THE UPPER CENTRAL SERIES IN SOLUBLE GROUPS

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1. Introduction

1.1. This paper has grown from an attempt to study relations between the periodic structure of a soluble group and its commutator structure. The discussion will centre primarily around the upper central series and the four Engel radicals that we introduced in an earlier paper in this journal. (This paper is listed as [EE] in the references at the end.) We recall the definition of these radicals. If x is an element in a group G such that $\mathrm{Gp}\{x\}$, the subgroup generated by x, can be linked to G by a series (i.e., a well-ordered ascending normal system), then x is called a serial element in G (we write $x \infty\triangleleft G$), and the set of all such elements forms a subgroup $\sigma(G)$, whatever the nature of G. If $\mathrm{Gp}\{x\}$ can be linked to G by a finite series, x is called finitely-serial (we write $x \triangleleft\triangleleft G$), and the set of these also forms a subgroup, $\bar{\sigma}(G)$. Further, if $\rho(G)$ denotes the set of all $a \in G$ such that $x \infty\triangleleft a_x$ for every x, where a_x is the subgroup generated by x and all conjugates of a, then $\rho(G)$ is a subgroup of G; while if $\bar{\rho}(G)$ is the set of all a with the property that $x \triangleleft\triangleleft a_x$ for every x, and where the length of the series can be taken independent of x, then $\bar{\rho}(G)$ also is a subgroup. These four characteristic subgroups satisfy the inclusion relations

$$\alpha(G) \leqq \rho(G) \leqq \sigma(G) \leqq \eta(G);$$
$$\alpha_\omega(G) \leqq \bar{\rho}(G) \leqq \bar{\sigma}(G),$$

where $\alpha(G)$ and $\alpha_\omega(G)$, are the final and ω^{th} terms, respectively, of the upper central series of G, and $\eta(G)$ is the unique maximal locally nilpotent normal subgroup of G. (The group $\eta(G)$ was denoted by $\varphi(G)$ in [EE] and called there the Fitting radical of G. However, it seems preferable to reserve the symbol φ for the Frattini subgroup, and the term Fitting radical for the union in G of all nilpotent normal subgroups. The group $\eta(G)$ might be called the *Hirsch-Plotkin radical*, after Hirsch and Plotkin who discovered its existence. It should be remarked that, with the exception of the switch from $\varphi(G)$ to $\eta(G)$, all the notation and terminology introduced in [EE] will continue to be used in the present paper.)

It was shown in [EE] that there exist countably generated metabelian groups in which no two of the six subgroups determined by σ, $\bar{\sigma}$, ρ, $\bar{\rho}$, α, α_ω coincide. Our example had the form $U \times V \times W$, where U and W were periodic (in fact, finite and of exponent 4, respectively), but V was not: V was an extension of an abelian group A of type 2^∞ by an infinite cyclic group with generator b, where $b^{-1}ab = a^3$ for $a \in A$. If we let V_1 be the split extension of A by a

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cyclic group of order 2 whose generator b_1 acts on A according to $b_1^{-1}ab_1 = a^{-1}$, then V_1 is easily seen to have precisely the same Engel structure as V, but is, of course, a 2-group. Hence $U \times V_1 \times W$ is a *periodic* metabelian group in which no two of our six subgroups coincide.

We may ask what happens if we insist either that the given group be a p-group (where p is a prime number), or else, at the other extreme, that it be a torsion-free group. The answer, except in one particular, is not encouraging:

THEOREM 1.1. (i) *Given any prime number p, there exists a countable metabelian p-group G in which no two of the following five subgroups coincide: $\sigma(G) = \rho(G)$, $\bar{\sigma}(G)$, $\bar{\rho}(G)$, $\alpha(G)$, $\alpha_\omega(G)$.*

(ii) *There exists a countable torsion-free metabelian group G in which no two of the following five subgroups coincide: $\sigma(G)$, $\rho(G)$, $\alpha(G)$, $\bar{\sigma}(G)$, $\bar{\rho}(G) = \alpha_\omega(G)$.*

The assertion that $\sigma(G) = \rho(G)$ in case (i) is, of course, a trivial consequence of the fact that G is locally nilpotent (since it is a soluble p-group) coupled with Theorem 4 [EE]. On the other hand, the equality $\bar{\rho}(G) = \alpha_\omega(G)$ in case (ii) is the one encouraging feature: this result is not trivial, and it leads, as we shall see later, to a considerably more general theorem.

1.2. Apart from the p-groups, there is another important class of periodic groups: this is the class of groups that satisfy the minimal condition on subgroups. Soluble groups of this type are much better behaved, and it turns out that their Engel structure can be well described. As a matter of fact, the relevant results will apply to a rather wider class of groups, and before stating them it will be convenient to introduce some further notation.

We shall denote classes of groups by German capitals and shall, in particular, employ the following alphabet (cf. P. Hall [7], §1.2):

$\mathfrak{A}$ is the class of all abelian groups;
$\mathfrak{F}$ the class of all finite groups;
$\mathfrak{G}$ the class of all finitely generated groups;
$\mathfrak{N}$ the class of all nilpotent groups;
$\mathfrak{S}$ the class of all soluble groups;
$\mathfrak{P}$ the class of all polycyclic groups.

A few further classes will be introduced later. Moreover, if $\mathfrak{X}$, $\mathfrak{Y}$ are any two classes, then we write $\mathfrak{X}\mathfrak{Y}$ for the class of all groups G having a normal subgroup N such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. This multiplication of classes is not associative, but it happens that in all the cases that we shall need, brackets can be omitted without causing ambiguity.

The trivial group belongs to every class of groups considered in this paper, and hence we always have $\mathfrak{X} \cup \mathfrak{Y} \leqq \mathfrak{X}\mathfrak{Y}$. Note also, as examples, the formulae:

$$\mathfrak{S} = \bigcup_{n=1}^{\infty} \mathfrak{A}^n, \qquad \mathfrak{P} = \bigcup_{n=1}^{\infty} (\mathfrak{G} \cap \mathfrak{A})^n.$$

Further, it will be convenient to use the following abbreviations:

Min for the phrase "the minimal condition on subgroups"; and
Max for "the maximal condition on subgroups".

The following theorem contains our information concerning soluble groups with Min, because it is well known that such groups are of type $\mathfrak{A}\mathfrak{F}$. Recall also that a group G is hypercentral (is a ZA-group) if, and only if, $\alpha(G) = G$.

THEOREM 1.2. *If $G \in \mathfrak{N}\mathfrak{P}\mathfrak{F}$, then*
(i) *$\sigma(G)$ is the unique maximal hypercentral normal subgroup;*
(ii) *$\bar{\sigma}(G)$ is the unique maximal nilpotent normal subgroup;*
(iii) *$\rho(G) = \alpha_{c\omega+k}(G)$ for some finite k and where c is the class of any nilpotent normal subgroup of G with factor group in $\mathfrak{P}\mathfrak{F}$; and, finally,*
(iv) *$\bar{\rho}(G) = \alpha_\omega(G)$.*

These conclusions concerning the Engel radicals also hold in another class of groups among which we again find the soluble groups with Min. To define these groups we first recall that the *spectrum* of an abelian group A consists, by definition, of all the prime numbers p such that $a \to a^p$ is *not* an automorphism of A (see [8], p. 39). If A is a torsion group, then p is in the spectrum if, and only if, the Sylow p-subgroup of A is nontrivial. By the *torsion spectrum* of an arbitrary abelian group we shall mean the spectrum of its torsion subgroup. Further, an abelian group has *finite rank* if there exists an integer n such that every finite subset lies in a subgroup generated by at most n elements. We add the following symbol to our alphabet of classes:

$\hat{\mathfrak{A}}$ is the class of all abelian groups of finite rank and finite torsion spectrum.

Every finitely generated abelian group is, of course, in $\hat{\mathfrak{A}}$; and so also is every abelian group with Min. Indeed, it is a result of Černikov that the soluble groups with Min form precisely the class of all the torsion groups in $\hat{\mathfrak{A}}(\mathfrak{S} \cap \mathfrak{F})$ ([9], p. 191).

Suppose the group H is poly-$\hat{\mathfrak{A}}$. This means, by definition, that there exists a finite series linking 1 to H in which the factors of successive terms are in $\hat{\mathfrak{A}}$. We shall be interested in the number of these factors which are *not* finitely generated, and we define $n(H)$ to be the smallest of these numbers, for all possible choices of $\hat{\mathfrak{A}}$-series in H. We shall find (Lemma 5.2) that $n(H)$ can always be obtained from a *characteristic* $\hat{\mathfrak{A}}$-series.

THEOREM 1.3. *If G is any group such that $\eta(G)$ is poly-$\hat{\mathfrak{A}}$, then*
(i) *$\sigma(G) = \eta(G)$ is the unique maximal hypercentral normal subgroup;*
(ii) *$\bar{\sigma}(G)$ is the unique maximal nilpotent normal subgroup;*
(iii) *$\rho(G) = \alpha_{n\omega+k}(G)$ for some finite k and where $n = n(\eta(G))$; and, finally,*
(iv) *$\bar{\rho}(G) = \alpha_\omega(G)$.*

An example of a group in $\hat{\mathfrak{A}}(\mathfrak{A} \cap \mathfrak{F})$ will show that the subgroups determined by σ, $\bar{\sigma}$, ρ, $\bar{\rho}$ may be all distinct (Proposition 6.1). Of course, if $\eta(G)$ is actually polycyclic (i.e., $n(\eta(G)) = 0$), then $\eta(G) = \sigma(G) = \bar{\sigma}(G)$ and $\rho(G) = \bar{\rho}(G) = \alpha_k(G)$. This occurs, in particular, *if G is a group satisfying* Max (Baer [2]).

Another special case of Theorem 1.3 arises *when $\eta(G)$ satisfies* Min: for $\eta(G)$ is then a locally nilpotent group with Min and thus is soluble by a theorem of Černikov ([9], p. 230), whence $\eta(G)$ has type $\hat{\mathfrak{A}}(\mathfrak{S} \cap \mathfrak{F})$. So here $n(\eta(G)) = 1$. The assertions concerning η and $\bar{\sigma}$ are known in this case (i.e., when $\eta(G)$ has Min): the hypercentrality of $\eta(G)$ follows from the work of Černikov ([9], p. 230), and the nilpotence of $\bar{\sigma}(G)$ from a result of Baer ([1], p. 432). But Baer's result can now be improved in another direction thanks to recent work of V. G. Vilyacer [11], who has shown that *an Engel group satisfying* Min *is locally nilpotent.* We assert:

If G is a group with Min *in which every element is bounded left Engel, then G is nilpotent.*

For, by Vilyacer's theorem, G is locally nilpotent, whence G is soluble by the theorem of Černikov quoted above, and thus $\bar{\sigma}(G) = G$ by Theorem 4 (ii) of [EE], so that, finally, G is nilpotent by Theorem 1.3 (ii) (or the result of Baer mentioned above).

1.3. The reader may have noticed that, while we assert the equality $\sigma(G) = \eta(G)$ in Theorem 1.3, we did not do the same in Theorem 1.2. This was not because it is false, but because its truth in the context of Theorem 1.2 is a consequence of a more general fact.

