MODULES OVER UNRAMIFIED REGULAR LOCAL RINGS[1]

BY

M. AUSLANDER

Introduction

Throughout this paper we assume that all rings are commutative, noetherian rings with unit and that all modules are finitely generated and unitary. The main object of study is what it means about two modules A and B over an unramified regular local ring to assert that the torsion submodule of $A \otimes B$ is zero. The basic fact established (see §3) is that if $A \otimes B$ is torsion-free and not zero, then

(a) A and B are torsion-free,

(b) $\mathrm{Tor}_i^R(A, B) = 0$ for all $i > 0$, and

(c) $\mathrm{hd}\, A + \mathrm{hd}\, B = \mathrm{hd}\,(A \otimes B) < \dim R$,

where hd A means the homological dimension of A (we refer the reader to [1] for notation and basic homological facts used). Using this result we give the following criteria for a module A over an unramified regular local ring R of dimension n to be free: (a) The tensor product of A with itself n-times is torsion-free; (b) $A \otimes A \otimes \mathrm{Hom}(A, R)$ is torsion-free. Section 3 concludes with some results which seem to indicate that the module theory of odd-dimensional unramified regular local rings is different from the module theory of even-dimensional unramified regular local rings.

The proofs of most of the results, including those just mentioned, are based in an essential way on the fact established in §2 that for an unramified regular local ring R and a torsion-free R-module A if $\mathrm{Tor}_i^R(A, B) = 0$ for some R-module B, then $\mathrm{Tor}_j^R(A, B) = 0$ for all $j \geqq i$. In fact, if this property of Tor can be established for arbitrary regular local rings, then almost all the results of this paper extend immediately to all regular local rings.

1. Some properties of Tor

Before proceeding to the main results of this section we review briefly some of the basic facts concerning the codimension of a module as can be found for instance in [1] or [2].

Let R be a local ring with maximal ideal $\mathfrak{m}$ and A a nonzero R-module. A sequence of elements $x_1, \cdots, x_t$ in $\mathfrak{m}$ is called an A-sequence if x_1 is not a zero-divisor in A and x_i is not a zero-divisor for $A/(x_1, \cdots, x_{i-1})A$ for all $i = 2, \cdots, t$. If $x_1, \cdots, x_t$ is an A-sequence, then it is easily seen that $t \leqq \dim R$ (where dim R means the Krull dimension of R). Thus it makes sense to talk about maximal A-sequences. It can be shown that all maximal

Received December 27, 1960.

[1] This work was done with the partial support of a National Science Foundation Grant.

A-sequences have the same length, and we call this common length the codimension of A (notation: codim A).

The codimension of a module has several important properties. Codim $A = 0$ if and only if there is an element in A whose annihilator is $\mathfrak{m}$, or what amounts to the same thing, there is an exact sequence $0 \to R/\mathfrak{m} \to A$. If the homological dimension of A (notation: hd A) is finite, then

$$\text{hd } A + \text{codim } A = \text{codim } R.$$

Finally we observe that it follows easily from primary decomposition theory of modules that if $A_1, \cdots, A_n$ are nonzero R-modules each of codimension greater than 0, then there is a single element x in $\mathfrak{m}$ which is not a zero-divisor for all the A_i.

PROPOSITION 1.1. *Let A and B be R-modules where R is a local ring with maximal ideal $\mathfrak{m}$. If* hd $A = s < \infty$ *and* codim $B = 0$, *then* $\mathrm{Tor}_s^R(A, B) \neq 0$ *and has codimension zero.*

Proof. It is well known that an R-module C is free if and only if $\mathrm{Tor}_1^R(C, R/\mathfrak{m}) = 0$. From this it follows that if $\mathrm{Tor}_i^R(C, R/\mathfrak{m}) = 0$ for some $i > 0$, then hd $C < i$.

Since codim $B = 0$, there is an exact sequence $0 \to R/\mathfrak{m} \to B \to B'' \to 0$. The fact that hd $A = s$ gives us the exact sequence

$$0 \to \mathrm{Tor}_s^R(A, R/\mathfrak{m}) \to \mathrm{Tor}_s^R(A, B).$$

By our previous observation we know that $\mathrm{Tor}_s^R(A, R/\mathfrak{m}) \neq 0$. Since $\mathfrak{m}\, \mathrm{Tor}_s^R(A, R/\mathfrak{m}) = 0$, it follows that codim $\mathrm{Tor}_s^R(A, R/\mathfrak{m}) = 0$ which gives the desired result.

THEOREM 1.2. *Let A and B be nonzero modules over the local ring R such that* hd $A = s < \infty$. *Let q be the largest integer such that* $\mathrm{Tor}_q^R(A, B) \neq 0$. *If either* codim $(\mathrm{Tor}_q^R(A, B)) \leqq 1$ *or* $q = 0$, *then we have*

$$\text{codim } B = \text{codim } (\mathrm{Tor}_q^R(A, B)) + \text{hd } A - q.$$

Proof. Part 1. Suppose codim $(\mathrm{Tor}_q^R(A, B)) \leqq 1$. Proceed by induction on codim B. If codim $B = 0$, then the result is nothing more than a restatement of Proposition 1.1.

Suppose codim $B = k > 0$. Let x in the maximal ideal of R be a nonzero divisor for B alone or for B and $\mathrm{Tor}_q^R(A, B)$ depending on whether codim $(\mathrm{Tor}_q^R(A, B))$ is zero or one. From the exact sequence

$$0 \to B \xrightarrow{x} B \to B/xB \to 0$$

we deduce the exact sequence

$$(*) \qquad \begin{aligned} \cdots \to \mathrm{Tor}_{q+1}^R(A, B) \to \mathrm{Tor}_{q+1}^R(A, B/xB) \to \mathrm{Tor}_q^R(A, B) \\ \xrightarrow{x} \mathrm{Tor}_q^R(A, B) \to \mathrm{Tor}_q^R(A, B/xB) \to \cdots. \end{aligned}$$

Thus we have that $\mathrm{Tor}_p^R(A, B/xB) = 0$ for $p > q + 1$ since $\mathrm{Tor}_p^R(A, B) = 0$ for $p > q$.

Now if codim $(\mathrm{Tor}_q^R(A, B)) = 0$, then $\mathrm{Tor}_{q+1}^R(A, B/xB) \neq 0$ and has codimension zero. Using the fact that $1 +$ codim $(B/xB) =$ codim B and that B/xB satisfies the inductive hypothesis, it is easily seen that the theorem has been established in the case codim $(\mathrm{Tor}_q^R(A, B)) = 0$.

Suppose codim $(\mathrm{Tor}_q^R(A, B)) = 1$. Since x is not a zero-divisor for $\mathrm{Tor}_q^R(A, B)$, it follows from the exact sequence $(*)$ that

(a) $0 \to \mathrm{Tor}_q^R(A, B) \xrightarrow{x} \mathrm{Tor}_q^R(A, B) \to \mathrm{Tor}_q^R(A, B/xB)$ is exact, and
(b) $\mathrm{Tor}_p^R(A, B/xB) = 0$ for $p > q$.

