ON THE RELATION BETWEEN THE DETERMINANT AND THE PERMANENT[1]

BY

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In this paper we consider the problem of determining whether or not there exist linear operations on matrices that change their permanents into their determinants. Recent interest in the permanent function stems from its application to certain combinatorial problems [4; p. 166] and from an unresolved conjecture of van der Waerden [2].

If X is an n-square matrix, then the permanent of X is defined by

$$\operatorname{per}(X) = \sum_\sigma \prod_{i=1}^n x_{i\sigma(i)} ,$$

where σ runs over all permutations of $1, \cdots, n$. We introduce a notation to simplify writing sets of indices: if $1 \leqq r \leqq n$, then $Q_{n,r}$ will denote the totality of increasing sequences $\omega : 1 \leqq i_1 < \cdots < i_r \leqq n$. If X is an $m \times n$ matrix and $\omega \in Q_{m,r}$, $\tau \in Q_{n,r}$, then $X^+_{\omega,\tau}$ will denote the permanent of the submatrix of X with row indices ω and column indices τ. The symbol $X_{\omega,\tau}$ will denote the determinant of this submatrix. If s is an index in ω, then ω'_s will denote the sequence ω with s omitted. In case $\omega \in Q_{m,r}$, $\tau \in Q_{n,r}$, we will reserve the special notation $E_{\omega,\tau}$ for the $\binom{m}{r} \times \binom{n}{r}$ *unit matrix* with 1 in the (ω, τ) position in the doubly lexicographic ordering, zero elsewhere. That is, we imagine the rows of $E_{\omega,\tau}$ indexed with the elements in $Q_{m,r}$ where the ordering is the lexicographic one, and similarly for the columns. If $u_1, \cdots, u_p$ are vectors in some space V, we denote by $\langle u_1, \cdots, u_p \rangle$ the subspace of V spanned by these vectors. Finally, $\rho(X)$ will denote the rank of the matrix X; $[u, v] \parallel [x, y]$ will mean that the two vectors are linearly dependent, and $[u, v] \perp [x, y]$ that they are orthogonal, i.e., that $ux + vy = 0$. Let $2 \leqq r \leqq \min(m, n)$, and denote by $M_{m,n}$ the vector space of $m \times n$ matrices over a field F of characteristic zero. To fix the notation assume henceforth that $m \leqq n$. Then $C_r(X)$ and $P_r(X)$ will denote the r^{th} *determinantal* and *permanental compound matrices* of X, respectively. That is, $C_r(X)$, $(P_r(X))$, is the matrix in $M_{\binom{m}{r},\binom{n}{r}}$ whose entries are the r-square subdeterminants (subpermanents) of X arranged in doubly lexicographic order. Let Δ be the map on $M_{2,2}$ into itself where $(\Delta(X))_{21} = -x_{21}$ and $(\Delta(X))_{ij} = x_{ij}$ otherwise. Note that $\operatorname{per}(X) = \det(\Delta(X))$.

THEOREM. *There is no linear transformation T of $M_{m,n}$ into itself such that*

$$P_r(T(X)) = S_r\, C_r(X) \tag{1}$$

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for all $X \in M_{m,n}$, *where* S_r *is a nonsingular map of* $M_{\binom{m}{r},\binom{n}{r}}$ *into itself, unless* $m = n = r = 2$.

In this case

$$\Delta T(X) = AXB, \quad \textit{or else} \quad \Delta T(X) = AX'B,$$

where A *and* B *are in* $M_{2,2}$ *and* $\det(AB) \neq 0$. *Here* X' *denotes the transpose of* X.

In case $m = n = r > 2$ and $S_n = 1$, we conclude that *there is no linear operation on n-square matrices that converts the determinant into the permanent uniformly*. This result constitutes a generalization of the result of Pólya [3], [5], which states that there is no uniform way of affixing + and − signs to the elements of a matrix so as to change the determinant into the permanent except in the 2-square case.

We prove the theorem by contradiction. Assume then that $m + n \geqq 5$ and T is a linear transformation satisfying (1). We show in a series of lemmas that T preserves rank 1 and conclude on the basis of a recent theorem [1] that $T(X) = AXB$ or $T(X) = AX'B$ for all X, a situation that leads us to a contradiction.

LEMMA 1. *T is nonsingular.*

Proof. Assume $T(A) = 0$. Then

$$\begin{aligned} S_r\, C_r(A + X) &= P_r(T(A + X)) \\ &= P_r(T(A) + T(X)) \\ &= P_r(T(X)) \\ &= S_r\, C_r(X), \end{aligned}$$

for any $X \in M_{m,n}$. Hence since S_r is nonsingular, we obtain

$$C_r(A + X) = C_r(X)$$

for all X. Let X be the matrix with $-t$ in positions (i, i), $i = 1, \cdots, r$, $-a_{ij}$ in positions (i, j), $r \geqq j > i \geqq 1$, zero elsewhere. Then clearly the $(1, 1)$ element of $C_r(A + X)$ is $\prod_{i=1}^{r} (a_{ii} - t)$, whereas the $(1, 1)$ of $C_r(X)$ is $(-1)^r t^r$. Since the ground field has at least r elements, it follows that $a_{ii} = 0$, $i = 1, \cdots, r$. By pre- and post-multiplication by permutation matrices we can conclude in the same way that all $a_{ij} = 0$, i.e., $A = 0$. Note here that this proof is valid for $m = n = r = 2$ as well.