If G is an arbitrary group, let us denote by $L(G)$, $\bar{L}(G)$, $R(G)$, $\bar{R}(G)$, respectively, the sets of all left, bounded left, right, and bounded right, Engel elements in G. The reader will recall from [EE] that these are the sets of all elements $g \in G$ satisfying, respectively, the conditions $G \mathfrak{e} g$, $G \mid\!\mathfrak{e}\, g$, $g \mathfrak{e} G$, $g \mathfrak{e}\!\mid G$. (If A, B are subsets of G, then $A \mathfrak{e} B$ means

$$[a, {}_k b] \equiv [a, \underbrace{b, \cdots, b}_{k}] = 1$$

for each $a \in A$, $b \in B$ and some $k = k(a, b)$. If k can be taken independent of a in A, we write $A \mid\!\mathfrak{e}\, B$, and if independent of b in B, then $A \mathfrak{e}\!\mid B$. Of course, $A \mid\!\mathfrak{e}\!\mid B$ then means that there exists a fixed integer k such that

$$[a, {}_k b] = 1$$

for all $a \in A$, $b \in B$. We shall frequently find it convenient to write, in this case, $A \mid\!\mathfrak{e}\!:\!k\!\mid B$.)

It is always, and trivially, true that

$$\sigma(G) \leqq L(G), \qquad \bar{\sigma}(G) \leqq \bar{L}(G), \qquad \rho(G) \leqq R(G), \qquad \bar{\rho}(G) \leqq \bar{R}(G).$$

For a given group G one is naturally interested in knowing which, if any, of these inequalities are in fact equalities. In general, this seems to be a very difficult problem.

Let us denote by $\mathfrak{E}$ the class of all groups for which each of the above inequalities is an equality. Since in any group, $\eta(G) \leqq L(G)$ (trivially) and $\sigma(G) \leqq \eta(G)$ (by Theorem 2 [EE]), it follows that for every $G \in \mathfrak{E}$, $\sigma(G) = \eta(G)$.

Baer has proved (in [2]) that $\mathfrak{E}$ contains all groups with Max, and Theorem 4 of [EE] asserted that $\mathfrak{E}$ also contains all soluble groups: $\mathfrak{S} \leqq \mathfrak{E}$. Both these results can be incorporated in the following:

If G has a soluble normal subgroup N such that G/N has Max, *then $G \in \mathfrak{E}$. In particular, $\mathfrak{S}\mathfrak{F} \leqq \mathfrak{E}$.*

This is an immediate consequence of the results concerning soluble groups and groups with Max and the following theorem.

THEOREM 1.4. *If G has a soluble normal subgroup N such that $\bar{\rho}(G/N)$ is soluble and $G/N \in \mathfrak{E}$, then $G \in \mathfrak{E}$.*

The major step in the proof of Baer's theorem was to show that a group with Max, which is generated by Engel elements, is necessarily nilpotent. The next theorem shows why this step was really the crucial one for his proof.

THEOREM 1.5. *If G is a group such that* $\mathrm{Gp}\{L(G) \cup R(G)\}$ *is soluble and locally nilpotent, then $G \in \mathfrak{E}$.*

If it could be established that a group with Min, generated by a finite number of (right or left) Engel elements, is nilpotent, then Theorem 1.5 would imply that any group with Min lies in $\mathfrak{E}$. (For then $\mathrm{Gp}\{L(G) \cup R(G)\}$ would be locally nilpotent, and since it has Min, it would be soluble.)

The above two theorems depend on a fact which was essentially proved in [EE], but was not stated there in the general form that we shall require here.

THEOREM 1.6. *If G is any group and a is a bounded right Engel element, whose normal closure a_a in G is soluble, then $a_a \,|\mathfrak{e}|\, G$ and $a \in \bar{\rho}(G)$.*

The following remarks will suffice as proof. Lemma 16 [EE] remains true if G is arbitrary but a, b lie in a soluble and locally nilpotent normal subgroup K of G provided one uses $t(K)$ instead of $t(G)$. Then Lemma 17 [EE] is valid if one takes $a_a = K$: for a_a is soluble by hypothesis and hence is also locally nilpotent since it is generated by (bounded) right Engel elements (Theorem 4 [EE]). Finally, the implication $a_a \,|\mathfrak{e}|\, G \Rightarrow a \in \bar{\rho}(G)$ follows exactly as in Lemma 18 [EE].

Observe that, in *any* group G, if $a \in \bar{\rho}(G)$, it follows immediately from the definition of $\bar{\rho}$, that $a_a \,|\mathfrak{e}|\, G$. But whether a_a need be soluble remains an open question.

1.4. We mentioned, in connexion with Theorem 1.1 (ii), that the functions

$\bar{\rho}$, α_ω always have the same value when applied to a torsion-free metabelian group. This fact is contained, as a special case, in the following result.

THEOREM 1.7. *Let $G \in \mathfrak{S}\mathfrak{F}$, let P be the torsion group of $\bar{\rho}(G)$, and C the centralizer of P in G. If there exists a nilpotent normal subgroup N of G such that G/NC is finitely generated, then $\bar{\rho}(G) = \alpha_\omega(G)$.*

When $P = 1$, $C = G$, and then the condition on G/NC is trivially satisfied. Thus $\bar{\rho}(G) = \alpha_\omega(G)$ *whenever* $\bar{\rho}(G)$ *is torsion-free and* $G \in \mathfrak{S}\mathfrak{F}$. Again, if $G \in \mathfrak{N}(\mathfrak{G} \cap \mathfrak{S})\mathfrak{F}$ and we choose for N a nilpotent normal subgroup of G with factor group in $(\mathfrak{G} \cap \mathfrak{S})\mathfrak{F}$, then $NC \geqq N$, and so $G/NC \in (\mathfrak{G} \cap \mathfrak{S})\mathfrak{F}$, whence G/NC is finitely generated. Hence *if* $G \in \mathfrak{N}(\mathfrak{G} \cap \mathfrak{S})\mathfrak{F}$, *then* $\bar{\rho}(G) = \alpha_\omega(G)$. This includes, of course, the relevant part of Theorem 1.2. We stress that all finitely generated soluble groups are also covered by this special case of Theorem 1.7. I have been unable to decide whether $\rho(G)$ is necessarily equal to $\alpha(G)$ for finitely generated soluble groups. This is certainly not the case for torsion-free soluble groups, since one of our examples in connexion with Theorem 1.1 (ii) has $\rho(G) = G$ and $\alpha(G) = 1$.

Theorem 1.7 is a simple consequence of the following two "hypercentrality criteria", which are our main results in this paper.

THEOREM 1.8. *Let H be a soluble normal subgroup of a group G, and let the image of G in the automorphism group of H be of type $\mathfrak{S}\mathfrak{F}$. If $H \,|\mathfrak{e}{:}n|\, G$, then there exists an integer k such that $[H, {}_kG]$ has finite exponent equal to an $n!$-number.*[1]

THEOREM 1.9. *Let T be a soluble group of finite exponent and normal in a group G, and let C_i be the centralizer in G of the i^{th} derived factor of T. If G contains a nilpotent normal subgroup N such that each $G/NC_i \in \mathfrak{G} \cap \mathfrak{S}\mathfrak{F}$, then $T \,|\mathfrak{e}\, G$ implies that $[T, {}_sG] = 1$ for some finite s; i.e., $T \leqq \alpha_s(G)$.*

We stop to prove Theorem 1.7 since we can do so now without further trouble. Let $G \in \mathfrak{S}\mathfrak{F}$, and choose $a \in \bar{\rho}(G)$. Then a_a is a locally nilpotent group in $\mathfrak{S}\mathfrak{F}$ and hence is soluble; also $a_a \,|\mathfrak{e}|\, G$. By Theorem 1.8, we can find k such that $T = [a_a, {}_kG]$ has finite exponent. Consequently $T \leqq P$, the torsion group of $\bar{\rho}(G)$. Now $C \leqq C_i$, the centralizer of the i^{th} derived factor of T, and since G/NC is given to be finitely generated, therefore the same holds for each G/NC_i. Because $G \in \mathfrak{S}\mathfrak{F}$, we have $G/NC_i \in \mathfrak{G} \cap \mathfrak{S}\mathfrak{F}$ for each i, and consequently Theorem 1.9 can be applied and shows $a_a \leqq \alpha_s(G)$. Thus $\bar{\rho}(G) \leqq \alpha_\omega(G)$, as required.

The special case of Theorem 1.8 when G itself is soluble is worth noting. We shall, indeed, have to establish this special case as a step in the proof of the theorem; and we may observe that it tells us, in particular, that if $G \,|\mathfrak{e}{:}n|\, G$

[1] If m is a positive integer, then another positive integer r is called an m number if r divides some power of m.

and G is $n!$-free, then G is nilpotent. However, this last fact can be improved by resorting to a little trick, and we shall actually prove

THEOREM 1.10. *If G is soluble of derived length t and $G \mid \mathfrak{e}:n \mid G$ with $n \geqq 2$, then $\gamma_k(G)$ (the k^{th} term of the lower central series of G) has exponent $e = e(n, t)$, where e is an $(n-1)!$-number and where*

$$k = k(n, t) = 1 + (1/2^{n-2})\{(1 + 2^{n-2})^t - 1\}.$$

That $(n-1)!$ cannot be replaced by $(n-2)!$ is shown by the example of a metabelian p-group W, with trivial centre, in which $W \mid \mathfrak{e}:p+1 \mid W$ ([EE], p. 166 and the footnote there). At any rate, we are able to add the torsion-free soluble groups to the (still very short) list of groups for which an Engel identity implies nilpotence.

1.5. In the situation of Theorem 1.8, if the group of automorphisms of H induced by G happens to be actually *finite*, then we can apply Theorem 1.9 to $T = [H, {}_kG]$ (with k chosen in accordance with Theorem 1.8) and we deduce $H \leqq \alpha_{k+s}(G)$.

It will be convenient to restate the special case of this when H is abelian in the language of module theory. If G is a given group and A is a G-module, then let us write as $\alpha_1(A:G)$ the set of all the G-invariant elements of A (i.e., all $a \in A$ such that $ag = a$ for every $g \in G$). We may then define, in the obvious manner, the *upper G-series of A*,

$$0 = \alpha_0(A:G) < \alpha_1(A:G) < \cdots < \alpha_\tau(A:G) = \alpha_{\tau+1}(A:G) = \alpha(A:G),$$

where, of course, τ may be transfinite. If $\bar{G}$ is the image of G in the automorphism group of A and we let K be the subgroup of the holomorph of A generated by A and $\bar{G}$, then $\alpha_\lambda(A:G)$ is nothing but $\alpha_\lambda(K) \cap A$. The result stated in the last paragraph (but with H abelian) may now be rewritten thus: *If A is a module over the finite group G such that $A \mid \mathfrak{e}\, G$, then $A = \alpha_s(A:G)$ for some finite s.*

This same result is also fundamental for the proof of Theorem 1.2; and a close relative of it is needed for Theorem 1.3. This latter is worth stating explicitly as

PROPOSITION 1.1. *Let G be a given group and A a G-module.*

(i) *If A, as additive group, has finite rank and finite torsion spectrum, then $A \mid \mathfrak{e}\, G$ implies $A = \alpha_k(A:G)$ for some finite k; while*

(ii) *if A, as additive group, has finite rank, then $A\, \mathfrak{e}\, G$ implies $A = \alpha_{\omega+k}(A:G)$ for some finite k.*

This proposition generalizes a number of known hypercentrality criteria, notably one due to Vilyacer [11], who showed, in the situation (ii), that $\alpha_1(A:G) \neq 0$.

The meaning of the statement $A \mid \mathfrak{e}\, G$ in the additive notation for modules is, of course, that, for each $g \in G$, the element $g - 1$ of the integral group algebra of G acts on A as a nilpotent endomorphism: i.e., $A(g-1)^n = 0$ for a suitable n; and $A \mathfrak{e} G$ is the same as saying that each $g-1$ acts as a locally nilpotent endomorphism: i.e., $a(g-1)^n = 0$ for each $a \in A$ and $n = n(a)$. If Γ is the additive subgroup, of the group ring of G, spanned by all $g - 1$, for $g \in G$, then Γ is actually an ideal, the "difference ideal", or "augmentation ideal" of G. Now A, being a G-module, is also a Γ-module and $A = \alpha_k(A:G)$ is equivalent to $A\Gamma^k = 0$. Thus our basic problem is to prove that, if A is a Γ-module, where Γ is given to be spanned by elements with nilpotent action on A, then, under certain circumstances, Γ itself has nilpotent action on A.