It follows from (a) that $\mathrm{Tor}_q^R(A, B/xB)$ contains a submodule of codimension zero, and thus that codim $(\mathrm{Tor}_q^R(A, B/xB)) = 0$. Noting that

$$1 + \text{codim } (B/xB) = \text{codim } B$$

and

$$1 + \text{codim } (\mathrm{Tor}_q^R(A, B/xB)) = \text{codim } (\mathrm{Tor}_q^R(A, B)),$$

and that B/xB satisfies the inductive hypothesis, we obtain the desired result in the case codim $(\mathrm{Tor}_q^R(A, B)) = 1$, and thus the proof of Part 1 is complete.

Part 2. Suppose that $q = 0$, i.e., $\mathrm{Tor}_p^R(A, B) = 0$ for $p > 0$. Then we want to show that codim $B =$ codim $(A \otimes B) +$ hd A. Proceed by induction on codim $(A \otimes B)$. If codim $(A \otimes B) = 0$, we are back in the part of the theorem already established in Part 1.

Suppose codim $(A \otimes B) = k > 0$. Then codim $B > 0$. For if codim $B = 0$, then we have by Proposition 1.1 that A is free, and thus codim $(A \otimes B) =$ codim $B = 0$, which is a contradiction. Let x in the maximal ideal of R be a nonzero divisor for B and $A \otimes B$. From the exact sequence

$$0 \to B \xrightarrow{x} B \to B/BxB \to 0$$

we deduce the usual exact Tor sequence $(*)$ of Part 1. It follows from $(*)$ that

(a) since $\mathrm{Tor}_p^R(A, B) = 0$ for $p > 0$, we have that $\mathrm{Tor}_p^R(A, B/xB) = 0$ for $p > 1$;

(b) $0 \to \mathrm{Tor}_1^R(A, B/xB) \to A \otimes B \xrightarrow{x} A \otimes B \to A \otimes B/xB \to 0$ is exact.

Since x is not a zero-divisor for $A \otimes B$, we have that $\mathrm{Tor}_1^R(A, B/xB) = 0$ and $1 +$ codim $(A \otimes B/xB) =$ codim $(A \otimes B)$. Therefore we can apply the inductive hypothesis to B/xB which tells us that

$$\text{codim } (B/xB) = \text{codim } (A \otimes B/xB) + \text{hd } A.$$

From this it follows that codim $(B) =$ codim $(A \otimes B) +$ hd A. This completes the proof of Part 2 of the theorem, which establishes the entire theorem.

COROLLARY 1.3. *Let R be a local ring, A and B R-modules of finite homological dimension. If* $\mathrm{Tor}_p^R(A, B) = 0$ *for all* $p > 0$, *then* $\mathrm{hd}\, A + \mathrm{hd}\, B = \mathrm{hd}\, (A \otimes B)$.

Proof. We first show that $\mathrm{hd}\, (A \otimes B) < \infty$. Let X and Y be finite projective resolutions of A and B respectively. Since $\mathrm{Tor}_p^R(A, B) = 0$ for all $p > 0$, the complex $X \otimes_R Y$ is acyclic and thus a finite projective resolution of $A \otimes B$. Therefore $\mathrm{hd}\, (A \otimes B) < \infty$.

Now suppose that codim $R = n$. Then as observed before for an R-module C we have that $\mathrm{hd}\, C + \mathrm{codim}\, C = n$ if $\mathrm{hd}\, C < \infty$. Since the homological dimensions of A, B, and $A \otimes B$ are finite, the relation

$$\mathrm{codim}\, B = \mathrm{codim}\, (A \otimes B) + \mathrm{hd}\, A$$

proved in Theorem 1.2 yields the desired result that

$$\mathrm{hd}\, A + \mathrm{hd}\, B = \mathrm{hd}\, (A \otimes B).$$

Remark. Theorem 1.2 can be established in the case R is a regular local ring by standard spectral sequence arguments.

Remark. It would be interesting to know if the formula $\mathrm{codim}\, B = \mathrm{codim}\, (\mathrm{Tor}_q^R(A, B)) + \mathrm{hd}\, A - q$ of Theorem 1.2 is valid without the additional assumptions made on q and $\mathrm{codim}\, (\mathrm{Tor}_q^R(A, B))$.

2. Rigid complexes

Let R be a ring, X a complex of R-modules, and $\mathcal{C}$ a set of R-modules. Then X is said to be *rigid with respect to the set* $\mathcal{C}$ if each module A in $\mathcal{C}$ has the property that $H_i(X \otimes A) = 0$ implies that $H_j(X \otimes A)) = 0$ for all $j \geqq i$. For the purposes of this paper the most important example of a rigid complex is that of a Koszul complex. Let R be a noetherian ring, and $x_1, \cdots, x_s$ elements in R. For each $i = 1, \cdots, s$ define C_i to be the complex

$$\cdots \to (C_i)_j \to \cdots \to (C_i)_1 \xrightarrow{x_i} (C_i)_0 \to 0,$$

where $(C_i)_j = 0$ for all $j > 1$ and all $i = 1, \cdots, s$, and where $(C_i)_1 = (C_i)_0 = R$ for all $i = 1, \cdots, s$. The complex $C_1 \otimes \cdots \otimes C_s$ is called the Koszul complex on $x_1, \cdots, x_s$ (see [2]). According to [2, 2.6] a Koszul complex is rigid with respect to the set of all finitely generated modules over the noetherian ring R. We now use this fact to show that over unramified regular local rings projective resolutions of finitely generated torsion-free R-modules are rigid with respect to all finitely generated modules where an unramified regular local ring is defined as follows. We say that a regular local ring R is *ramified* if R has characteristic zero while its residue class field has characteristic $p \neq 0$ and p is in $\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of R. If R is not ramified, we say that it is *unramified*. It is obvious that if the characteristic of R is the same as its residue field (i.e., if R contains a field), then R is unramified.

Now we observe that if R is a local ring, C a complex of R-modules, and A an R-module, then C is rigid with respect to A if and only if $\hat{C}$ is rigid with respect to $\hat{A}$, where $\hat{E}$ denotes the completion of the R-module E with respect to the $\mathfrak{m}$-adic topology of R. This follows from the well known fact that the completion of $H_i(C \otimes A)$ is isomorphic to $H_i(\hat{C} \otimes \hat{A})$ for all i and the fact that the completion of a module is the zero module if and only if the module is the zero module.

THEOREM 2.1. *Let R be a regular local ring of equal characteristic. If X is a projective resolution for a module A, then X is rigid with respect to all R-modules (i.e., with respect to all finitely generated R-modules).*

Suppose R is an unramified regular local ring of characteristic 0 whose residue class field has characteristic p. Then a projective resolution of an R-module A for which p is not a zero-divisor, is rigid with respect to all R-modules (finitely generated).