LEMMA 2. *If* (1) *holds, then there exists a nonsingular linear transformation* S_{r-1} *of* $M_{\binom{m}{r-1},\binom{n}{r-1}}$ *into itself such that*

$$P_{r-1}(T(X)) = S_{r-1}\, C_{r-1}(X). \tag{2}$$

Proof. Let $Y = T(X)$. By Lemma 1 there exist constants $g_{p,q}^{u,v}$, $u, p = 1, \cdots, m$; $v, q = 1, \cdots, n$ such that

$$x_{u,v} = \sum_{p=1,\ q=1}^{m\quad n} g_{p,q}^{u,v}\, y_{p,q}\,. \tag{3}$$

Moreover (1) asserts the existence of scalars $s_{\omega,\tau}^{\alpha,\beta}$, $\alpha, \omega \in Q_{m,r}$, $\beta, \tau \in Q_{n,r}$ for which

(4) $$Y^{+}_{\omega,\tau} = \sum_{\alpha,\beta} s_{\omega,\tau}^{\alpha,\beta} X_{\alpha,\beta} .$$

By (3) we can regard (4) as a polynomial identity in the variables y_{ij}. Suppose in (4) that s is an integer in the sequence ω, and t is an integer in the sequence τ. Then using (3) and (4),

$$\begin{aligned} Y^{+}_{\omega'_s,\tau'_t} &= \frac{\partial Y^{+}_{\omega,\tau}}{\partial y_{s,t}} = \sum_{\alpha,\beta} s_{\omega,\tau}^{\alpha,\beta} \frac{\partial X_{\alpha,\beta}}{\partial y_{s,t}} \\ &= \sum_{\alpha,\beta} s_{\omega,\tau}^{\alpha,\beta} \sum_{u=1,v=1}^{m\quad n} \frac{\partial X_{\alpha,\beta}}{\partial x_{u,v}} \frac{\partial x_{u,v}}{\partial y_{s,t}} \\ &= \sum_{\alpha,\beta} s_{\omega,\tau}^{\alpha,\beta} \sum_{u,v} X_{\alpha'_u,\beta'_v}\, g_{s,t}^{u,v} . \end{aligned}$$

Hence the $(r - 1)$-order permanental minors of $Y = T(X)$ are fixed (depending only on T and S_r) linear homogeneous functions of the $(r - 1)$-order determinantal minors of X. That is, there exists S_{r-1} mapping

$$M_{\binom{m}{r-1}, \binom{n}{r-1}}$$

into itself such that (2) holds. It remains to prove that S_{r-1} is nonsingular. By Lemma 1, $X = T^{-1}(Y)$, $C_r(X) = S_r^{-1}P_r(Y)$, and we may calculate the partial with respect to x_{ij} to obtain

(5) $$C_{r-1}(X) = R_{r-1}\, P_{r-1}(T(X)),$$

where R_{r-1} is a mapping of $M_{\binom{m}{r-1}, \binom{n}{r-1}}$ into itself. Hence combining (2) and (5) we obtain

(6) $$C_{r-1}(X) = R_{r-1}(P_{r-1}(T(X))) = R_{r-1}\, S_{r-1}(C_{r-1}(X)).$$

Now we assert that there exists a basis in $M_{\binom{m}{r-1}, \binom{n}{r-1}}$ of matrices of the form $C_{r-1}(X)$ for $X \in M_{m,n}$. For let $X = \sum_{t=1}^{r-1} E_{i_t j_t}$, and then $C_{r-1}(X) = \pm E_{\omega,\tau}$ where

$$\omega = (i_1, \cdots, i_{r-1}) \in Q_{m,r-1}, \qquad \tau = (j_1, \cdots, j_{r-1}) \in Q_{n,r-1} .$$

Hence by (6), $R_{r-1}\, S_{r-1}$ is the identity on a basis of $M_{\binom{m}{r-1}, \binom{n}{r-1}}$, and thus S_{r-1} is nonsingular.

Reducing r to $r - 1$ etc. we finally obtain

(7) $$P_2(T(X)) = S_2(C_2(X)) \qquad \text{for all } X \in M_{m,n} .$$

We observe that (7) implies

LEMMA 3. *If $\rho(X) = 1$, then $P_2(T(X)) = 0$.*

For all second order subdeterminants of A are 0, and hence $C_2(A) = 0$.