This problem is no longer one of group theory but is purely a ring-theoretical one. Our main result in this direction is Theorem 1.11; and from this Proposition 1.1 will follow, as well as the result on modules over finite groups mentioned above.

THEOREM 1.11. *If A is an additive group of finite rank and finite torsion spectrum, and Λ is a ring of endomorphisms of A, additively spanned by nilpotent elements, then Λ is a nilpotent ring.*

The basic fact needed for the proof of this theorem is an old result of Wedderburn [12], that a finite-dimensional algebra over a field is nilpotent if it possesses a basis of nilpotent elements. Theorem 1.11, as it stands, is not a generalization of this theorem of Wedderburn since the ring of scalars in our theorem (namely, the integers) cannot be specialized to a field. It so happens however, that *our proof of Theorem 1.11 applies, with almost no change, when the scalars form a Dedekind domain*, and in this form Theorem 1.11 does represent a genuine generalization of Wedderburn's theorem.

1.6. The paper is arranged so that the proofs of the theorems stated in this introduction come in almost precisely the reverse order to that of the theorems. Thus, §2 deals with the results discussed in §1.5; in §3 we prove Theorem 1.9 and also another hypercentrality criterion (Theorem 3.1) needed later; §4 is primarily devoted to the proof of Theorem 1.8 but contains also the proof of Theorem 1.10; §5 deals with the results on Engel structure stated in §1.2 and §1.3; and, finally, in §6, we present the various examples needed for Theorem 1.1.

1.7. For the convenience of the reader we add here an index of the most frequently used (nonstandard) symbols and terms.

$\alpha_\lambda(G)$ is the λ^{th} term of the upper central series of G, and
$\gamma_\lambda(G)$ is the λ^{th} term of its lower central series;
$\alpha(G)$ (the hypercentre) is the limit of the upper central series, and
G is hypercentral if $\alpha(G) = G$;
$\alpha_\lambda(A:G)$ is the λ^{th} term of the upper G-series of the G-module A: definition at the beginning of §1.5 (p. 442); and
$\alpha(A:G)$ is the limit of the upper G-series of A;

$\eta(G)$ is the unique maximal locally nilpotent normal subgroup (the Hirsch-Plotkin radical);
$\mathrm{Gp}\{S\}$ is the subgroup generated by the set S;
series means a well-ordered ascending normal system ([9], p. 171), and *finite series* a series of finite length;
$H \infty\triangleleft G$ (H is serial in G) means that there exists a series from H to G, and $H \triangleleft\triangleleft G$ (H is finitely-serial in G) that there is a finite series from H to G;
$x \infty\triangleleft G$ is the same as $\mathrm{Gp}\{x\} \infty\triangleleft G$ (x is a serial element), and $x \triangleleft\triangleleft G$ as $\mathrm{Gp}\{x\} \triangleleft\triangleleft G$ (x is a finitely-serial element);
$\sigma(G)$, $\bar{\sigma}(G)$, $\rho(G)$, $\bar{\rho}(G)$ are the four *Engel radicals*: definitions in §1.1;
$\mathfrak{e}$ is used for the *Engel relation* between two subsets of a group: the notation is explained at the beginning of §1.3 (p. 439);
$L(G)$, $\bar{L}(G)$, $R(G)$, $\bar{R}(G)$ are the sets of the four types of Engel elements in G (p. 439);
German capitals denote classes of groups;
G is in poly-$\mathfrak{X}$ means that there exists a finite series linking 1 to G with factors of successive terms in $\mathfrak{X}$;
$\mathfrak{A}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{N}$, $\mathfrak{S}$, $\mathfrak{P}$ are defined at the beginning of §1.2;
$\hat{\mathfrak{A}}$ is the class of abelian groups of finite rank and finite torsion spectrum (p. 438);
$\mathfrak{E}$ is the class of groups G in which $\sigma(G) = L(G)$, $\bar{\sigma}(G) = \bar{L}(G)$, $\rho(G) = R(G)$, $\bar{\rho}(G) = \bar{R}(G)$;
$n(H)$, for $H \in$ poly-$\hat{\mathfrak{A}}$, is a numerical invariant defined immediately before the statement of Theorem 1.3 (p. 438);
the integer r is an m-number means that r divides some power of m.

2. Nilpotent rings

2.1. Our first task is to prove Theorem 1.11. Throughout this section we shall write Z for the ring of rational integers, and we shall speak of Z-modules and Z-algebras instead of additive groups and rings, respectively. We do this in order to ease the task of any reader who may wish to translate our proof to the case where Z is replaced by a Dedekind domain. We shall, anyway, insert remarks concerning the transition to the Dedekind case whenever necessary.

Perhaps we should begin with one such remark and explain what "finite rank" and "finite torsion spectrum" mean for a module A over a Dedekind domain. If T denotes the torsion submodule of A, then the requirement of *finite rank* shall mean that A/T has finite rank (in the sense usual for torsion-free modules over an integral domain) and that every finite subset of T is contained in a submodule of T having a set of at most n generators, where n is a fixed integer. Concerning the torsion spectrum of A, we recall that torsion modules over Dedekind domains are always direct sums of $\mathfrak{p}$-primary modules, where $\mathfrak{p}$ runs through the prime ideals in the Dedekind domain. We may thus say that A has finite torsion spectrum if T has only a finite number of nonzero $\mathfrak{p}$-primary components. It is however nicer to employ a generalization of the definition of spectrum given in §1.2. So, let R be an arbitrary commutative ring with identity, let A be an R-module, and let $o(a)$ denote the order ideal of the element a in A. We define the *spectrum* of A to be the set consisting of all the prime ideals $\mathfrak{p}$ of R that do *not* satisfy the following condition: for each $a \in A$, we have $a \in A\mathfrak{p}$ when $o(a) = 0$, while $o(a) + \mathfrak{p} = R$ when $o(a) \neq 0$. This definition reduces to the usual one when R is a principal ideal domain, while if R is a Dedekind domain and A is a torsion module, one can prove that $\mathfrak{p}$ is in the spectrum if, and only if, the $\mathfrak{p}$-component of A is nontrivial.

Our plan is to reduce the proof of Theorem 1.11 to a consideration of three cases, in each of which the Z-module A has a particularly simple form.

So let A be a Z-module in the class $\hat{\mathfrak{A}}$, and T its torsion submodule. Then

$$T = T_1 \oplus \cdots \oplus T_r,$$

where T_i is p_i-primary and the primes $p_1, \cdots, p_r$ are distinct. Suppose further that A is a Λ-module for a given Z-algebra Λ, and that Λ is Z-generated by a set S whose elements have nilpotent action on A. Since each T_i is fully invariant in A, so each T_i is Λ-invariant. Hence $T_1, \cdots, T_r$ and also A/T are Λ-modules, and the elements of S have nilpotent action on each of these modules. If we can prove that Λ itself has nilpotent action on each of A/T, $T_1, \cdots, T_r$, then it will clearly also have nilpotent action on A. It is therefore sufficient to prove Theorem 1.11 for Z-modules A that are either torsion-free of finite rank or p-primary of finite rank.

The second case can be reduced still further. For if A is p-primary of finite rank, then there exists a smallest integer $n \geqq 0$ such that $Ap^n = Ap^{n+1}$ and

$$A > Ap > \cdots > Ap^n = A_0$$

is a series of fully invariant Z-submodules. Moreover, A_0 is a direct sum of a finite number of modules of type p^∞, and each Ap^i/Ap^{i+1} is finitely Z-generated (see, e.g., [4], Chapter 7, §4, Exercise 22). Thus to prove our Z-algebra Λ has nilpotent action on A, it suffices to show it has this on each Ap^i/Ap^{i+1} $(i = 0, \cdots, n - 1)$ and on A_0. In other words, instead of having to consider arbitrary p-primary modules of finite rank, we may confine attention to the divisible ones (like A_0 above) and to those annihilated by p (the elementary p-groups).

(When Z is replaced by a Dedekind domain, we have the same reduction to the torsion-free and $\mathfrak{p}$-primary cases. To deal with the latter one goes to the local situation, where the theory of principal ideal domains is available and where one may apply all the following arguments on p-groups.)

The three cases to be discussed are then the following:

A is torsion-free and of finite rank (Lemma 2.2);

A is a finite elementary p-group (Lemma 2.3 (i)); and

A is a direct sum of a finite number of groups each of type p^∞ (Lemma 2.3 (ii)).

The discussion of each of these cases depends on the following result.

LEMMA 2.1. *Let C be a Z-module such that* End C, *the ring of Z-endomorphisms of C, is an integral domain, and suppose $A = C_1 \oplus \cdots \oplus C_n$, where each $C_i = C$. If Λ is a Z-subalgebra of* End A *which is Z-generated by nilpotent elements, then Λ is nilpotent.*

Proof. Write $E =$ End C, and let F be the quotient field of E. Clearly, End A is isomorphic (as Z-algebra) to E_n, the Z-algebra of all $n \times n$ matrices with coefficients in E. If we make A into an E-module in the natural way, i.e., by defining

$$(c_1 + \cdots + c_n)e = c_1 e + \cdots + c_n e,$$

for $c_i \in C$ and $e \in E$, then the image of E in End A corresponds, in End $A \cong E_n$, to the set of all multiples of the unit matrix. Hence End A is actually an E-algebra, and the isomorphism with E_n is an isomorphism of E-algebras. Since E_n is E-torsion-free, so therefore is End A. Hence the E-algebra homomorphism

$$\text{End } A \to \text{End } A \otimes_E F = \Gamma$$

given by $\theta \to \theta \otimes 1$ is actually an injection (see, e.g., [4], Chapter 3, §3, No. 4) and, of course, Γ is an n^2-dimensional algebra over the field F. If the image of Λ generates the F-subalgebra Λ' of Γ, then Λ' is a finite-dimensional algebra and has a basis of nilpotent elements. By Wedderburn's theorem [12], Λ' is nilpotent, and consequently so also is Λ.

(We remark that Lemma 2.1 remains true if Z is replaced by any commutative ring with unit. In this form, Lemma 2.1 formally includes the theorem of Wedderburn.)

LEMMA 2.2. *If A is a torsion-free Z-module of finite rank, and Λ is a Z-algebra of endomorphisms of A which is Z-generated by locally nilpotent endomorphisms, then Λ is nilpotent.*

Proof. We write Q for the field of rational numbers and put

$$A_{(Q)} = A \otimes_Z Q.$$

Then $a \to a \otimes 1$ is a Z-monomorphism of A in $A_{(Q)}$, and this yields a Z-algebra monomorphism ζ:

$$\text{End } A \to \text{End}_Q\, A_{(Q)}$$

given by

$$\theta\zeta : a \otimes r \to a\theta \otimes r,$$

where $\theta \in$ End A, $a \in A$, and $r \in Q$. If now $a_1, \cdots, a_n$ is a maximal linearly independent set in A and λ is a locally nilpotent endomorphism of A, we can find $m = m(\lambda)$ such that $a_i \lambda^m = 0$ for each i. Consequently

$(a_i \otimes 1)(\lambda\zeta)^m = 0$ for each i, and since $a_1 \otimes 1, \cdots, a_n \otimes 1$ form a Q-basis of $A_{(Q)}$, so $A_{(Q)}(\lambda\zeta)^m = 0$, i.e., $\lambda\zeta$ is a nilpotent linear transformation of $A_{(Q)}$. Thus $\Lambda\zeta$ is Z-generated by nilpotent elements. The result now follows by applying Lemma 2.1, with C, A, Λ replaced by Q, $A_{(Q)}$, $\Lambda\zeta$, respectively.