Proof. Let R be a regular local ring of equal characteristic. As observed before, it suffices to prove the theorem in the case that R is a complete regular local ring of equal characteristic. By Cohen's well known structure theorem $R = k[[X_1 \cdots X_n]]$, the ring of formal power series over the residue class field k of R. Now Serre has shown in [7; Chapter V, Part B] that

$$\operatorname{Tor}_i^{k[[X]]}(A, B) \approx \operatorname{Tor}_i^{k[[X,Y]]}(k[[X]], A \hat{\otimes}_k B) \qquad \text{for all } i,$$

where $k[[X]]$ denotes $k[[X_1, \cdots, X_n]]$, where $k[[X, Y]]$ denotes $k[[X_1, \cdots, X_n, Y_1, \cdots, Y_n]]$, where $k[[X]]$ is considered a $k[[X, Y]]$-module by means of the ring epimorphism $X_i \to X_i$ and $Y_i \to X_i$, and where $A \hat{\otimes}_k B$ is a finitely generated $k[[X, Y]]$-module called the complete tensor product of the $k[[X]]$-modules A and B. Since the kernel of $k[[X, Y]] \to k[[X]]$ is generated by the $k[[X, Y]]$-sequence $X_1 - Y_1, \cdots, X_n - Y_n$, we know by [2, 2.8] that the Koszul complex of $X_1 - Y_1, \cdots, X_n - Y_n$ is a $k[[X, Y]]$-projective resolution of $k[[X]]$. Therefore if $\operatorname{Tor}_i^{k[[X,Y]]}(k[[X]], A \hat{\otimes}_k B) = 0$, then as observed before

$$\operatorname{Tor}_j^{k[[X,Y]]}(k[[X]], A \hat{\otimes}_k B) = 0 \qquad \text{for all } j \geqq i.$$

Thus if $\operatorname{Tor}_i^{k[[X]]}(A, B) = 0$, then $\operatorname{Tor}_j^{k[[X]]}(A, B) = 0$ for all $j \geqq i$. Therefore we have that a projective resolution of A is rigid with respect to any $k[[X]]$-module B, which establishes the first part of the theorem.

Now suppose R is an unramified regular local ring of characteristic zero whose residue class field is of characteristic $p \neq 0$. Since p is a nonzero divisor for an R-module E if and only if p in $\hat{R}$ is a nonzero divisor for $\hat{E}$ (this follows from the fact that $\hat{R}$ is R-flat and $\hat{E} = E \otimes \hat{R}$), it follows that it suffices to prove the second part of the theorem also in the case R is complete. Again by a well known structure theorem of Cohen we know that $R \approx k[[X_1, \cdots, X_n]]$ where k is a complete discrete rank-one valuation ring. If A and B are R-modules such that p is not a zero-divisor for A, then Serre has shown [7, Chapter V, Part B]

$$\mathrm{Tor}_i^{k[[X]]}(A, B) \approx \mathrm{Tor}_i^{k[[X, Y]]}(k[[X]], A \hat{\otimes}_k B),$$

where $k[[X]] = k[[X_1, \cdots, X_n]]$, where

$$k[[X, Y]] = k[[X_1, \cdots, X_n, Y_1, \cdots, Y_n]],$$

where $k[[X]]$ is considered a $k[[X, Y]]$-module by means of the ring epimorphism $k[[X, Y]] \to k[[X]]$ given by $X_i \to X_i$ and $Y_i \to X_i$, and where $A \hat{\otimes}_k B$ is a finitely generated $k[[X, Y]]$-module called the complete tensor product of A and B. Since the kernel of $k[[X, Y]] \to k[[X]]$ is generated by the $k[[X, Y]]$-sequence $X_1 - Y_1, \cdots, X_n - Y_n$ we can now proceed to establish the second part of the theorem using similar arguments to those used in the equal characteristic case.

We have as an immediate consequence of this theorem

COROLLARY 2.2. *Let R be an unramified regular local ring, and A a torsion-free R-module. Then a projective resolution of A is rigid with respect to all R-modules.*

Remark. It would be very interesting to know for arbitrary local rings (or at least for arbitrary regular local rings) which modules have projective resolutions which are rigid with respect to all finitely generated modules. In particular it would be nice to know if Corollary 2.2 holds for arbitrary regular local rings.

3. Criteria for freeness

Throughout this section we assume for ease of exposition that all local rings are unramified regular local rings (unless stated to the contrary), even though some of the results can be stated in slightly more general terms.

The principal tool of this section is the following:

LEMMA 3.1. *Let R be an unramified regular local ring* ($\dim R > 0$), *and let A and B be nonzero R-modules such that $A \otimes B$ is torsion-free. Then*

(a) *A and B are torsion-free,*

(b) $\mathrm{Tor}_i^R(A, B) = 0$ *for all* $i > 0$,

(c) $\mathrm{hd}(A) + \mathrm{hd}(B) = \mathrm{hd}(A \otimes B) < \dim R$.

Proof. We first prove (a) and (b) under the additional hypothesis that A is torsion-free.

Let $0 \to B' \to B \to B'' \to 0$ be exact with B' the torsion submodule of B. Then

$$(**) \qquad \mathrm{Tor}_1^R(A, B'') \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$$

is exact. Since B' is a finitely generated torsion module, it has a nontrivial annihilator, and thus so does $A \otimes B'$. But by hypothesis $A \otimes B$ is torsion-free. Therefore the map $A \otimes B' \to A \otimes B$ is the zero map, or what is the same thing, $A \otimes B \approx A \otimes B''$. From this we have that $A \otimes B''$ is torsion-free. Since we are assuming that A is a torsion-free, finitely generated R-module, we know that there exists an exact sequence $0 \to A \to F \to F/A \to 0$ with F a finitely generated free R-module. From this exact sequence we deduce the exact sequence $0 \to \operatorname{Tor}_1^R(F/A, B'') \to A \otimes B'' \to F \otimes B''$. Since $A \otimes B''$ is torsion-free and $\operatorname{Tor}_1^R(F/A, B'')$ is a torsion module (all $\operatorname{Tor}_i(\ ,\ )$ are torsion modules for $i > 0$) it follows that the map

$$\operatorname{Tor}_1^R(F/A, B'') \to A \otimes B''$$

is the zero map. Therefore we have that $\operatorname{Tor}_1^R(F/A, B'') = 0$. The fact that B'' is torsion free, enables us to apply Corollary 2.2 to conclude that $\operatorname{Tor}_i^R(F/A, B'') = 0$ for all $i > 0$. But $\operatorname{Tor}_j^R(A, B'') \approx \operatorname{Tor}_{j+1}^R(F/A, B'')$ for all $j \geqq 1$ since F is free and $0 \to A \to F \to F/A \to 0$ is exact. Therefore we have that $\operatorname{Tor}_j^R(A, B'') = 0$ for all $j > 0$.

Going back to the exact sequence $(**)$, we have that $A \otimes B' = 0$ since $\operatorname{Tor}_1^R(A, B'') \to A \otimes B'$ is an epimorphism and $\operatorname{Tor}_1^R(A, B'') = 0$. But $A \neq 0$ by hypothesis. Therefore $B' = 0$ since the tensor product of two finitely generated modules over a local ring can be zero if and only if one of the modules is the zero module. Therefore we have established that if A and B are nonzero R-modules such that A and $A \otimes B$ are torsion-free, then B is torsion-free, and $\operatorname{Tor}_j^R(A, B) = 0$ for all $j > 0$.

Let us now return to the proof of the desired proposition. Assume that $A \otimes B$ is a nonzero torsion-free R-module, but make no further assumptions regarding A and B.

