \in S \setminus (\partial\tilde{D})^n$. In Section 2.2 we will define, for every rooted tree $\mathbb{D}$, a function $\rho_x(\mathbb{D}, z^n)$ on $(\partial\tilde{D})^n$ depending on the parameter $x \in \tilde{D}$. For $\mu \in \mathcal{M}_c(\tilde{D})$ we put

$$\rho_\mu(\mathbb{D}, z^n) = \int \rho_x(\mathbb{D}, z^n)\mu(dx). \tag{2.1}$$

For a diagram $\mathbb{D}$ which is a union of trees $\mathbb{D}_1, \dots, \mathbb{D}_k$ we put

$$\rho_\mu(\mathbb{D}, z^n) = \prod_{i=1}^{k} \rho_\mu(\mathbb{D}_i, (z^n). \tag{2.2}$$

Finally, we define

$$\rho_\mu(z^n) = \sum \rho_\mu(\mathbb{D}, z^n), \tag{2.3}$$

where $\mathbb{D}$ runs over all diagrams with n leaves.

2.2. To define $\rho_x(\mathbb{D}, \cdot)$ for a tree $\mathbb{D}$ we label the sites and the arrows of $\mathbb{D}$ by certain functions.

Put $\ell(x) = \psi_1[x, V_D(\phi)(x)]$ and consider a sequence

$$q_r(x) = (-1)^r \psi_r[x, \ell(x)] \quad \text{for } r = 2, 3, \dots, \tag{2.4}$$

where ψ_r is the r-th derivative of ψ with respect to u. (For the functions (1.6) subject to the conditions (1.7), ℓ and q_r are strictly positive.) Denote by $g(x,y)$ the Green function and by $k(x,y)$ the Poisson kernel of the operator $Lu - \ell u$ in $\tilde{D}$.

Denote by $\mathcal{V}$ the set of all sites of $\mathbb{D}$ different from leaves and roots. Mark $v \in \mathcal{V}$ by a $\tilde{D}$-valued variable y_v, the root by a $\tilde{D}$-valued variable x and the leaf i by a $\partial\tilde{D}$-valued variable z_i. Mark every arrow by the marks of its beginning and end. For instance, $(y_v, y_{v'})$ is the mark of the arrow leading from v to v'.

We attach a label $q_r(y_v)$ to $v \in \mathcal{V}$ if r is the number of arrows starting from v. The leaves and the root are labeled by the constant 1. The labels of the arrows are:

$$g(y_v, y_{v'}) \quad \text{for } (y_v, y_{v'}), \quad k(y_v, z_i) \quad \text{for } (y_v, z_i),$$
$$g(x, y_v) \quad \text{for } (x, y_v), \quad k(x, z_1) \quad \text{for } (x, z_1).$$

(The last type appears only for the tree with one leaf.)

Denote by $\mathcal{L}(\mathbb{D})$ the product of the labels of all sites and all arrows and put

$$\rho_x(\mathbb{D}, z^n) = \int \mathcal{L}(\mathbb{D}) \prod_{v \in \mathcal{V}} dy_v \quad \text{for } z^n \in (\partial\tilde{D})^n. \tag{2.5}$$

Examples. For the first diagram in Figure 1,

$$\rho_x(\mathbb{D}, z^3) = \int g(x,y) q_3(y) k(y, z_1) \gamma(dz_1) k(y, z_2) k(y, z_3) dy.$$

For the second diagram,

$$\rho_x(\mathbb{D}, z^3) = \int g(x, y_1) q_2(y_1) k(y_1, z_3) g(y_1, y_2) q_2(y_2) k(y_2, z_1) k(y_2, z_2) dy_1 dy_2.$$

(In contrast to the leaves, the enumeration of the sites in $\mathcal{V}$ is of no importance.)