(Lemma 2.2 remains true when Z is replaced by an arbitrary integral domain.)

LEMMA 2.3. *Let A be a p-primary Z-module of finite rank such that either* (i) $Ap = 0$, *or* (ii) $Ap = A$. *If Λ is a Z-algebra of endomorphisms of A which is Z-generated by nilpotent elements, then Λ is nilpotent.*

Proof. Both parts of this lemma are immediate consequences of Lemma 2.1. In case (i), we take the C of Lemma 2.1 to be the integers modulo p (so that End C is a field of p elements), and in case (ii), C is a group of type p^∞ (whence End C is isomorphic to the ring of p-adic integers).

The proof of Theorem 1.11 is now complete. An important special case of the theorem is stated in

PROPOSITION 2.1. *If Λ is a Z-algebra which can be Z-generated by a finite number of nilpotent elements, then Λ is nilpotent.*

For let us regard Λ as a (right) Λ-module by using the multiplication in Λ. Then Theorem 1.11 can be applied with $A = \Lambda$, since a finitely Z-generated module is certainly of finite Z-rank and finite Z-torsion spectrum.

(Proposition 2.1 is also true if Z is replaced by a Dedekind domain R. For if A is a finitely generated module over R, then A satisfies the maximal condition on R-submodules, in view of the fact that R is a Noetherian ring, and thus the torsion module of A is also finitely generated. Hence A has finite rank and finite torsion spectrum. But, as a matter of fact, Proposition 2.1, even with R in the place of Z, is still a special case of a more general result.)

2.2. We turn now to the group-theoretical applications of Theorem 1.11. First we shall prove Proposition 1.1. Since part (i) is an immediate consequence of Theorem 1.11, we shall confine attention to part (ii).

Thus A is a G-module, of finite Z-rank, and such that $A \mathfrak{e} G$. The difference ideal Γ of G is spanned by the elements $g - 1$, all of which have locally nilpotent action on A. Hence, by Lemma 2.2, there exists $k \geqq 1$ such that $A' = A\Gamma^k$ is contained in the torsion submodule of A. It remains to prove that $A' = \alpha_\omega(A':G)$. Take any $a \in A'$, and let M be the G-submodule generated by a. Because A' has finite Z-rank, therefore a lies in a finite characteristic Z-submodule, whence M is finite, and consequently $M \mid\mathfrak{e}\, G$. Now applying Theorem 1.11 with A, Λ replaced by M, Γ, respectively, we conclude $M\Gamma^s = 0$ and thus $a \in \alpha_s(A':G)$.

In Proposition 1.1 the given group is arbitrary, but the module has restrictions on its additive structure. In the next result the situation is reversed: the module is unrestricted but the group must be finite.

PROPOSITION 2.2. *Let G be a finite group, and A a G-module.*
(i) *If $A \mid \mathfrak{e}\, G$, then $A = \alpha_k(A:G)$ for some finite k; while*
(ii) *if $A\ \mathfrak{e}\ G$, then $A = \alpha_\omega(A:G)$.*

Proof. We first prove (i). Let Γ be the difference ideal of G, and K the kernel of the representation of Γ on A. Then some power of each $g - 1$ is contained in K, and thus Γ/K is additively spanned by the (finite number of) nilpotent elements $(g - 1) + K$, $g \in G$. By Proposition 2.1, Γ/K is nilpotent, say $\Gamma^k \leqq K$, and thus $A\Gamma^k = 0$, i.e., $A = \alpha_k(A:G)$.

Now part (ii) follows simply. Take any $a \in A$, and let M be the G-submodule generated by a. Then M is finitely Z-generated, and hence $M\ \mathfrak{e}\ G$ implies $M \mid \mathfrak{e}\, G$. By part (i), $M = \alpha_k(M:G)$ for some finite k. Thus $a \in \alpha_k(A:G)$, and so $A = \alpha_\omega(A:G)$.

We shall show later (Theorem 3.1) that Proposition 2.2 remains true when G is in the class $\mathfrak{P}\mathfrak{F}$.

3. Hypercentrality criteria

3.1. Theorem 1.9 is essentially a consequence of Proposition 2.2 (i), since we shall find that our hypotheses force each of the groups G/NC_i to be finite.

LEMMA 3.1. *If A is a G-module of finite additive exponent and such that $A \mid \mathfrak{e}\, G$, then G induces on A a periodic group of automorphisms.*

Proof. If G induces a periodic group of automorphisms on each Sylow p-subgroup of A, then G does likewise on A itself. We may therefore assume, without loss of generality, that A is a p-group, say of exponent p^n.

Suppose first that $n = 1$. If x lies in the image of G in End A, then $(x - 1)^m = 0$ for some $m = m(x)$, and hence $(x - 1)^{p^s} = 0$ for a suitable $s = s(x)$. But the additive group of End A is itself of exponent p, and consequently

$$(x - 1)^{p^s} = x^{p^s} - 1,$$

whence $x^{p^s} = 1$. Thus the action of G on A is periodic.

We now use induction on n to prove the result in general. If $g \in G$, we assume there exists $m = m(g)$ such that, for all $a \in A$,

$$ag^m \equiv a \pmod{B},$$

where B is the characteristic subgroup of A consisting of all the elements of order p. Thus

$$ag^m = a + b,$$

where $b \in B$, and hence

$$ag^{mk} = a + b(1 + g^m + \cdots + g^{m(k-1)})$$

for any $k \geqq 1$. By the case $n = 1$, we know that g^m has finite order, say s, in its action on B. If we set

$$\eta = 1 + g^m + \cdots + g^{m(s-1)},$$

then

$$\begin{aligned} ag^{msp} &= a + b(\eta + g^{ms}\eta + \cdots + g^{ms(p-1)}\eta) \\ &= a + pb\eta \\ &= a. \end{aligned}$$

Consequently g has finite order in its action on A, and the induction is complete.

LEMMA 3.2. *Let A be abelian of finite exponent and normal in a group G, and let C be the centralizer of A in G. If N is a nilpotent normal subgroup of G such that $G/NC \in \mathfrak{G} \cap \mathfrak{S}\mathfrak{F}$, then $A \mid \mathfrak{e}\, G$ implies $A \leqq \alpha_k(G)$, for some finite k.*

Proof. By Lemma 3.1, G/NC is periodic and hence is finite (since a finitely generated, periodic, soluble group is finite). Consequently in the case when $N = 1$, the lemma reduces to Proposition 2.2 (i). An obvious induction on the class of N then yields our result in general.

Proof of Theorem 1.9. By an induction on the derived length of T, we may suppose

$$[T, {}_{s_1}G] \leqq A,$$

where A is the last nontrivial term of the derived series of T. Then

$$[A, {}_{s_2}G] = 1,$$

by Lemma 3.2, whence $T \leqq \alpha_{s_1+s_2}(G)$.

3.2. For our proof of Theorem 1.2 (in §5) we shall need a hypercentrality criterion which generalizes Proposition 2.2.

THEOREM 3.1. *Let H be a normal subgroup of a group G, contained in a nilpotent normal subgroup N of G and such that $G/NC \in \mathfrak{P}\mathfrak{F}$, where C is the centralizer of H in G. Then*

(i) *$H \mid \mathfrak{e}\, G$ implies $H \leqq \alpha_s(G)$ for some finite s; while*

(ii) *$H\, \mathfrak{e}\, G$ implies $H \leqq \alpha_{c\omega}(G)$, where c is the class of N.*

LEMMA 3.3. *If $G \in \mathfrak{P}\mathfrak{F}$ and A is a G-module such that $A \mid \mathfrak{e}\, G$, then $A = \alpha_k(A:G)$, for some finite k.*

Proof. Let

$$1 = P_0 < P_1 < \cdots < P_{s+1} = P < G,$$

where each term is normal in the next, G/P is finite, and P_{i+1}/P_i is cyclic $(i = 0, \cdots, s)$. If we can prove

$$A = \alpha_m(A:P), \tag{1}$$

then, since each $\alpha_{j+1}(A:P)/\alpha_j(A:P)$ is a G/P-module, our result will follow by an easy induction on m and an application of Proposition 2.2 (i).

It remains to prove (1), and we do this by an induction on s. So we suppose

$$A = \alpha_n(A:P_s),$$

and we have to show that, for each j, the P/P_s-module

$$B_j = \alpha_{j+1}(A:P_s)/\alpha_j(A:P_s)$$

satisfies

$$B_j = \alpha_r(B_j:P/P_s)$$

for some $r = r(j)$. But this is obvious because P/P_s is cyclic.

LEMMA 3.4. *If* $G \in \mathfrak{P}\mathfrak{F}$ *and* A *is a* G*-module such that* $A \in G$, *then* $A = \alpha_\omega(A:G)$.

Proof. We use the same notation for G as in Lemma 3.3, and our first aim is to prove

$$(2) \qquad A = \alpha(A:G).$$

Since every G-image of A satisfies the same condition as A, we obtain (2) if we show $\alpha_1(A:G) \neq 0$.

Suppose we already know that $\alpha_1(A:P_i) \neq 0$, with $i \leqq s$. Take any $a \neq 0$ in $\alpha_1(A:P_i)$ and a generator x of P_{i+1} modulo P_i. If l is the least integer such that $a' = [a, {}_l x] \neq 0$, then $a' \in \alpha_1(A:P_{i+1})$. Hence by an induction on i, we conclude $\alpha_1(A:P) \neq 0$, whence $\alpha_1(A:G) \neq 0$ by Proposition 2.2 (ii).

Finally, we assert (2) implies our required result, because G is finitely generated. For let $G = \mathrm{Gp}\{g_1, \cdots, g_n\}$, and choose $a \in \alpha_{\omega+1}(A:G)$. Then for each i, $[a, g_i] \in \alpha_{k_i}(A:G)$ for some finite k_i, and hence we can find k such that *each* $[a, g_i] \in \alpha_k(A:G)$. But then $a \in \alpha_{k+1}(A:G)$, and thus $\alpha_{\omega+1}(A:G) = \alpha_\omega(A:G)$.

Proof of Theorem 3.1. Let $H_i = \alpha_i(N) \cap H$, so that

$$1 = H_0 < H_1 < \cdots < H_r = H,$$

where $r \leqq c$. Suppose we already know

$$H_i \leqq \alpha_s(G) \quad \text{in case (i)}$$

and

$$H_i \leqq \alpha_{i\omega}(G) \quad \text{in case (ii)},$$

where $0 \leqq i < r$. Now H_{i+1}/H_i is a G/NC-module, whence

$$H_{i+1}/H_i \leqq \alpha_k(G/H_i) \quad \text{in case (i)}, \qquad \text{by Lemma 3.3,}$$

and

$$H_{i+1}/H_i \leqq \alpha_\omega(G/H_i) \quad \text{in case (ii)}, \qquad \text{by Lemma 3.4.}$$

Consequently,

$$H_{i+1} \leqq \alpha_{s+k}(G) \qquad \text{in case (i)},$$

and

$$H_{i+1} \leqq \alpha_{(i+1)\omega}(G) \quad \text{in case (ii)}.$$

The theorem now follows by an induction on i.

4. Proof of Theorem 1.8

4.1. Theorem 1.8 is proved by a series of inductions, all of which rest ultimately on the following lemma.

LEMMA 4.1. *Let B be an abelian group, and let A be a B-module such that $A \,|\mathfrak{e}{:}n|\, B$. Then*

$$n^*[A, {}_{2^n-1}B] = 0,$$

where $1^* = 1$ *and, when* $n > 1$,

$$n^* = n(n-1)^2(n-2)^{2^2} \cdots 2^{2^{n-2}}.$$

Proof. Let θ be the homomorphism of B into the automorphism group of A. Then we are given that $(x - 1)^n = 0$ for every $x \in B\theta$.