Let $0 \to A' \to A \to A'' \to 0$ be exact with A' the torsion submodule of A. Tensoring this exact sequence with B we have that $A \otimes B \approx A'' \otimes B$ since $A' \otimes B$ is a torsion module and $A \otimes B$ is torsion-free. Therefore $A'' \otimes B$ is a nontrivial torsion-free module. Since A'' and $A'' \otimes B$ are nontrivial torsion-free modules, it follows from what has already been established that B is torsion-free. Hence we have that B and $A \otimes B$ are nontrivial torsion-free R-modules. Applying again what has been established before, we have that A is torsion-free and $\operatorname{Tor}_i^R(A, B) = 0$ for $i > 0$. Thus parts (a) and (b) of the lemma have been established.

(c) From the fact that $\operatorname{Tor}_i^R(A, B) = 0$ for $i > 0$, we have by Corollary 1.3 that $\operatorname{hd} A + \operatorname{hd} B = \operatorname{hd}(A \otimes B)$. Since $A \otimes B$ is torsion-free, we know that $\operatorname{codim}(A \otimes B) > 0$, and therefore that $\operatorname{hd}(A \otimes B) < \dim R$. Thus the proof of the lemma is complete.

Remark. In the proof of parts (a) and (b) the only hypothesis on R that was used was that R is a domain and that projective resolutions of torsion-free R-modules are rigid. Thus if this fact about torsion-free R-modules holds for arbitrary regular local rings, then Lemma 3.1 and all of the consequences of it derived in the rest of this section are valid for arbitrary regular local rings.

THEOREM 3.2. *Let R be an unramified regular local ring of dimension $n > 0$. An R-module A is free if and only if the n-fold tensor product of A (i.e., $A \otimes \cdots \otimes A$ n-times) is torsion-free.*

Proof. If A is R-free, then certainly the n-fold tensor product of A is torsion-free. Suppose the n-fold tensor product of A is torsion-free. If it is zero, then $A = 0$ and is thus free. If the n-fold tensor product is not zero, then we have by Lemma 3.1 that $n(\text{hd } A) < \dim R = n$. Since hd A is an integer greater than or equal to zero, we have that hd $A = 0$ or A is free.

It should be observed that Theorem 3.2 is the best result along these lines that is possible. For let R be an arbitrary regular local ring of dimension $n \geqq 3$, let C be an R-module such that hd $C = n$, and let

$$0 \to A \to X_{n-2} \to \cdots \to X_0 \to C \to 0$$

be an exact sequence with the X_i projective R-modules. Then it is not difficult to see that hd $A = 1$ and that the $(n - 1)$-fold tensor product of A is torsion-free. Of course the n-fold tensor product is not torsion-free. Thus the dimension of R is the smallest integer n which can be used to test whether or not a module is free by looking at the torsion submodule of the n-fold tensor product. However it is possible to give other criteria for freeness by looking at the torsion submodule of the tensor product of modules which are independent of the dimension of R if one makes additional assumptions about the module A. Before going on with these other criteria we need a few observations.

Let R be a ring and A an R-module. We denote the R-module $\text{Hom}_R(A, R)$ which we will call the dual of A by A^*. We shall say that an R-module A is reflexive if the natural map $f: A \to A^{**}$ given by $f(a)(g) = g(a)$ for all a in A and g in A^* is an isomorphism. It is well known that if A is projective, then A is reflexive.

PROPOSITION 3.3. *Let R be a noetherian, integrally closed domain, A a torsion-free R-module. Then A is projective if and only if $A \otimes A^*$ is reflexive.*

Proof. It is easily seen that if A is projective, then so is A^* and $A \otimes A^*$. Thus if A is projective, then $A \otimes A^*$ is reflexive.

Suppose $A \otimes A^*$ is reflexive. Then $A \otimes A^*$ is torsion-free. Since A is torsion-free, $\text{Hom}_R(A, A)$ is torsion-free. Now it is well known that the mapping $h: A \otimes A^* \to \text{Hom}_R(A, A)$ given by $h(a \otimes f)(x) = f(x)a$ for all a and x in A and f in A^* is an isomorphism if and only if A is projective (see for instance [4, A.1]). Now if $V = A \otimes K$ where K is the field of quotients of R we have a commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ A \otimes A^* & \rightarrow & \mathrm{Hom}_R(A, A) \\ \downarrow & & \downarrow \\ V \otimes V^* & \approx & \mathrm{Hom}_K(V, V) \end{array}$$

with exact columns since $A \otimes A^*$ and $\mathrm{Hom}_R(A, A)$ are torsion-free and Hom and $\otimes$ commute with passing to rings of quotients. Therefore we have that $h: A \otimes A^* \rightarrow \mathrm{Hom}_R(A, A)$ is a monomorphism and $A \otimes A^*$ and $\mathrm{Hom}_R(A, A)$ have equal ranks (i.e., after tensoring with K the derived vector spaces have the same dimension over K). Since R is a normal and $A \otimes A^*$ is reflexive, it follows from [3, 3.4] that in order to show that the monomorphism h is onto, it suffices to show that h induces an isomorphism when one passes to the ring of quotients $R_\mathfrak{p}$ for each minimal ideal $\mathfrak{p}$ in R. But A is a torsion-free R-module. Therefore $A_\mathfrak{p}$ is a finitely generated torsion-free module over the principal ideal ring $R_\mathfrak{p}$ and is thus $R_\mathfrak{p}$-free. Hence h induces the desired isomorphism when one passes to $R_\mathfrak{p}$ for each minimal prime ideal $\mathfrak{p}$ in R. Thus h is an isomorphism, i.e., A is a projective R-module.

LEMMA 3.4. *If R is an unramified regular local ring, then $R_\mathfrak{p}$ is an unramified regular local ring for each prime ideal $\mathfrak{p}$ in R.*

Proof. It suffices to consider only the case that R has characteristic zero and the residue class field has characteristic $p \neq 0$. Let $\mathfrak{p}$ be a prime ideal in R. If $\mathfrak{p}$ does not contain p, then $R_\mathfrak{p}$ contains the field of rational numbers and is thus unramified. Suppose p is in $\mathfrak{p}$. Since p is in $\mathfrak{m} - \mathfrak{m}^2$, we know that $R/(p)$ is a regular local ring. Therefore $R_\mathfrak{p}/pR_\mathfrak{p}$ is a regular local ring, which shows that p is not in the square of the maximal ideal of $R_\mathfrak{p}$. Therefore $R_\mathfrak{p}$ is unramified.

PROPOSITION 3.5. *Let A and B be nonzero modules over the unramified regular local ring R which satisfy the conditions:* (a) *for each prime ideal $\mathfrak{p}$ in R such that $A_\mathfrak{p}$ is $R_\mathfrak{p}$-free we have that $B_\mathfrak{p}$ is $R_\mathfrak{p}$-free, and* (b) *$A \otimes B$ is torsion-free. Then B is reflexive.*

Proof. We proceed by induction on $\dim R$. Suppose that $\dim R \leqq 2$. Then we claim that B is free and thus reflexive. For if A is free, then by the hypothesis B is free. If A is not free, then A is torsion-free by Lemma 3.1 and thus has homological dimension 1. But since by Lemma 3.1 we have that $\mathrm{hd}\, A + \mathrm{hd}\, B = \mathrm{hd}(A \otimes B) < 2$, it follows that B is free.