3. The measure $\mathbb{P}$

3.1. The measures $\mathcal{R}_\mu$. It follows from (1.3) that, for every $\mu \in \mathcal{M}(E)$ and every $f \in \mathcal{B}(E)$,

$$\log P_\mu e^{-\langle f, X_D \rangle} = \int_E \mu(dz) \log P_z e^{-\langle f, X_D \rangle}, \tag{3.1}$$

which implies that, for every n,

$$P_\mu e^{-\langle f, X_D \rangle} = \left[P_{\mu/n} e^{-\langle f, X_D \rangle} \right]^n.$$

Hence (X_D, P_μ) is an infinitely divisible measure on ∂D and, since $P_\mu\{X_D = 0\} > 0$ for $\mu \neq 0$, there exists a finite measure $\mathcal{R}_\mu$ on $\mathcal{M}(\partial D)$ such that

$$-\log P_\mu e^{-\langle f, X_D\rangle} = \int [1 - e^{-\langle f,\nu\rangle}]\mathcal{R}_\mu(d\nu) \tag{3.2}$$

for all $f \in \mathcal{B}(E)$ (see, e.g., [Dyn04b, p. 37]). The right side in (3.2) does not depend on the value of $\mathcal{R}_\mu\{0\}$. If we put $\mathcal{R}_\mu\{0\} = 0$, then the measure $\mathcal{R}_\mu$ is determined uniquely. Put $\mathcal{R}_z = \mathcal{R}_{\delta_z}$. Formula (3.1) implies

$$\mathcal{R}_\mu = \int_D \mathcal{R}_z \mu(dz) \tag{3.3}$$

and (3.2) implies

$$c(\mu) = P_\mu\{X_D = 0\} = e^{-\mathcal{R}_\mu(\mathcal{M})}. \tag{3.4}$$

If $\mu \neq 0$, then $c(\mu) > 0$.

3.2. Definition of $\mathbb{P}$. Fix $\tilde{D} \Subset D$ and denote by γ the surface area on $\partial\tilde{D}$. Consider a measure Q on $S = D \times \mathcal{M}$ concentrated on $\partial\tilde{D} \times \mathcal{M}$ and given on $\partial\tilde{D} \times \mathcal{M}$ by the formula

$$Q(dz, d\nu) = \gamma(dz)\mathcal{R}_z(d\nu). \tag{3.5}$$

The total mass of Q is equal to $\mathcal{R}_\gamma(\mathcal{M})$. For every n, we consider a measure Q^n on $\mathcal{Z}_S$ concentrated on Z_S^n and defined by the formula

$$Q^n(dz^n, d\nu^n) = Q(dz_1, d\nu_1)\dots Q(dz_n, d\nu_n) = \gamma^n(dz^n)\mathcal{R}_{z^n}(d\nu^n). \tag{3.6}$$

The formula

$$\mathbb{P} = c(\gamma)\sum_0^\infty \frac{1}{n!}Q^n \tag{3.7}$$

defines a probability measure on Z_S depending on D and $\tilde{D}$.

4. Proof of Theorem 1.1

4.1. For the sake of brevity we put

$$\begin{aligned}
\bar{\nu} &= N(\nu^n) = \nu_1 + \dots + \nu_n \quad \text{for } \nu^n = (\nu_1, \dots, \nu_n),\\
\mathcal{R}_{z^n}(d\nu^n) &= \mathcal{R}_{z_1}(d\nu_1)\dots\mathcal{R}_{z_n}(d\nu_n),\\
\mu^n(dz^n) &= \mu(dz_1)\dots\mu(dz_n),\\
\mathcal{R}_\mu^n(d\nu^n) &= \mathcal{R}_\mu(d\nu_1)\dots\mathcal{R}_\mu(d\nu_n).
\end{aligned}$$

We define a linear operator C^n mapping positive Borel functions f on $\mathcal{M} = \mathcal{M}(\partial D)$ to functions on D^n by the formula

$$C^n f(z^n) = \int_{\mathcal{M}^n} e^{-\langle 1,\bar{\nu}\rangle} f(\bar{\nu}))\mathcal{R}_{z^n}^n(d\nu^n). \tag{4.1}$$

Put

$$A^n(\mu, f) = \int C^n f(z^n)\mu^n(dz^n) = \int_{\mathcal{M}^n} e^{-\langle 1,\bar\nu\rangle} f(\bar\nu)\mathcal{R}^n_\mu(d\nu^n) \quad \text{for } \mu \in \mathcal{M}_c(D).$$