Suppose for the moment that $n = m + 1$, and choose any $x, y \in B\theta$. Then

$$\begin{aligned} 0 &= (xy - 1)^{m+1} \\ &= \{(x-1)(y-1) + (x-1) + (y-1)\}^{m+1} \\ &= \sum_{\substack{i+j \geqq m+1 \\ i,j \leqq m}} r(i,j)(x-1)^i(y-1)^j. \end{aligned}$$

Since $i + j \geqq m + 1$ and $j \leqq m$, therefore $i > 0$ (and similarly $j > 0$). Now

$$\begin{aligned} 0 &= (xy-1)^{m+1}(x-1)^{m-1} \\ &= \sum r(i,j)(x-1)^{m+i-1}(y-1)^j, \end{aligned}$$

and in the sum on the right-hand side all terms vanish except those for which $m + i - 1 \leqq m$, i.e., $i = 1$. Hence j must equal m, and we have

$$0 = r(1, m)(x-1)^m(y-1)^m.$$

But clearly $r(1, m) = m + 1$, and consequently

$$(m+1)(x-1)^m(y-1)^m = 0. \tag{3}$$

We shall prove the lemma by an induction on n. When $n = 1$, it is, of course, trivially true. So let us assume it for $n = m$ and consider the case $n = m + 1$. Then we know by the argument in the last paragraph that the relation (3) holds for all x, y in $B\theta$.

Let A_1 be the additive subgroup of A generated by all

$$(m+1)a(x-1)^m,$$

for $a \in A$, $x \in B\theta$. Then A_1 is B-invariant, and we have $A_1 \,|\mathfrak{e}{:}m|\, B$, by (3). Hence, by the induction hypothesis,

$$m^*[A_1, {}_{2^{m-1}}B] = 0,$$

and consequently

$$m^*(m+1)a(y_1 - 1) \cdots (y_{2^{m-1}} - 1)(x-1)^m = 0, \tag{4}$$

for all $a \in A$ and all y_i, $x \in B\theta$. If we set

$$A_2 = (m+1)m^*[A, {}_{2^{m-1}}B],$$

then (4) asserts that $A_2 \,|\mathfrak{e}{:}m|\, B$, and now, again by the induction hypothesis,

$$m^*[A_2, {}_{2^{m-1}}B] = 0.$$

So, finally,

$$(m+1)(m^*)^2[A, {}_{(2^{m-1}+2^{m-1})}B] = 0,$$

and we have established the result for $n = m+1$, since $(m+1)^* = (m+1)(m^*)^2$.

We may now rapidly dispose of Theorem 1.10. The next lemma embodies the trick referred to immediately before the statement of Theorem 1.10.

LEMMA 4.2. *If A is an abelian normal subgroup of a group G and A is contained in the centralizer of $G' = \gamma_2(G)$ (the commutator group of G), then $G \,|\mathfrak{e}{:}n|\, G$ with $n \geqq 2$ implies*

$$[A, {}_{(1+2^{n-2})}G]^{(n-1)^*} = 1.$$

Proof. For any $a \in A$ and any $x, y, \in G$,

$$1 = [x, {}_n ya] = [x, ya, {}_{(n-1)}y] = [[x, a][x, y], {}_{(n-1)}y]$$
$$= [x, a, {}_{(n-1)}y] = [a, x, {}_{(n-1)}y]^{-1}.$$

If now $A_1 = [A, G]$, then the centralizer of A_1 contains G', whence A_1 is a G/G'-module, and we have $A_1 \,|\mathfrak{e}{:}n-1|\, G$. Hence, by Lemma 4.1, with A, B, n replaced by A_1, G/G', $n-1$, respectively, we have

$$[A_1, {}_{2^{n-2}}G]^{(n-1)^*} = 1,$$

as required.

LEMMA 4.3. *If Y is normal in a given group X and is such that $[Y, {}_cX] = 1$ and $[Y^m, X] = 1$,[2] then $[Y, X]^{m^{c-1}} = 1$.*

Proof. If we set $Y_i = [Y, {}_iX]$, then $[Y_i^m, X] = 1$ for all i because $Y \lhd X$. Now

$$[Y_i^r, X] \equiv Y_{i+1}^r \pmod{Y_{i+2}}$$

for each $i \geqq 0$, and hence

[2] Y^m means the subgroup generated by all elements y^m, for $y \in Y$.

$$Y_{i+1}{}^{m} \leqq Y_{i+2}$$

for each $i \geqq 0$, whence $Y_1{}^{m^{c-1}} \leqq Y_c = 1$, as required.

LEMMA 4.4 *If $G' = \gamma_2(G)$ is nilpotent of class c, then $G\,|\mathfrak{e}{:}n|\,G$ with $n \geqq 2$ implies $\gamma_r(G)^s = 1$, where*

$$r = r(n, c) = 2 + c(1 + 2^{n-2}),$$

and

$$s = s(n, c) = \{(n-1)^*\}^{c'},$$

$$c' = 1 + (c-1)!\left\{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(c-2)!}\right\}.$$

Proof. We shall use an induction on c and note that the result for $c = 1$ is precisely Lemma 4.2 with $A = G'$.

Suppose then that the class of G' is $c + 1$ and that

$$\gamma_r(G)^s \leqq \alpha_1(G'),$$

where $r = r(n, c)$, $s = s(n, c)$. Now by Lemma 4.3, taking $X = G'$, $Y = \gamma_r(G)$, and noting that $Y \leqq X$ because $r \geqq 2$, we have

$$(5) \qquad [\gamma_r(G), G']^{s^c} = 1.$$

If we set $C = [\gamma_r(G), G']$, then $\gamma_r(G)/C$ lies in the centre of G'/C; hence by Lemma 4.2,

$$[\gamma_r(G), {}_{(1+2^{n-2})}G]^{(n-1)^*} \leqq C,$$

and thus, by (5),

$$\{\gamma_{r+1+2^{n-2}}(G)\}^{(n-1)^* s^c} = 1.$$

But

$$r(n, c+1) = r(n, c) + 1 + 2^{n-2},$$

and

$$s(n, c+1) = (n-1)^* s(n, c)^c,$$

and so the induction is complete.

Proof of Theorem 1.10. We use an induction on t, the derived length of our group G, and observe that by definition $k(n, 1) = 2$, so that the result is certainly true when $t = 1$.

Assume therefore that the result holds for the commutator group G' of G: i.e.,

$$\gamma_k(G')^e = 1,$$

where $k = k(n, t-1)$ and $e = e(n, t-1)$ is an $(n-1)!$-number. Now $G/\gamma_k(G')$ has its commutator group nilpotent of class $\leqq k - 1$, and hence, by Lemma 4.4,

$$\gamma_r(G)^s \leqq \gamma_k(G'),$$

where $r = r(n, k-1)$, $s = s(n, k-1)$. We deduce that

$$\gamma_r(G)^{se} = 1,$$

where

$$r = 2 + (k - 1)(1 + 2^{n-2}) = 1 + (1/2^{n-2})\{(1 + 2^{n-2})^t - 1\},$$

by using the induction hypothesis on $k(n, t - 1)$; and

$$se = s(n, k - 1)e(n, t - 1)$$

depends on n and t only, and it is an $(n - 1)!$-number since, on the one hand, s is such a number by Lemma 4.4 and, on the other, e is an $(n - 1)!$-number by the induction hypothesis.

This concludes the proof of Theorem 1.10.

4.2. We return to our main task which is to prove Theorem 1.8. The next lemma occupies a place in the proof of Proposition 4.1 analogous to that taken by Lemma 4.4 in the proof of Theorem 1.10.

LEMMA 4.5. *G is a soluble group, and H is a normal subgroup contained in the commutator group G'. If $H \,|\mathfrak{e}{:}n|\, G$ with $n \geqq 2$, and $[H, {}_cG'] = 1$, then*

$$[H, {}_rG]^s = 1,$$

where

$$r = r(n, c) = 2^{n-1}c,$$

and

$$s = s(n, c) = (n^*)^{c'},$$

with n^* as in Lemma 4.1 and c' as in Lemma 4.4.

Proof. When $c = 1$, the result reduces to Lemma 4.1. We use an induction on c and may assume, therefore, that

$$[H, {}_rG]^s \leqq [H, {}_{(c-1)}G'],$$

where $r = r(n, c - 1)$, $s = s(n, c - 1)$.

Now $[H, {}_{(c-1)}G']$ lies in the centre of G', and hence, by Lemma 4.3 (with $X = G'$, $Y = [H, {}_rG]$, and $m = s$),

$$[H, {}_rG, G']^{s^{c-1}} = 1.$$

If $C = [H, {}_rG, G']$, then $[H, {}_rG]/C$ lies in the centre of G'/C, and thus, by Lemma 4.1,

$$[H, {}_rG, {}_{2^{n-1}}G]^{n^*} \leqq C.$$

It follows that

$$[H, {}_{(r+2^{n-1})}G]^{n^* s^{c-1}} = 1,$$

and because

$$r(n, c) = r(n, c - 1) + 2^{n-1}$$

and

$$s(n, c) = n^* s(n, c - 1)^{c-1},$$

the induction is complete.

PROPOSITION 4.1. *Let G be a soluble group of derived length t, and let H be a normal subgroup such that $H\,|\mathfrak{e}{:}n|\,G$, with $n \geqq 2$. Then $[H, {}_kG]$ has exponent $e = e(n, t)$, where e is an n!-number, and*

$$k = k(n, t) = (2^{(n-1)t} - 1)/(2^{n-1} - 1).$$

Proof. When $t = 1$, $k = 1$, and so the proposition is certainly true in this case. If we write $K = [H, G]$, then $K \leqq G'$ and thus, using induction on t, we suppose

$$[K, {}_kG']^e = 1,$$

where $k = k(n, t - 1)$ and $e = e(n, t - 1)$ is an n!-number. By Lemma 4.5,

$$[K, {}_rG]^s \leqq [K, {}_kG'],$$

where $r = r(n, k)$, $s = s(n, k)$. Thus

$$[H, {}_{(1+r)}G]^{se} = 1$$

and

$$1 + r = 1 + 2^{n-1}k(n, t - 1) = k(n, t),$$

while se depends only on n, t and is an n!-number.

LEMMA 4.6. *Let H be a soluble normal subgroup of a group G such that G/C is also soluble, where C is the centralizer of H in G. If $H\,|\mathfrak{e}{:}n|\,G$, then there exists an integer k such that $[H, {}_kG]$ has finite exponent equal to an n!-number.*

Proof. Since H is soluble and $H\,|\mathfrak{e}{:}n|\,H$, therefore $\gamma_m(H)^l = 1$, for a suitable m and where l is an n!-number (by Proposition 4.1 or Theorem 1.10). The required result will follow if we can establish that

$$[H, {}_{k(i)}G]^{e(i)} \leqq \gamma_i(H)$$

for each $i \geqq 1$ and where $e(i)$ is an n!-number. We do this by an induction on i. So let us write $K = [H, {}_{k(i)}G]$ and assume $K^{e(i)} \leqq \gamma_i(H)$. Then

$$[K^{e(i)}, H] \equiv 1 \pmod{\gamma_{i+1}(H)},$$

and

$$[K, {}_iH] \equiv 1 \pmod{\gamma_{i+1}(H)},$$

whence

$$[K, H]^{e(i)^{i-1}} \leqq \gamma_{i+1}(H),$$

by Lemma 4.3.