Suppose $\dim R = k > 2$ and the proposition is valid for $\dim R < k$. Since $A \otimes B$ is torsion-free, we know that B is torsion-free, and we therefore have the exact sequence $0 \rightarrow B \rightarrow B^{**} \rightarrow B^{**}/B \rightarrow 0$. Since $R_\mathfrak{p}$ is unramified for each prime ideal $\mathfrak{p}$ in R (see Lemma 3.4), we have by the inductive hypothesis that $B_\mathfrak{p}$ is reflexive for each prime $\mathfrak{p}$ which is not the maximal ideal of R.

Therefore $(B^{**}/B)_{\mathfrak{p}} = 0$ for each prime ideal $\mathfrak{p}$ in R other than the maximal ideal of R. Thus B^{**}/B has finite length. If $B^{**}/B \neq 0$, then

$$\mathrm{hd}(B^{**}/B) = k.$$

Since $\mathrm{hd}(B^{**}) \leqq k - 2$ (see [4, 4.7]), it follows that $\mathrm{hd}\, B = k - 1$. But by Lemma 3.1 we know that $\mathrm{hd}\, A + \mathrm{hd}\, B < k$ since $A \otimes B$ is torsion-free. Therefore $\mathrm{hd}\, A = 0$, which in view of the hypothesis means that $\mathrm{hd}\, B = 0$, which is a contradiction. Thus $B^{**}/B = 0$, or in other words B is reflexive. Combining Propositions 3.3 and 3.5 we obtain

THEOREM 3.6. *Let R be an unramified regular local ring and A an R-module. Then we have that*

(a) *If $A \otimes A$ is torsion-free, then A is reflexive.*

(b) *If $A \otimes A \otimes A^*$ is torsion-free and $A^* \neq 0$, then A is free.*

(c) *If $A \approx A^*$ and the three-fold tensor product of A is torsion-free, then A is free.*

Proof. (a) We obtain the desired result immediately from Proposition 3.5 if we set $A = A$ and $B = A$.

(b) Setting $A = A$ and $B = A \otimes A^*$ in Proposition 3.5 we see immediately that $A \otimes A^*$ is reflexive and A is torsion-free. However since A is torsion-free and $A \otimes A^*$ is reflexive, we have by Proposition 3.3 that A is projective and thus free.

(c) This is an immediate consequence of (b).

We now give an example to show that parts (b) and (c) of Theorem 3.6 cannot be improved in general. We do this by showing that if R is a regular local ring of odd dimension greater than 1, then there exists an R-module A such that $A \approx A^*$ and $A \otimes A$ is torsion-free, but A is not free. In fact, the R-module A which we construct will have the additional property that $A_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free for each prime ideal $\mathfrak{p}$ other than the maximal ideal of R. Suppose the dimension of R is n, and $\mathfrak{m}$ is the maximal ideal of R. Let

$$(***) \qquad 0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to R/\mathfrak{m} \to 0$$

be a minimal resolution of $R/\mathfrak{m}$ (see [6] for definition of minimal resolutions and their basic properties). Now it is well known[2] that $\mathrm{Ext}^i_R(R/\mathfrak{m}, R) = 0$ for $i = 0, \cdots, n-1$ and $\mathrm{Ext}^n_R(R/\mathfrak{m}, R) \approx R/\mathfrak{m}$. Therefore we have that

$$(****) \qquad 0 \to X_0^* \to X_1^* \to \cdots \to X_n^* \to R/\mathfrak{m} \to 0$$

is exact and thus a projective resolution of $R/\mathfrak{m}$. Actually $(****)$ is a minimal resolution of $R/\mathfrak{m}$. For if it were not minimal, then $(****)$ would be a direct

[2] In [5] it is shown that if E is an R-module (R an arbitrary noetherian ring) whose annihilator contains an R-sequence $x_1, \cdots, x_t$ of length t but no R-sequence of longer length, then $\mathrm{Ext}^i_R(E, R) = 0$ for all $i < t$ and $\mathrm{Ext}^t(E, R) = \mathrm{Hom}_R(E, R/\mathfrak{a})$ where $\mathfrak{a} = (x_1, \cdots, x_t)$. Since the maximal ideal of a regular local ring of dimension n is generated by an R-sequence of length n, we have the desired assertion.

sum as a complex of a minimal resolution and a nontrivial resolution of (0). But since taking the dual of (****) gives us (***) back again, we would have that (***) was not minimal, which is a contradiction. Since (***) and (****) are both minimal resolutions of $R/\mathfrak{m}$, they are isomorphic complexes. Now let $A = \operatorname{Im}(X_{q+1} \to X_q)$ and $B = \operatorname{Im}(X_q^* \to X_{q+1}^*)$, where $2q = n - 1$. It follows from the fact that the complexes (***) and (****) are isomorphic that $A \approx B$. From the exact sequence $X_{q+2} \to X_{q+1} \to A \to 0$ we deduce the exact sequence $0 \to A^* \to X_{q+1}^* \to X_{q+2}^*$. Since (****) is exact, it follows that $A^* \approx B$. Therefore we have that $A \approx A^*$. Also we have that $\operatorname{hd} A = q$ since $0 \to X_n \to X_{n-1} \to \cdots \to X_{q+1} \to A \to 0$ is a minimal resolution of A. Thus $\operatorname{Tor}_i^R(A, R/\mathfrak{m}) = 0$ for $i > q$. Now from the exact sequence

$$0 \to A \to X_q \to C \to 0$$

we deduce the exact sequence $0 \to \operatorname{Tor}_1^R(A, C) \to A \otimes A \to X_q \otimes A$. Since $\operatorname{Tor}_1^R(A, C) \approx \operatorname{Tor}_{q+1}^R(A, R/\mathfrak{m})$, we have that $0 \to A \otimes A \to X_q \otimes A$ is exact, and therefore $A \otimes A$ is torsion-free. Also we have that if $\mathfrak{p} \neq \mathfrak{m}$, then $A_\mathfrak{p}$ is a module of relations in a free resolution over $R_\mathfrak{p}$ of $R/\mathfrak{m} \otimes R_\mathfrak{p} = 0$. Hence we have that $A_\mathfrak{p}$ is $R_\mathfrak{p}$-free for $\mathfrak{p} \neq \mathfrak{m}$. Therefore the module A gives us our desired example. The following theorem shows that the assumption that R is of odd dimension is essential in the example just given. This gives a strong indication that the module theory of even- and odd-dimensional regular local rings are different.

THEOREM 3.7. *Let R be an unramified regular local ring* ($\dim R > 0$), *and A an R-module satisfying the following conditions*:

(a) $\operatorname{hd} A = \operatorname{hd} A^*$,

(b) *$A \otimes A^*$ is torsion-free,*

(c) *$A_\mathfrak{p}$ is $R_\mathfrak{p}$-free for each nonmaximal prime ideal $\mathfrak{p}$ in R.*

Then $\operatorname{hd} A = 0$ *or* $(n - 1)/2$ *where n is the dimension of R.*

Therefore if n is even, A must be projective. If n is odd, then there are modules A satisfying (a), (b), (c) *and such that* $\operatorname{hd} A = (n - 1)/2$.