It follows from formula (3.6) in [Dyn04b, Chapter 5, p. 58] that

$$P_\mu e^{-\langle 1,X_D\rangle} f(X_D) = c(\mu)\sum_0^\infty \frac{1}{n!} A^n(\mu, f). \tag{4.2}$$

4.2. We claim that, for every $\tilde D \Subset D$,

$$P_\mu e^{-\langle 1,X_D\rangle} f(X_D) = \sum_{n=0}^\infty \frac{1}{n!} P_\mu\{X_D = 0, A^n(X_{\tilde D}, f)\}. \tag{4.3}$$

Indeed, by the Markov property 1.2.A,

$$P_\mu e^{-\langle 1,X_D\rangle} f(X_D) = P_\mu P_{X_{\tilde D}} e^{-\langle 1,X_D\rangle} f(X_D). \tag{4.4}$$

By (4.2),

$$P_{X_{\tilde D}} e^{-\langle 1,X_D\rangle} f(X_D) = c(X_{\tilde D})\sum_0^\infty \frac{1}{n!} A^n_D(X_{\tilde D}, f). \tag{4.5}$$

By the Markov property and (3.4),

$$P_\mu\{X_D = 0, A^n(X_{\tilde D}, f)\} = P_\mu c(X_{\tilde D}) A^n(X_{\tilde D}, f)\}. \tag{4.6}$$

Formula (4.3) follows from (4.4), (4.5) and (4.6).

4.3. Put

$$B^n(F) = \int F(z^n) X^n_{\tilde D}(dz^n) \quad \text{for } F \in \mathcal{B}(\mathcal{Z}^n_{\tilde D}). \tag{4.7}$$

It follows from Theorem 1.2 and Theorem 3.1 in [Dyn04b, Chapter 5] that, for $\mu \in \mathcal{M}_c(\tilde D)$,

$$P_\mu e^{-\langle \Phi, X_{\tilde D}\rangle} B^n(F) = e^{-\langle V_{\tilde D}(\Phi),\mu\rangle} \int F(z^n)\rho_\mu(z^n)\gamma^n(dz^n) \tag{4.8}$$

if $\Phi \in \mathcal{B}(\partial\tilde D)$ is the subject to the condition $r_1 < \Phi < r_2$ with $0 < r_1 < r_2 < \infty$. Here ρ_μ is the function defined in Section 2 and γ is the surface area on $\partial\tilde D$ (as in Section 3.2).

Choose a constant $\lambda > 0$ and put $\Phi = V_D(\lambda)$. By the Markov property and (1.3),

$$\begin{aligned} P_\mu\{B^n(F) e^{-\langle \lambda, X_D\rangle}\} &= P_\mu\{B^n(F) P_{X_{\tilde D}} e^{-\langle \lambda, X_D\rangle}\} \\ &= P_\mu\{B^n(F) e^{-\langle \Phi, X_{\tilde D}\rangle}\} \end{aligned} \tag{4.9}$$

and $V_{\tilde D}(\Phi) = \Phi$. By (4.8) and (4.9),

$$P_\mu\{B^n(F) e^{-\langle \lambda, X_D\rangle}\} = e^{-\langle \Phi,\mu\rangle} \int F(z^n)\rho_\mu(z^n)\gamma^n(dz^n). \tag{4.10}$$

Note that, as $\lambda \to \infty$, $\Phi = V_D(\lambda)$ tends to the minimal solution u of (1.14) and therefore (4.10) implies

$$P_\mu\{X_D = 0, B^n(F)\} = e^{-\langle u,\mu\rangle} \int F(z^n)\rho_\mu(z^n)\gamma^n(dz^n). \tag{4.11}$$