If we now write $\bar{X}$ for a subgroup X taken modulo $[K, H]$, then $\bar{D}$, the centralizer of $\bar{K}$ in $\bar{G}$, contains $\bar{H}\bar{C}$, whence $\bar{G}/\bar{D}$ is soluble. If B is the image of $\bar{G}$ in the automorphism group of $\bar{K}$, then $\bar{K}B$ is a soluble subgroup of the holomorph of $\bar{K}$, and $\bar{K}\,|\mathfrak{e}{:}n|\,\bar{G}$ implies $\bar{K}\,|\mathfrak{e}{:}n|\,\bar{K}B$ because $\bar{K}$ is an *abelian* normal subgroup of $\bar{K}B$. Hence we may apply Proposition 4.1 to conclude

$$[K, {}_kG]^e \leqq [K, H],$$

for $k = k(n, t(\bar{K}B))$ and e an n!-number. Hence

$$[H, {}_{(k(i)+k)}G]^{e\cdot e(i)^{i-1}} \leqq \gamma_{i+1}(H)$$

and $e\cdot e(i)^{i-1}$ is an n!-number.

Proof of Theorem 1.8. Let C be the centralizer of H in G and $C \leqq D \leqq G$, where $D \lhd G$, G/D is finite, and D/C is soluble. By Lemma 4.6, there exists k_1 such that $[H \cap D, {}_{k_1}D]$ has exponent an n!-number.

Now $H/H \cap D$ is a finite soluble normal subgroup of $G/H \cap D$, and hence its centralizer has finite index in $G/H \cap D$. By an induction on the derived length of $H/H \cap D$ and the use of Proposition 2.2 (i), we see that for some finite k_2,

$$[H, {}_{k_2}G] = H_1 \leqq H \cap D.$$

Hence $[H_1, {}_{k_1}D]$ has exponent an n!-number.

We shall have finished if we can show that

$$[H_1, {}_{r(i)}G] \leqq [H_1, {}_iD] \tag{6}$$

for each $i \geqq 0$; we naturally do this by an induction on i. Set $H_2 = [H_1, {}_{r(i)}G]$, so that, by (6), H_2 lies in the centre of D, modulo $[H_1, {}_{(i+1)}D]$. In other words, H_2 is a G/D-module, modulo $[H_1, {}_{(i+1)}D]$, and thus, by Proposition 2.2 (i),

$$[H_2, {}_{k_3}G] \leqq [H_1, {}_{(i+1)}D],$$

from which we conclude that $r(i + 1)$ exists and can be taken to be $r(i) + k_3$.

5. Engel structure

5.1. *Proof of Theorem* 1.5. Since $\mathrm{Gp}\{L(G) \cup R(G)\}$ is normal in G, it must equal $\eta(G)$. Thus $L(G) = \eta(G)$, and moreover $\eta(G)$ is soluble, whence $\eta(G) = \sigma\eta(G)$ by Theorem 4 (i) [EE]. But $\sigma\eta(G) = \sigma(G)$ (Theorem 2 [EE]) and consequently $\sigma(G) = L(G)$. Further,

$$\begin{aligned}\bar{L}(G) = \bar{L}\eta(G) &= \bar{\sigma}\eta(G), \qquad \text{by Theorem 4 (ii) [EE],}\\ &= \bar{\sigma}(G).\end{aligned}$$

For any $a \in R(G)$, and any $x \in G$,

$$a_x = \mathrm{Gp}\{x \text{ and } g^{-1}ag \text{ for all } g \in G\}$$

is soluble, because $a_a \leqq \eta(G)$. Moreover, $R(G) \leqq \eta(G)$ implies that $R(G)$ is a subgroup (Lemma 14 [EE]), and thus $a_a \mathfrak{e}\, G$. In particular $a_a \mathfrak{e}\, x$, whence $a_x \mathfrak{e}\, x$ and so $x \;\infty\!\lhd\; a_x$ by Theorem 4 (i) [EE]. We conclude $R(G) = \rho(G)$.

Finally, if $a \in \bar{R}(G)$, a_a is soluble (because $a \in \eta(G)$), and hence $a \in \bar{\rho}(G)$ by Theorem 1.6. Thus $\bar{R}(G) = \bar{\rho}(G)$.

Proof of Theorem 1.4. The argument depends heavily on Theorem 4 [EE], and we shall use this result without further explicit mention.

If $G \mathfrak{e}\, a$, then $aN \;\infty\!\lhd\; G/N$, and therefore $\mathrm{Gp}\{a, N\} = M \;\infty\!\lhd\; G$. Now

M is soluble, and so $M \mathfrak{e} a$ implies $a \infty\!\lhd M$, whence $a \infty\!\lhd G$. On the other hand, when $G \,|\mathfrak{e}\, a$, then $M = \mathrm{Gp}\{a, N\} \lhd\lhd G$ and $a \lhd\lhd M$, from which we conclude $a \lhd\lhd G$.

Now suppose $a \mathfrak{e} G$. Then $aN \in \rho(G/N) \leqq \sigma(G/N)$, and thus $G/N \mathfrak{e} aN$. So for any $x \in G$, we can find n such that $x' = [x, {}_n a] \in N$. But $M = \mathrm{Gp}\{a, N\}$ is soluble and $a \mathfrak{e} M$, whence $M \mathfrak{e} a$, and $[x', {}_m a] = 1$ for some m. Hence $G \mathfrak{e} a$, and consequently $R(G) \leqq L(G) = \sigma(G)$. But this implies that $R(G)$ is a subgroup (Lemma 14 [EE]), so that $a_a \leqq R(G)$. For any $x \in G$, $a_a \mathfrak{e} x$, and so also $a_x \mathfrak{e} x$. Further, $xN \infty\!\lhd a_x N/N$ (since $G/N \in \mathfrak{E}$), and hence

$$H = \mathrm{Gp}\{x, N\} \cap a_x \infty\!\lhd a_x .$$

As $\mathrm{Gp}\{x, N\}$ is soluble and $a_x \mathfrak{e} x$, so x is a left Engel element in the soluble group H, and thus $x \infty\!\lhd H$. We conclude $x \infty\!\lhd a_x$, and as this works for any x, $a \in \rho(G)$.

Finally, take $a \in \bar{R}(G)$. Then $a_a N/N$ is soluble because $aN \in \bar{\rho}(G/N)$ (note that this is the first and only use of the hypothesis that $\bar{\rho}(G/N)$ is soluble), and consequently a_a itself is soluble. But then $a \in \bar{\rho}(G)$ by Theorem 1.6.

5.2. In order to prove Theorem 1.3, we must show that any given finite series from 1 to $\eta(G)$ with factors in $\hat{\mathfrak{A}}$ yields the existence of a similar series in which all the terms are *normal in G*. This we do in Lemma 5.2, which also implies, incidentally, that powers of $\hat{\mathfrak{A}}$ may be written without the need of brackets. First, however, we establish a very simple fact.

LEMMA 5.1. *If G is poly-$\hat{\mathfrak{A}}$ with series of length l and n factors that are not finitely generated, then every subgroup H is also poly-$\hat{\mathfrak{A}}$ with corresponding integers l', n' satisfying $l' \leqq l$, $n' \leqq n$.*

Proof. Let

$$1 = G_0 < G_1 < \cdots < G_l = G$$

be a series in which factors of successive terms are in $\hat{\mathfrak{A}}$. Setting $H_i = G_i \cap H$, we obtain the series

$$1 = H_0 \leqq H_1 \leqq \cdots \leqq H_l = H,$$

where

$$H_{i+1}/H_i \cong (G_{i+1} \cap H)G_i/G_i \tag{7}$$

for each $i = 0, \cdots, l - 1$. The right-hand side of (7) is a subgroup of G_{i+1}/G_i and, as such, is in $\hat{\mathfrak{A}}$.[3] Hence H is poly-$\hat{\mathfrak{A}}$ with $l' \leqq l$. If G_{i+1}/G_i is finitely generated, so is every subgroup, and hence H_{i+1}/H_i is finitely generated. Thus also $n' \leqq n$.

[3] While the class $\hat{\mathfrak{A}}$ is (clearly) closed with respect to the operation of taking subgroups, it is unfortunately not closed with respect to homomorphic images. For example, the additive group of the rational numbers is in $\hat{\mathfrak{A}}$, but not so the rationals modulo one.

LEMMA 5.2. *If G has a series*

$$1 = G_0 < G_1 < \cdots < G_l = G,$$

where, for each i, $G_{i+1}/G_i \in \hat{\mathfrak{A}}$ and exactly n of these factors are not finitely generated, then G possesses a characteristic series (i.e., a series of characteristic subgroups)

$$1 = C_0 < C_1 < \cdots < C_{l'} = G,$$

where $l' \leqq l$, each $C_{i+1}/C_i \in \hat{\mathfrak{A}}$ and at most n of these factors are not finitely generated.

Proof. We shall prove the lemma by an induction on l. So let $l = s + 1$ and set

$$C_s = \bigcap G_s^{\alpha},$$

where the intersection is taken over all automorphisms α of G. Then C_s is characteristic in G, and, as a subgroup of G_s, it is poly-$\hat{\mathfrak{A}}$ with series of length $\leqq s$ and the number of nonfinitely generated factors at most equal to that between 1 and G_s (Lemma 5.1). By the induction hypothesis, and because characteristic subgroups of C_s are characteristic in G, we shall have completed the proof of the lemma if we can verify that $G/C_s \in \hat{\mathfrak{A}}$ and that $G/C_s \in \mathfrak{G}$ whenever $G/G_s \in \mathfrak{G}$.

Suppose $x \notin C_s$, but that $x^p \in C_s$ for some prime p. Then there exists α such that $x \notin G_s^{\alpha}$, and hence p is in the torsion spectrum of G/G_s^{α}. But $G/G_s^{\alpha} \cong G/G_s$, and thus p is in the torsion spectrum of G/G_s. This last is finite, and consequently so also is the torsion spectrum of G/C_s.

To show that G/C_s has finite rank, in view of the fact that C_s contains G', the commutator group of G, it will be sufficient to show G/G' has finite rank. To do this, we set $A_i = G_i G'$, so that

$$G' = A_0 \leqq A_1 \leqq \cdots \leqq A_{s+1} = G.$$

By the Zassenhaus Lemma,

$$A_{i+1}/A_i \cong G_{i+1}/(G_{i+1} \cap G')G_i,$$

and the right-hand side, being a homomorphic image of G_{i+1}/G_i, has finite rank. Thus each A_{i+1}/A_i has finite rank, and thus G/G' also has finite rank. We conclude that $G/C_s \in \hat{\mathfrak{A}}$.

Suppose, finally, that G/G_s is finitely generated, and let S/G_s be the torsion group of G/G_s. Clearly T/C_s is the torsion group of G/C_s, where $T = \bigcap S^{\alpha}$. If e is the exponent of the (finite) group S/G_s, then T/C_s has exponent at most e. But T/C_s has finite rank, and thus T/C_s is finite. Hence to prove that G/C_s is finitely generated, it only remains to show this of G/T.

Now G/T, as a torsion-free group of finite rank, has a countable group of automorphisms, and hence the family (G/S^{α}) is countable. But each group in this family is a direct product of the same finite number of infinite cyclic groups, and hence their cartesian product P is the cartesian product of a countable number of infinite cyclic groups. On the other hand, G/T is countable and is isomorphic to a subgroup of P, whence G/T is free abelian by a theorem of Specker [10]. The fact that G/T also has finite rank now ensures that G/T is finitely generated.

The reader will note that the proofs of both Lemmas 5.1 and 5.2 work equally well if, instead of $\hat{\mathfrak{A}}$, one uses the class of all abelian groups of finite rank. We state this as

LEMMA 5.3. *The statements of* (i) *Lemma* 5.1 *and* (ii) *Lemma* 5.2 *remain true when* $\hat{\mathfrak{A}}$ *is replaced by the class of all abelian groups of finite rank.*

We are now in a position to prove Theorem 1.3. It turns out that parts (i), (iii) and (ii), (iv) go naturally together; and the first of these pairs is contained in the following more general result.