Proof. The conclusions of the last paragraph are either obvious or they have been established already. The first part of the theorem will follow easily from

PROPOSITION 3.8. *Let R be a regular local ring (not necessarily unramified) of dimension $n > 0$, and A an R-module which is not projective. Let j be the smallest strictly positive integer*[3] *such that* $\operatorname{Ext}^j(A, R) \neq 0$. *If* $\operatorname{hd}(\operatorname{Ext}^j(A, R)) = n$, *then*

(a) *There exists an exact sequence*

$$0 \to A^* \to Y_j \to \cdots \to Y_1 \to Y_0 \to L \to 0$$

with the Y_i free R-modules and $\operatorname{hd} L = n$.

[3] Since A is not projective and has finite homological dimension, we know by [4, 4.10] that $\operatorname{Ext}^p(A, R) \neq 0$ where $p = \operatorname{hd} A$. Therefore it makes sense to talk about the smallest strictly positive integer such that $\operatorname{Ext}^j(A, R) \neq 0$.

(b) $\operatorname{Hd} A^* = n - (j + 1)$ *if* $j < n$, *and* $\operatorname{hd} A^* = 0$ *if* $j = n$.

(c) *If B is an R-module, then* $\operatorname{Tor}_i^R(A^*, B) = 0$ *for all* $i > 0$ *if and only if* $\operatorname{hd} B \leqq j + 1$.

(d) *If B is torsion-free and* $\operatorname{hd} B \leqq j$, *then* $A^* \otimes B$ *is torsion-free.*

If we assume in addition that R is unramified, then we have

(e) *If B is torsion-free, then $A^* \otimes B$ is torsion-free if and only if* $\operatorname{hd} B \leqq j$.

(f) *If A is torsion-free, then the following statements are equivalent:*

(i) $A \otimes A^*$ *is torsion-free*;

(ii) $\operatorname{hd} A = j$;

(iii) $\operatorname{hd} A + \operatorname{hd} A^* = n - 1$.

Proof. (a) Let $0 \to X_t \to \cdots \to X_{j+1} \to X_j \to \cdots \to X_1 \to X_0 \to A \to 0$ be exact with the X_i free R-modules. Since $\operatorname{Ext}^i(A, R) = 0$ for all i such that $j \geqq i > 0$, we have that

$$0 \to A^* \to X_0^* \to X_1^* \to \cdots \to X_j^* \to L \to 0$$

is exact where $L = \operatorname{Coker}(X_{j-1}^* \to X_j^*)$. Now each X_i^* is free since the dual of a free module is free. From the definition of $\operatorname{Ext}^j(A, R)$ we know that there is a monomorphism $\operatorname{Ext}^j(A, R) \to L$. Since $\operatorname{hd}(\operatorname{Ext}^j(A, R)) = n$, we know that $\operatorname{hd} L = n$, which establishes part (a) of the proposition.

(b) follows immediately from (a).

(c) From (a) it follows that $\operatorname{Tor}_i^R(A^*, B) \approx \operatorname{Tor}_{i+j+1}^R(L, B)$ for any R-module B. Since $\operatorname{hd} L = n$, we know that $\operatorname{codim} L = 0$. Thus we are in a position to apply Proposition 1.1 from which we deduce that

$$\operatorname{Tor}_t^R(L, B) \neq 0$$

where $t = \operatorname{hd} B$. Therefore $\operatorname{hd} B \leqq j + 1$ if and only if $\operatorname{Tor}_i^R(A^*, B) = 0$ for all $i > 0$.

(d) and (e) Let $0 \to A^* \to Y_j \to \cdots \to Y_0 \to L \to 0$ be the exact sequence with the Y_i free and $\operatorname{hd} L = n$ whose existence was established in (a).

Suppose $0 \to A^* \to Y_j \to C \to 0$ is exact. We now show that $A^* \otimes B$ is torsion-free if and only if $\operatorname{Tor}_1^R(C, B) = 0$.

Since $\operatorname{Tor}_1^R(C, B)$ is a torsion module, we deduce from the exact sequence

$$0 \to \operatorname{Tor}_1^R(C, B) \to A^* \otimes B \to Y_j \otimes B \to C \otimes B \to 0$$

and the fact that B is torsion-free, that $A^* \otimes B$ is torsion-free if and only if $\operatorname{Tor}_1^R(C, B) = 0$. Now $\operatorname{Tor}_i^R(C, B) = \operatorname{Tor}_{j+i}^R(L, B)$ for all $i > 0$ since $0 \to C \to Y_{j-1} \to \cdots \to Y_0 \to L \to 0$ is exact with the Y_i free. Therefore if $\operatorname{hd} B \leqq j$, then $\operatorname{Tor}_1^R(C, B) = 0$, which means that $A^* \otimes B$ is torsion-free.

On the other hand if $A^* \otimes B$ is torsion-free, then

$$\operatorname{Tor}_1^R(C, B) = \operatorname{Tor}_{j+1}^R(1, B) = 0$$

which in the case that R is unramified means that $\operatorname{Tor}_p^R(L, B) = 0$ for all

$p > j$. Since $\operatorname{hd} L = n$, we have by Proposition 1.1 that $\operatorname{hd} B \leqq j$. Thus (d) and (e) are established.

(f) By parts (d) and (e) we know that $A \otimes A^*$ is torsion-free if and only if $\operatorname{hd} A \leqq j$. Since $\operatorname{Ext}^j(A, R) \neq 0$, we know that $\operatorname{hd} A \geqq j$. Thus we have that $A \otimes A^*$ is torsion-free if and only if $\operatorname{hd} A = j$. Now A is torsion-free, and thus $\operatorname{hd} A < n$. Therefore by (a) we know that $\operatorname{hd} A^* = n - j - 1$. Therefore $\operatorname{hd} A = j$ if and only if $\operatorname{hd} A + \operatorname{hd} A^* = n - 1$.

Remark. It should be observed that part (f) can be used to give another proof of the fact that if $A \otimes A \otimes A^*$ is torsion-free and A is torsion-free, then A is projective (see Theorem 3.6).

We now return to the proof of Theorem 3.7. Suppose that A satisfies the conditions of the hypothesis of Theorem 3.7. Then for each nonmaximal prime ideal $\mathfrak{p}$ in R we have that $A_\mathfrak{p}$ is $R_\mathfrak{p}$-free. Since

$$R_\mathfrak{p} \otimes \operatorname{Ext}^i_R(A, R) \approx \operatorname{Ext}^i_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p}) \qquad \text{for all } i$$

and

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p}) = 0 \qquad \text{for all } i > 0,$$

we have that $\operatorname{Ext}^i(A, R)$ has finite length. Therefore if A is not projective, then A satisfies the hypothesis of Proposition 3.8 since certainly a nonzero module of finite length has homological dimension equal to the dimension of the ring R. Since we are also assuming that $A \otimes A^*$ is torsion-free, we have by (f) of Proposition 3.8 that $\operatorname{hd}(A) + \operatorname{hd}(A^*) = n - 1$. Therefore imposing the last condition that $\operatorname{hd} A = \operatorname{hd} A^*$ we have that

$$\operatorname{hd}(A) = (n - 1)/2,$$

which establishes Theorem 3.7.

We conclude this section with another consequence of Proposition 3.8.