By (4.1) and (4.7), $A^n(X_{\tilde{D}}, f) = B^n(C^n f)$. Thus (4.11), (4.1) and (3.6) imply

$$\begin{aligned} P_\mu\{X_D = 0, A^n(X_{\tilde{D}}, f)\} &= e^{-\langle u,\mu\rangle} \int\limits_{Z_S^n} e^{-\langle 1,\bar{\nu}\rangle} f(\bar{\nu})\rho_\mu(z^n)Q^n(dz^n, d\nu^n) \\ &= e^{-\langle u,\mu\rangle} \int\limits_{Z_S^n} e^{-N} f(N)\rho_\mu dQ^n. \end{aligned} \tag{4.12}$$

By (4.3), (4.12), (3.7) and (1.13),

$$\begin{aligned} P_\mu e^{-\langle 1,X_D\rangle} f(X_D) &= ce^{-\langle u,\mu\rangle}\mathbb{P}e^{-N} f(N)\rho_\mu \\ &= ce^{-\langle u,\mu\rangle}\mathbb{P}e^{-N} f(N)\varphi_\mu(N), \end{aligned} \tag{4.13}$$

where $c = c(\gamma)^{-1}$. We obtain (1.15) by applying (4.13) to the function $f(\nu)e^{\nu(\mathcal{M})}$. □

5. Proof of Theorem 1.2

By (1.12),

$$\int \mathcal{P}_D(\mu, d\nu) f(\nu) = \int \mathcal{P}_D(d\nu) f(\nu) H_D^\nu(\mu) \tag{5.1}$$

for all $f \in \mathcal{B}(\mathcal{M})$. It follows from (1.15) that

$$\int \mathcal{P}_D(d\nu) f(\nu) H_D^\nu(\mu) = ce^{-u(a)}\mathbb{P} f(N) H_D^N(\mu)\varphi_a(N). \tag{5.2}$$

By (5.1), (1.15) and (5.2),

$$e^{-\langle u,\mu\rangle}\mathbb{P} f(N)\varphi_\mu(N) = e^{-u(a)}\mathbb{P} f(N) H_D^N(\mu)\varphi_a(N). \tag{5.3}$$

By (1.15),

$$\mathbb{P}e^{-\langle u,\mu\rangle}\varphi_\mu(N) = c^{-1}\mathcal{P}_D(\mu, \mathcal{M}) < \infty.$$

Therefore (5.3) implies

$$e^{-\langle u,\mu\rangle}\varphi_\mu(N) = e^{-u(a)} H_D^N(\mu)\varphi_a(N) \quad \mathbb{P}\text{-a.s.}. \tag{5.4}$$

Since $N(\nu) = \nu$ on $\mathcal{Z}_S^1$ and since the restriction of $\mathbb{P}$ to $\mathcal{Z}_S^1$ is $cQ(dz, d\nu) = \gamma(dz)\mathcal{R}_z(d\nu)$, we conclude from (5.4) that

$$H_D^\nu(\mu) = e^{u(a)-\langle u,\mu\rangle} \frac{\varphi_\mu(\nu)}{\varphi_a(\nu)} \quad \mathcal{R}_\gamma\text{-a.s.} \tag{5.5}$$

We have

$$\mathcal{P}_D(A) = ce^{-u(a)} \int_A \varphi_a(\nu)\mathcal{R}_\gamma(d\nu) = 0$$

if $\mathcal{R}_\gamma(A) = 0$. Hence (1.16) follows from (5.5). □

References

[Dyn02] E. B. Dynkin, *Diffusions, superdiffusions and partial differential equations*, American Mathematical Society Colloquium Publications, vol. 50, American Mathematical Society, Providence, RI, 2002. MR 1883198 (2003c:60001)

[Dyn04a] ———, *Harmonic functions and exit boundary of superdiffusion*, J. Funct. Anal. **206** (2004), 33–68. MR 2024345 (2004k:60143)

[Dyn04b] ———, *Superdiffusions and positive solutions of nonlinear partial differential equations*, University Lecture Series, vol. 34, American Mathematical Society, Providence, RI, 2004. MR 2089791 (2005i:60004)

[Dyn05] ———, *On extreme X-harmonic functions*, Math. Res. Lett. **13** (2006), 59–69. MR 2199566

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail address: `ebd1@cornell.edu`