PROPOSITION 5.1. *If G is any group such that $\eta(G)$ is poly-"abelian of finite rank", then $\sigma(G) = \eta(G)$ is the unique maximal hypercentral normal subgroup, and $\rho(G) = \alpha_{n\omega+k}(G)$, where k is finite and n is the number of non-finitely generated factors in any given series for $\eta(G)$.*

Proof. Suppose we have already proved that $\rho(G) = \alpha_{n\omega+k}(G)$. If $Y = \eta(G)$, then $Y = \eta(Y)$, and thus we may apply the fact concerning ρ to Y itself to obtain $\rho(Y) = \alpha(Y)$. But as Y is soluble, every right Engel element is in $\rho(Y)$ (by Theorem 4 (iii) [EE]), and thus, as Y is locally nilpotent, $\rho(Y) = Y$. Consequently $Y = \alpha(Y)$, i.e., Y is hypercentral. That every other hypercentral normal subgroup of G is contained in Y follows from the simple fact that every element in such a normal subgroup is serial in G. Moreover, $\sigma(Y) = Y$ and $\sigma(Y) = \sigma(G)$, so that $\sigma(G) = Y$.

Let $H = \rho(G)$ and observe that, by Lemma 5.3 (i), H is also poly-"abelian of finite rank" and has a series with at most n factors that are not finitely generated. By Lemma 5.3 (ii) we can find a series of characteristic subgroups of H,

$$1 = H_0 < H_1 < \cdots < H_s = H,$$

such that each H_{i+1}/H_i is abelian of finite rank and at most n are not finitely generated.

Now, $H_1 \lhd G$ and $H_1 \mathfrak{e}\, G$ together imply $H_1 \leqq \alpha_{\omega+r}(G)$ by Proposition 1.1 (ii). But if, in fact, $H_1 \in \mathfrak{G} \cap \mathfrak{A}$, then $H_1 \mathfrak{e}\, G$ yields $H_1 |\mathfrak{e}\, G$ (for $[ab, {}_rx] = [a, {}_rx][b, {}_rx]$ whenever $a, b \in H_1$, since H_1 is an abelian normal subgroup of G); and then $H_1 \leqq \alpha_r(G)$, by Proposition 1.1 (i). An easy induction on s now shows that $H \leqq \alpha_{n\omega+k}(G)$, whence actually $H = \alpha_{n\omega+k}(G)$, because, anyway, $\alpha(G) \leqq H$.

Parts (ii), (iv) of Theorem 1.3 depend on the following result.

LEMMA 5.4. *If M is a subgroup of a given group X and it contains a series*

$$1 = M_0 < M_1 < \cdots < M_m = M$$

of subgroups normal in X, *such that each* $M_{i+1}/M_i \in \hat{\mathfrak{A}}$ *and* $M \mid\mathfrak{e}\, X$, *then* $M \leqq \alpha_k(X)$.

The proof is by an induction on m and the use of Proposition 1.1 (i).

Proof of Theorem 1.3, *parts* (ii), (iv). By Lemma 5.1, $\bar{\sigma}(G)$ is poly-$\hat{\mathfrak{A}}$, and hence, by Lemma 5.2, $\bar{\sigma}(G)$ can be linked to 1 by a finite characteristic series with factors of successive terms in $\hat{\mathfrak{A}}$. Further, $\bar{\sigma}(G) \mid\mathfrak{e}\, \bar{\sigma}(G)$, and thus we may apply Lemma 5.4 with both M and X replaced by $\bar{\sigma}(G)$. We conclude $\bar{\sigma}(G) \leqq \alpha_k\, \bar{\sigma}(G)$, i.e., $\bar{\sigma}(G)$ is nilpotent. To complete part (ii) of Theorem 1.3, we note that, naturally, every nilpotent normal subgroup of G must lie in $\bar{\sigma}(G)$.

Finally, choose $a \in \bar{\rho}(G)$. Then $a_a \mid\mathfrak{e}\, G$, and Lemmas 5.1, 5.2 show that we may apply Lemma 5.4 with a_a, G taking the places of M, X, respectively. Hence $a_a \leqq \alpha_k(G)$, whence $\bar{\rho}(G) \leqq \alpha_\omega(G)$, as required.

5.3. We turn now to the proof of Theorem 1.2. The first two parts of that theorem will be proved in a somewhat more general setting as follows.

PROPOSITION 5.2. *If* G *has a nilpotent normal subgroup* N *such that* G/N *has* Max, *then* $\sigma(G)$ *is hypercentral, and* $\bar{\sigma}(G)$ *is nilpotent.*

Proof. If $Y = \sigma(G)$, then $N \cap Y$ is a nilpotent normal subgroup of Y and $Y/N \cap Y \in \mathfrak{G} \cap \mathfrak{N}$, because Y is locally nilpotent. Hence there exists k such that

$$\gamma_k(Y) = H \leqq N \cap Y.$$

Since $H\, \mathfrak{e}\, Y$, we have $H \leqq \alpha_{c\omega}(Y)$ by Theorem 3.1 (ii) (where c is the class of $N \cap Y$). Thus $Y = \alpha_{c\omega+k}(Y)$.

The proof that $\bar{\sigma}(G)$ is nilpotent is exactly similar, but uses part (i) of Theorem 3.1 instead of part (ii).

Let us now prove the remaining two parts of Theorem 1.2.

Suppose N is a nilpotent normal subgroup of our group G and such that $G/N \in \mathfrak{P}\mathfrak{F}$. Since G/N has Max, Theorem 1.3 (iii) shows $\rho(G/N) = \alpha_k(G/N)$ for some finite k. Then

$$H = [\rho(G), {}_kG] \leqq N,$$

and thus Theorem 3.1 (ii) yields part (iii) of our present theorem.

For the last part, take $a \in \bar{\rho}(G)$, and observe that $a_a\, N/N \leqq \alpha_k(G/N)$, whence

$$H = [a_a\,, {}_kG] \leqq N.$$

Now $a_a \mid\mathfrak{e}\, G$ and thus Theorem 3.1 (i) applied with $H = [a_a, {}_kG]$ yields $a \in \alpha_{k+s}(G)$, whence $\bar{\rho}(G) \leqq \alpha_\omega(G)$, as required.

Of course, part (iv) of Theorem 1.2 is already implied by Theorem 1.7, but the virtue of the present proof is that it bypasses Theorem 1.8.

To conclude this section we shall show what happens when one confines attention to the *finitely generated* groups in $\mathfrak{N}\mathfrak{P}\mathfrak{F}$.

THEOREM 5.1. *If* $G \in \mathfrak{G} \cap \mathfrak{N}\mathfrak{P}\mathfrak{F}$, *then* $\sigma(G) = \bar{\sigma}(G)$ *and* $\rho(G) = \bar{\rho}(G)$.

In any finitely generated group, α and α_ω have the same value (cf. the last paragraph of the proof of Lemma 3.4), and hence the equality $\rho(G) = \bar{\rho}(G)$ is an immediate corollary of Theorem 1.2. The first part of the theorem is a consequence of the following more general result, which was effectively found by Philip Hall in connexion with his investigation [7]. The proof that we give is also due to him.

PROPOSITION 5.3. *If G is finitely generated and has a nilpotent normal subgroup N such that G/N has* Max *and is finitely related, then $\eta(G)$ is nilpotent.*

The reader will recall that a finitely related group is one that can be defined by a finite number of generators subject to a finite number of relations. The property of being finitely related is a poly property, whence all groups in $\mathfrak{P}\mathfrak{F}$ are finitely related. (See [5], Lemma 1.)

Proof of Proposition 5.3. We shall use an induction on c, the class of N and, for the moment, shall assume the result true when $c = 1$.

Let us write $Y = \eta(G)$ and note that $Y \geqq N$. We assume $c > 1$ and, by the induction hypothesis, $\eta(G/\gamma_c(N))$ is nilpotent. Hence $Y/\gamma_c(N)$ is nilpotent, from which we conclude $Y/\gamma_2(N)$ is nilpotent (because $c \geqq 2$). But then Y is nilpotent by a result of P. Hall ([6], Theorem 7).

This leaves the case $c = 1$. As G is finitely generated and G/N is finitely related, therefore N is finitely generated as a normal subgroup of G, say by $x_1, \cdots, x_m$. Since G/N has Max, Y/N is finitely generated, and thus $Y = \mathrm{Gp}\{N, y_1, \cdots, y_n\}$, for suitable y_j's. The local nilpotence of Y implies that the groups $\mathrm{Gp}\{x_i, y_1, \cdots, y_n\}$ are nilpotent, for each $i = 1, \cdots, m$. If r is the maximal among their classes, then

$$[x_i, g_1, \cdots, g_r] = 1 \tag{8}$$

for all $i = 1, \cdots, m$ and all $g_j \in Y$. (Recall that now N is abelian!) Because every element in N is a word in conjugates of the x_i's and as Y is normal in G, we may replace (8) by the stronger assertion that

$$[x, g_1, \cdots, g_r] = 1 \tag{9}$$

for all $x \in N$ and all $g_j \in Y$. But (9) is equivalent to saying that $N \leqq \alpha_r(Y)$. Since Y/N is nilpotent, we may conclude that Y itself is nilpotent.

6. Examples

6.1. The two groups, whose existence will establish Theorem 1.1, are to be constructed on the same plan as the counterexample in [EE]. Thus, each of the groups will be a direct product of three groups whose Engel structures intertwine in the required manner. Of course, this leaves open the problem of the existence of *indecomposable* soluble groups of the prescribed types.

We shall construct, below, two countable metabelian p-groups, called U and V, with the following Engel structures:

$$\sigma(U) = \bar{\sigma}(U) = U, \qquad \alpha(U) = \alpha_\omega(U) = \bar{\rho}(U) = 1;$$

and

$$\sigma(V) = \alpha(V) = V, \qquad \bar{\sigma}(V) = \bar{\rho}(V) = \alpha_\omega(V) = D,$$

where D is a nontrivial normal subgroup strictly smaller than V. Further, let W be the group $F/(F')^p F''$, where F is the free product of a countable number of cyclic groups, each of order p. We showed in [EE] (pp. 166–167) that

$$\sigma(W) = \bar{\sigma}(W) = \bar{\rho}(W) = W,$$

and

$$\alpha(W) = \alpha_\omega(W) = 1.$$

(As a matter of fact, we only proved the equalities when $p = 2$, but, as we remarked in the footnote on p. 166, essentially the same proof works for arbitrary p.)

If we set $G = U \times V \times W$, then G is a countable metabelian p-group and, by Proposition 5 [EE], we have

$$\begin{aligned} \sigma(G) &= U \times V \times W, & \alpha(G) &= 1 \times V \times 1, \\ \bar{\sigma}(G) &= U \times D \times W, & \alpha_\omega(G) &= 1 \times D \times 1. \\ \bar{\rho}(G) &= 1 \times D \times W, \end{aligned}$$

Thus G satisfies all the conditions required in Theorem 1.1 (i).

6.2. The groups U, V are both constructed from an abelian group A of type p^∞ and a cyclic group B of order p.

Let Γ be the difference ideal of the group algebra of A over a field F of characteristic p. Then $(a - 1)^p = a^p - 1$, and hence, as every element in A has order a power of p, Γ is spanned (over F) by nilpotent elements. But Γ is commutative and thus is actually a nil-algebra. However, $\Gamma = \Gamma^{(p)}$ where $\Gamma^{(p)}$ is the F-subspace spanned by all $(a - 1)^p$, with $a \in A$: for if $a' \in A$, then there exists $a \in A$ such that $a' = a^p$.