PROPOSITION 3.9. *Let R be an unramified regular local ring, and A a nonzero reflexive R-module such that $A_\mathfrak{p}$ is $R_\mathfrak{p}$-free for each nonmaximal prime ideal $\mathfrak{p}$ in R. If B is a torsion-free R-module, then $A \otimes B$ is torsion-free if and only if* $\operatorname{hd} A + \operatorname{hd} B \leqq n - 1$, *where n is the dimension of R ($n > 0$).*

Proof. Since $A_\mathfrak{p}$ is $R_\mathfrak{p}$-free for all nonmaximal prime ideals $\mathfrak{p}$ in R, we know that $A^*_\mathfrak{p}$ is $R_\mathfrak{p}$-free for all nonmaximal ideals $\mathfrak{p}$ in R. Thus for each $i > 0$ we have that $\operatorname{Ext}_R(A^*, R)$ has finite length, since

$$0 = \operatorname{Ext}_{R_\mathfrak{p}}(A^*_\mathfrak{p}, R_\mathfrak{p}) = R_\mathfrak{p} \otimes \operatorname{Ext}_R(A, R)$$

for all nonmaximal prime ideals $\mathfrak{p}$ in R. Since the proposition is obviously true if A^* is projective, we might as well assume A^* is not projective. Then A^* satisfies the hypothesis of Proposition 3.8. Let j be the smallest strictly positive integer such that $\operatorname{Ext}^j_R(A, R) \neq 0$. Since $A = A^{**}$, we have by (a) of Proposition 3.8 that $\operatorname{hd}(A) = n - j - 1$ (observe that $j \neq n$ since A is torsion-free). Now by (e) of Proposition 3.8 we have that $A^* \otimes B$ is torsion-free if and only if $\operatorname{hd} B \leqq j$. Using the fact that $j = (n-1) - \operatorname{hd} A$ we have that $A^* \otimes B$ is torsion-free if and only if $\operatorname{hd} A + \operatorname{hd} B \leqq n-1$.

4. Further applications

In this section we give some applications of the notion of a rigid complex in directions different from those already given.

PROPOSITION 4.1. *Let R be an arbitrary local ring, X a free positive complex over R (i.e., each component X_i of X is free and $X_i = 0$ for $i < 0$) such that (a) $H_0(X) = A \neq 0$, (b) if C is an R-module and $H_1(X \otimes C) = 0$, then $H_i(X \otimes C) = 0$ for all $i \geqq 1$. If Y is a positive complex such that $X \otimes Y$ is acyclic, then Y is acyclic.*

Proof. Filtering the double complex $X \otimes Y$ in the standard manner by defining $F_p(X \otimes Y) = \sum_{p \geqq r} \sum_q X_r \otimes Y_q$, we obtain a spectral sequence with $E^2_{p,q} = H_p(X \otimes H_q(Y))$. Since $H_n(X \otimes Y) = 0$ for all $n > 0$ by hypothesis, we have that $E^\infty_{p,q} = 0$ for all p and q greater than zero. Now $d^2_{p,q}: E^2_{p,q} \to E^2_{p-2,q+1}$ and $E^2_{p,q} = 0$ for $p < 0$ and $q < 0$. Therefore

$$E^2_{1,0} = E^i_{1,0}$$

for all $i \geqq 2$. But $E^\infty_{1,0} = 0$, which means that $0 = E^2_{1,0} = H_1(X \otimes H_0(Y))$. Since X is a rigid complex we know that $0 = H_p(X \otimes H_0(Y)) = E^2_{p,0}$ for all $p \geqq 1$. Proceeding by induction we have that

$$0 = E^2_{p,q} = H_p(X \otimes H_q(Y))$$

for all $p \geqq 1$ and all q. Combining this with the fact that $E^\infty_{p,q} = 0$, for all p and q greater than zero, we have that

$$0 = E^2_{0,q} = H_0(X \otimes H_q(Y)) = A \otimes H_q(Y) \qquad \text{for} \quad q > 0.$$

Since R is a local ring and $A \neq 0$, we conclude that $H_q(Y) = 0$ for all $q > 0$, i.e., Y is acyclic.

PROPOSITION 4.2. *Let R be a local ring, X a positive free complex, and i an integer greater than zero such that if $H_i(X \otimes C) = 0$ for some R-module C, then $H_j(X \otimes C) = 0$ for all $j \geqq i$. Suppose $x_1, \cdots, x_t$ is an A-sequence where $A = H_0(X) \neq 0$; then $X \otimes K$, where K is the Koszul complex on $x_1, \cdots, x_t$, has the property that if $H_i(X \otimes K \otimes C) = 0$ for a module C, then*

$$H_j(X \otimes K \otimes C) = 0 \qquad \textit{for all} \quad j \geqq i.$$

Proof. Since the Koszul complex is free, the complex $X \otimes K$ is free. Also since the tensor product of complexes is associative, it is easily seen that it is sufficient to prove the proposition in the case $t = 1$. By [2, 1.1] we have the exact sequence

$$\cdots \to H_{i+1}(X \otimes K \otimes C) \to H_i(X \otimes C) \xrightarrow{\partial_i} H_i(X \otimes C) \to H_i(X \otimes K \otimes C) \to \cdots,$$

where ∂_i is multiplication by $(-1)^i x_1$. Now if $H_i(X \otimes K \otimes C) = 0$, then the map

$$H_i(X \otimes C) \xrightarrow{x_1} H_i(X \otimes C),$$

multiplication by x_1, is an epimorphism. Therefore, since R is a local ring, $H_i(X \otimes C)$ is a finitely generated R-module, and x_1 is in the radical of R, we know that $H_i(X \otimes C) = 0$. Consequently we have that $H_j(X \otimes C) = 0$ for all $j \geqq i$, which means in view of the above exact sequence that

$$H_j(X \otimes K \otimes C) = 0 \qquad \text{for all } j \geqq i.$$

THEOREM 4.3. *Let R be a local ring, A a nonzero R-module satisfying the condition that if C is an R-module such that* $\operatorname{Tor}_1^R(A, C) = 0$, *then*

$$\operatorname{Tor}_i^R(A, C) = 0$$

for all $i \geqq 1$. Then every A-sequence is an R-sequence.

Proof. Let $x_1, \cdots, x_t$ be an A-sequence, and let K_i be the Koszul complex on $x_1, \cdots, x_i$ for each $i = 1, \cdots, t$. By Proposition 4.1 we have that if X is a projective resolution of A and $X \otimes K_t$ is acyclic, then K_t is acyclic. However by [2, 2.8] we know that if K_t is acyclic, then $x_1, \cdots, x_t$ is an R-sequence. Therefore it suffices to show that $X \otimes K_t$ is acyclic in order to establish the theorem.

Suppose $t = 1$. Then by [2, 1.1] we have the exact sequence

$$\cdots \to H_{i+1}(X \otimes K_1) \to H_i(X) \xrightarrow{\partial_i} H_i(X) \to H_i(X \otimes K_1) \to \cdots,$$

where ∂_i is multiplication by $(-1)^i x_1$. Since X is a projective resolution of A, we know that $H_i(X) = 0$ for all $i > 0$ and thus $H_i(X \otimes K) = 0$ for all $i > 1$. From the exact sequence

$$0 \to H_1(X \otimes K_1) \to H_0(X) \xrightarrow{x_1} H_0(X) \to H_0(X \otimes K_1) \to 0$$

and the fact that x_1 is not a zero-divisor for $H_0(X) = A$, we conclude that $H_1(X \otimes K_1) = 0$ and $H_0(X \otimes K_1) = A/x_1 A$. Therefore $X \otimes K_1$ is a projective resolution of $A/x_1 A$. It follows easily by induction that $X \otimes K_t$ is a projective resolution of $A/(x_1, \cdots, x_t)A$, which gives the desired result.