Now take any Γ-module M. The fact that Γ is nil implies $M \mid\mathfrak{e}\, A$, and the equality $\Gamma = \Gamma^{(p)}$ shows that if x in M satisfies $x\ \mathfrak{e}\!\mid A$, then $x \in \alpha_1(M:A)$: for then $x(a - 1)^{p^s} = 0$ for all $a \in A$ and some fixed s, whence $x(a' - 1) = 0$ for all $a' \in A$.

Suppose U is a group containing a p-elementary abelian normal subgroup M with $U/M \cong A$. Then U is a metabelian p-group so that

$$\sigma(U) = \rho(U) = U.$$

As M is a Γ-module (with F the field of the integers modulo p), $M \mid\mathfrak{e}\, U$ and hence $U \mid\mathfrak{e}\, U$, i.e., $\bar{\sigma}(U) = U$. Moreover, if $u \in \bar{\rho}(U)$, then for any $u' \in U$,

$[u, u'] \in M \cap \bar{\rho}(U)$, i.e., $[u, u'] \mathrel{\mathfrak{e}|} A$, whence, by the remarks above, $[u, u'] \in \alpha_1(U)$. *If U has trivial centre*, then $[u, u'] = 1$, from which we conclude $u \in \alpha_1(U)$, i.e., $\bar{\rho}(U) = \alpha_1(U) = 1$.

A group of the required type is provided by the wreath product of B by A, and we define U to be this group. The diagonal group of U is the direct product of a countable number of copies of B and thus has all the properties of the subgroup M above. The fact that U has trivial centre is a corollary of a result of Baumslag ([3], Corollary 3.2).

Finally, we define V to be the wreath product of A by B. Then V is another locally nilpotent metabelian p-group, so that $\sigma(V) = \rho(V) = V$. But V satisfies Min, whence, by Theorem 1.2 or Theorem 1.3, $\rho(V) = \alpha(V)$, $\bar{\rho}(V) = \alpha_\omega(V)$, and $\bar{\sigma}(V)$ is nilpotent. Further, if D is the diagonal group of V, $D \mathrel{\mathfrak{e}} V$ implies $D \leqq \alpha_\omega(V)$ by Proposition 2.2 (ii), and, since D is a maximal subgroup of V, we shall have

$$D = \alpha_\omega(V) = \bar{\rho}(V) = \bar{\sigma}(V)$$

provided only that we are able to check V is not nilpotent. This, however, follows from a theorem of Baumslag [3] to the effect that the wreath product of a group X by a group Y is nilpotent if, and only if, Y is a finite p-group and X is a nilpotent p-group of *finite exponent*.

Aside from its use in Theorem 1.1 (i), the group V has another part to play. If S denotes the symmetric group on three symbols and C is its normal subgroup of order 3, then

$$\sigma(S \times V) = C \times V, \qquad \bar{\sigma}(S \times V) = C \times D,$$
$$\rho(S \times V) = 1 \times V, \qquad \bar{\rho}(S \times V) = 1 \times D.$$

Since $S \times V$ has Min, we have established

PROPOSITION 6.1. *There exists a countable metabelian group with* Min *in which the subgroups determined by σ, ρ, $\bar{\sigma}$, $\bar{\rho}$ are all distinct.*

6.3. We turn now to torsion-free groups. Our example for Theorem 1.1 (ii) has the form $L \times M \times N$, where

$$1 = \rho(L) < \sigma(L) = \bar{\sigma}(L) < L;$$
$$1 < \bar{\rho}(M) = \bar{\sigma}(M) < \alpha_{\omega+1}(M) = M;$$

and

$$1 = \bar{\rho}(N) < \bar{\sigma}(N) = \rho(N) = N.$$

The fact that σ, ρ, α, $\bar{\sigma}$, $\bar{\rho}$ really do determine five distinct subgroups of $L \times M \times N$ follows, as usual, from Proposition 5 [EE]. Thus Theorem 1.1 will be completely proved once we have determined the groups L, M, N.

The first group, L, is the extension of Q, the additive group of the rational numbers, by an infinite cyclic group on x defined by $1^x = r$, where $r \notin \{-1, 0, +1\}$. For any integer $k \neq 0$ and any positive integer m, we have

$$[1, {}_m x^k] = (r^k - 1)^m,$$

so that $x^k \notin \sigma(L)$. But $\bar{\sigma}(L) \geqq Q$, and hence $\sigma(L) = \bar{\sigma}(L) = Q$. If $y \in \alpha_1(L)$, then $y \in \sigma(L)$, i.e., $y \in Q$. Then $y = y^x = yr$ implies $y = 0$, whence $\alpha_1(L) = 1$. As $\eta(L)$ is of type $\mathfrak{A}$, we know $\alpha(L) = \rho(L)$ from Theorem 1.3, and thus $\rho(L) = 1$.

The second group, M, is the extension of a group C, free abelian on a countable set $c_1, c_2, \cdots$, by an infinite cyclic group on y defined by

$$c_i^{\,y} = c_i \, c_{i-1}$$

for all $i \geqq 1$ and where $c_0 = 1$. We assert

$$(10) \qquad \alpha_r(M) = \mathrm{Gp}\{c_1, \cdots, c_r\}.$$

If $cy^k \in \alpha_1(M)$, then

$$1 = [cy^k, c_2]^{-1} = [c_2, y^k] = c_1[c_2, y^{k-1}] = \cdots = c_1^{\,k},$$

so that $k = 0$ and $c \in \alpha_1(M)$. But then $c = c_1^{m_1} \cdots c_s^{m_s}$, whence

$$1 = [c, y] = c_1^{m_2} \cdots c_{s-1}^{m_s},$$

and consequently $m_2 = \cdots = m_s = 0$, i.e., $c \in \mathrm{Gp}\{c_1\}$. This has established (10) when $r = 1$. Assume it for $r = s$, and let $g \to \bar{g}$ be the epimorphism $M \to M/\alpha_s(M)$. Then

$$\alpha_1(M/\alpha_s(M)) = \mathrm{Gp}\{\bar{c}_{s+1}\}$$

by the case $r = 1$, and hence $\alpha_{s+1}(M) = \mathrm{Gp}\{c_1, \cdots, c_{s+1}\}$, thus completing the induction argument. We conclude from equation (10) that $\alpha_\omega(M) = C$, and thus $\alpha_{\omega+1}(M) = M$.

Suppose $y^k \in \bar{\sigma}(M)$. Then $[c, {}_n y^k] = 1$ for all $c \in C$ and some fixed n. By (10),

$$[c_{n+1}, y^k] \equiv [c_{n+1}, y]^k \equiv c_n^{\,k} \pmod{\alpha_{n-1}(M)},$$

whence

$$\begin{aligned}[c_{n+1}, {}_n y^k] &= [c_n^{\,k} v, {}_{(n-1)} y^k] \qquad \text{with } v \in \alpha_{n-1}(M)\\ &= [c_n, {}_{(n-1)} y^k]^k,\end{aligned}$$

and consequently

$$1 = [c_{n+1}, {}_n y^k] = c_1^{\,k^n}.$$

Thus $\mathrm{Gp}\{y\} \cap \bar{\sigma}(M) = 1$, and as $\bar{\sigma}(M) \geqq C$, we must have $\bar{\sigma}(M) = C$.

Finally, we define *the third group*, N. Let S be the set of all sequences of integers $s = (s_i) = (s_1, s_2, \cdots)$, with $0 \leqq s_i \leqq i$ for each $i \geqq 1$, but with $s_i = 0$ for all but a finite number of i's. Purely in order to prevent possible confusion later, we exclude the zero sequence from S. We set X to be the free (additively written) abelian group on S.

If $s = (s_i) \in S$ and j, n are any positive integers, then we define $s(j{:}n)$ to be

(i) the zero element of X when $s_j + n > j$; and

(ii) the element (t_i) of S, when $s_j + n \leqq j$, where $t_j = s_j + n$ and $t_i = s_i$ if $i \neq j$.

For each positive integer i, the mapping

$$s \rightarrow s + s(i:1)$$

extends to an endomorphism of X, say c_i. We assert that c_i is actually an automorphism. For if $s \in S$, then

$$s = (s - s(i:1) + s(i:2) - \cdots)c_i,$$

where the series must terminate as $s(i:n) = 0$ when $s_i + n > i$, and thus $Xc_i = X$. Further, if

$$(m_1 s^1 + \cdots + m_k s^k)c_i = 0,$$

where $m_1, \cdots, m_k$ are nonzero integers and $s^1, \cdots, s^k$ distinct elements of S, then

$$\sum_j m_j s^j + \sum_j m_j s^j(i:1) = 0.$$

Supposing s^1 to be one of those among $s^1, \cdots, s^k$ with smallest i^{th} coordinate, then $s^1 \neq s^j(i:1)$ for any j, whence $m_1 = 0$. This is a contradiction, and consequently c_i is one-one.

We show next that the subgroup C, of the automorphism group of X, generated by $c_1, c_2, \cdots$ is the free abelian group on these elements. That C is abelian is obvious. Consider next an element

$$c = c_1^{m_1} \cdots c_r^{m_r}$$

of C. By an induction on $|m_1| + \cdots + |m_r|$, one may prove quite easily that

$$sc_1^{m_1} \cdots c_r^{m_r} = s + m_1 s(1:1) + \cdots + m_r s(r:1) + x,$$

where x is 0 or is a sum of basis elements t for which $t_i \geqq s_i$ always and $t_i > s_i$ at least twice or $t_i > s_i + 1$ at least once. If c is the identity automorphism, then

$$\sum_j m_j s(j:1) + x = 0,$$

and, as no basis element in the expression x is the same as any of $s(1:1), \cdots, s(r:1)$, we must have, for each j, $m_j = 0$ or $s(j:1) = 0$, i.e., $m_j = 0$ or $s_j = j$. If we choose s such that $s_j < j$ for all $j = 1, \cdots, r$, then $sc = s$ implies $m_1 = \cdots = m_r = 0$. Hence $c_1, c_2, \cdots$ form a basis of C.

We define our group N to be XC, regarded as a subgroup of the holomorph of X. Clearly, N is a torsion-free metabelian group.

If $x \in \alpha_1(N)$, then $x \in X$ since $\alpha_1(N) \leqq X$. Suppose

$$x = m_1 s^1 + \cdots + m_k s^k,$$

and choose any positive integer r such that $(s^i)_r = 0$ for each i. Then $s^i(r:1) \neq 0$, and

$$[x, c_r] = x(c_r - 1) = \sum_i m_i s^i(r{:}1) \neq 0$$

unless $m_1 = \cdots = m_k = 0$. We conclude $\alpha_1(N) = 1$. But, by Theorem 1.7, $\alpha_\omega(N) = \bar{\rho}(N)$, and hence

$$\alpha(N) = \bar{\rho}(N) = 1.$$

It remains to check that $\bar{\sigma}(N) = N$. Since $C \leqq \bar{\sigma}(N)$ anyway, we need only prove each $c_i \in \bar{\sigma}(N)$. Now for any $xc \in N$ (where $x \in X$, $c \in C$), $[xc, {}_rc_i] = [x, {}_rc_i]^c$ and thus, to show $N \mid \mathfrak{e}\, c_i$, it is sufficient to prove $X \mid \mathfrak{e}\, c_i$, while this, in turn, follows from $S \mid \mathfrak{e}\, c_i$. But this last fact has been built into the definition of S: $[s, {}_{(i+1)}c_i] = s(i{:}i + 1) = 0$ always. We conclude that

$$\bar{\sigma}(N) = \rho(N) = N,$$

as was required.

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