Remark. It is clear that the hypothesis of Theorem 4.3 is satisfied in the following situations:

(a) R is a regular local ring of equal characteristic, and A is an R-module.

(b) R is an unramified regular local ring, and A is an R-module such that the characteristic of the residue class field of R is not a zero-divisor for A.

(c) R is an arbitrary local ring, and A is an R-module of homological dimension 1.

Theorem 4.3 shows that not all projective resolutions of modules satisfy the hypothesis of the theorem. For let R be a local ring whose codimension is zero but whose dimension is not zero. Let $\mathfrak{p}$ be a nonmaximal prime ideal in R. Then codim $R/\mathfrak{p} > 0$. Thus a projective resolution of $R/\mathfrak{p}$ does not satisfy the hypothesis of 4.3 and thus in particular is not rigid.

We conclude this paper with an application of these ideas to ideal theory.

LEMMA 4.4. *Let R be a noetherian ring, $\mathfrak{a}$ an ideal in R containing a nonzero divisor such that* hd $R/\mathfrak{a} < \infty$. *If B is a nonzero R-module and*

$$\operatorname{Tor}_i^R(R/\mathfrak{a}, B) = 0$$

for all $i > 0$, then $(0):\mathfrak{a} = (0)$ in B (i.e., there is some element in $\mathfrak{a}$ which is not a zero-divisor in B).

Proof. It is sufficient to show that if a prime ideal $\mathfrak{p}$ in R belongs to (0) in B, then $\mathfrak{p}$ does not contain $\mathfrak{a}$. Suppose some prime ideal $\mathfrak{p}$ belonging to (0) in B does contain $\mathfrak{a}$. Then $B_\mathfrak{p} \neq (0)$ and $\operatorname{codim}_{R_\mathfrak{p}} B_\mathfrak{p} = 0$. Also we have that

$$\operatorname{Tor}_i^{R_\mathfrak{p}}((R/\mathfrak{a})_\mathfrak{p}, B_\mathfrak{p}) = R_\mathfrak{p} \otimes \operatorname{Tor}_i^R(R/\mathfrak{a}, B) = 0 \qquad \text{for all } i > 0.$$

Therefore by Proposition 1.1 we have that $\operatorname{hd}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}) = 0$, which means that $\mathfrak{a}R_\mathfrak{p} = 0$ since $R_\mathfrak{p}$ is a local ring. But $\mathfrak{a}R_\mathfrak{p} \neq 0$ since $\mathfrak{a}$ contains a nonzero divisor. Therefore $\mathfrak{p}$ does not contain $\mathfrak{a}$, which gives us the desired result.

PROPOSITION 4.5. *Let R be a noetherian ring, $\mathfrak{a}$ an ideal in R containing a nonzero divisor, and $\mathfrak{b} \neq (0)$ an ideal in R.*

(a) *If $\mathfrak{a}$ is projective and $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, then $\mathfrak{b}:\mathfrak{a} = \mathfrak{b}$.*

(b) *If $\mathfrak{b}$ contains a nonzero divisor,* hd $R/\mathfrak{b} < \infty$, *the ideal $\mathfrak{a}$ is projective, and $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, then $\mathfrak{a}:\mathfrak{b} = \mathfrak{a}$ and $\mathfrak{b}:\mathfrak{a} = \mathfrak{b}$.*

(c) *If $\mathfrak{b}$ contains a nonzero divisor,* hd $R/\mathfrak{b} < \infty$, *the ideal $\mathfrak{a} = (x)$, and $\mathfrak{b}:(x) = \mathfrak{b}$, then $(x):\mathfrak{b} = x$.*

(d) *If R is a regular domain*[4] *containing a field and $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, then $\mathfrak{a}:\mathfrak{b} = \mathfrak{a}$ and $\mathfrak{b}:\mathfrak{a} = \mathfrak{b}$.*

Proof. We first observe that $\operatorname{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{b}) = \mathfrak{a} \cap \mathfrak{b}/\mathfrak{a}\mathfrak{b}$. Then (a) and (b) are immediate consequences of Lemma 4.4.

(c) Since $\mathfrak{a}$ contains a nonzero divisor, x must be a nonzero divisor. The fact that $\mathfrak{b}:x = \mathfrak{b}$ easily shows that $\mathfrak{b} \cap (x) = \mathfrak{b}x$. Therefore

$$\operatorname{Tor}_i^R(R/(x), R/\mathfrak{b}) = 0$$

for all $i > 0$. Applying Lemma 4.4. we have that $(x):\mathfrak{b} = (x)$.

(d) If R is a regular domain containing a field, then $R_\mathfrak{p}$ is a regular local ring of equal characteristic for each prime ideal $\mathfrak{p}$ in R. Suppose we assume that $\operatorname{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b}) = 0$. Then for each prime ideal $\mathfrak{p}$ in R, we have that

$$0 = R_\mathfrak{p} \otimes \operatorname{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{b}) = \operatorname{Tor}_1^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}).$$

By Theorem 2.1 we know therefore that $\operatorname{Tor}_1^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}) = 0$ for

[4] A noetherian domain R is said to be regular if $R_\mathfrak{p}$ is a regular local ring for each prime ideal $\mathfrak{p}$ in R.

all $i > 0$. Thus $R_{\mathfrak{p}} \otimes \mathrm{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b}) = 0$ for all $i > 0$ and all prime ideals $\mathfrak{p}$ in R, which means that $\mathrm{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b}) = 0$ for all $i > 0$.

We are now in position to apply Lemma 4.4 and obtain the desired result.

Remark. We show by example that the results of Proposition 4.5 require some homological hypothesis and are not just formal results. It is well known that there exist local domains R of dimension 2 but whose codimension is one. Let $\mathfrak{p}$ be a prime ideal of rank 1 in R, and let $x \neq 0$ be a non-unit not in $\mathfrak{p}$. Therefore we have that $\mathfrak{p}:(x) = \mathfrak{p}$. However, since the codimension of R is one, we know that the maximal ideal of R belongs to (x), and therefore $(x):\mathfrak{p} \neq (x)$. This shows that part (c) of Proposition 4.5 can be false if the hypothesis that hd $R/\mathfrak{b} < \infty$ is dropped, which is what has happened in this example.

Bibliography

1. M. Auslander and D. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc., vol. 85 (1957), pp. 390–405.
2. ———, *Codimension and multiplicity*, Ann. of Math. (2), vol. 68 (1958), pp. 645–657.
3. ———, *On ramification theory in noetherian rings*, Amer. J. Math., vol. 81 (1959), pp. 749–765.
4. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc., vol. 97 (1960), pp. 1–24.
5. D. Rees, *The grade of an ideal or module*, Proc. Cambridge Philos. Soc., vol. 53 (1957), pp. 28–42.
6. S. Eilenberg, *Homological dimension and syzygies*, Ann. of Math. (2), vol. 64 (1956), pp. 328–336.
7. J-P. Serre, *Algèbre locale-multiplicités*, mimeographed notes by P. Gabriel, Collège de France, 1958.

Brandeis University
Waltham, Massachusetts