LEMMA 4. *If $P_2(Y) = 0$ and $Y \neq 0$, then*
(a) *Y has exactly one nonzero row (i.e., Y is a row matrix), or*

(b) *Y has exactly one nonzero column (i.e., Y is a column matrix), or*
(c) *by permutation of rows and columns Y may be brought to the form*

$$\alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22}, \qquad \alpha\beta\gamma\delta \neq 0, \quad \alpha\delta + \beta\gamma = 0.$$

Proof. Unless (a) or (b) holds, we can assume without loss of generality that there exist $y_{r,s}$ and $y_{u,v}$, $r < u$, $s < v$, such that $y_{r,s}\, y_{u,v} \neq 0$. Hence $y_{r,v}\, y_{u,s} \neq 0$. By permuting rows and columns these four entries may be taken to be in the top left 2-square submatrix of Y. We show that no other entry of Y is nonzero. Now if $j \geqq 3$, then

$$\alpha y_{2j} + \gamma y_{1j} = 0, \qquad \beta y_{2j} + \delta y_{1j} = 0,$$

and since $\alpha\delta - \gamma\beta \neq 0$, $y_{1j} = y_{2j} = 0$. Similarly $y_{i1} = y_{i2} = 0$, $i \geqq 3$. If $i \geqq 3, j \geqq 3$, then $\delta y_{ij} + y_{2j}\, y_{i2} = 0$, $y_{ij} = 0$.

LEMMA 5. *If $\rho(X) = 1$, then $\rho(T(X)) = 1$.*

Proof. By Lemmas 3 and 4 either $\rho(T(X)) = 1$, or $T(X)$ has the form given in (c) of Lemma 4. Note that $P_r(QYR) = P_r(Q)P_r(Y)P_r(R)$, where Q and R are permutation matrices in $M_{m,m}$ and $M_{n,n}$ respectively. Thus we may assume that

$$T(X) = \begin{bmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \gamma & \delta & 0 & \cdots & 0 \\ 0 & 0 & & & \cdot \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdot & \cdots & 0 \end{bmatrix}, \qquad \alpha\delta\beta\gamma \neq 0, \quad \alpha\delta + \beta\gamma = 0. \tag{8}$$

Let $G_1 = \alpha E_{11} + \gamma E_{21}$, $G_2 = \beta E_{12} + \delta E_{22}$, and $G_3 = \alpha E_{11} + \beta E_{12}$. Clearly if $G_4 = \gamma E_{21} + \delta E_{22}$, then $G_4 = G_1 + G_2 - G_3$. Let Z be such a matrix that $\rho(X + Z) = 1$. Then, by Lemma 4, $T(Z)$ is (a) a row matrix; (b) a column matrix; (c) a 2-square matrix. Since $P_2(T(X) + T(Z)) = 0$, we can conclude from arguments given in the proof of Lemma 4 that if $W = T(Z)$ and (a) or (b) hold, then $w_{ij} = 0$ for $i \geqq 3$, or $j \geqq 3$. Suppose then that $W = w_{11}\, E_{11} + w_{12}\, E_{12}$. Then

$$(\alpha + w_{11})\delta + (\beta + w_{12})\gamma = 0, \qquad w_{11}\, \delta + w_{12}\, \gamma = 0.$$

Since $\alpha\delta + \beta\gamma = 0$, we conclude that W is a multiple of G_3. By similar arguments we see that if W is either a row or column matrix, it is in the space $\langle G_1, G_2, G_3 \rangle$. Next assume that $T(Z) = W$ has the form given in (c) to within permutation. Since $P_2(T(X) + W) = 0$, we know from arguments identical to those in the proof of Lemma 4 that $w_{ij} = 0$, $i \geqq 3, j \geqq 3$. Since $\alpha\beta\gamma\delta \neq 0$, $w_{11}\, w_{12}\, w_{21}\, w_{22} \neq 0$, $\alpha\delta + \beta\gamma = 0$, $w_{11}\, w_{22} + w_{12}\, w_{21} = 0$, and $P_2(T(X) + W) = 0$, we conclude that

$$\alpha w_{22} + \delta w_{11} + \beta w_{21} + \gamma w_{12} = 0. \tag{9}$$

Now let $\gamma/\alpha = c$ and $w_{21}/w_{11} = d$; then

$$\gamma = c\alpha, \qquad \delta = -c\beta, \qquad w_{21} = dw_{11}, \qquad w_{22} = -dw_{12},$$

and substituting in (9) we have

$$(c - d)(\alpha w_{12} - \beta w_{11}) = 0. \tag{10}$$

Hence if $c = d$,

$$w_{11}/w_{21} = -w_{12}/w_{22} = \alpha/\gamma = -\beta/\delta,$$

and we have $[w_{11}, w_{21}] \parallel [\alpha, \gamma]$, $[w_{12}, w_{22}] \parallel [\beta, \delta]$. Thus

$$W \in \langle G_1, G_2 \rangle \subseteq \langle G_1, G_2, G_3 \rangle .$$

Next, if in (10), $\alpha w_{12} - \beta w_{11} = 0$,

$$[w_{11}, w_{12}] \parallel [\alpha, \beta] \quad \text{and} \quad [w_{22}, w_{21}] \perp [\alpha, \beta] \perp [\delta, \gamma].$$

Hence $[w_{21}, w_{22}] \parallel [\gamma, \delta]$, and we conclude in this case that

$$W \in \langle G_3, G_4 \rangle \subseteq \langle G_1, G_2, G_3 \rangle .$$

Hence we see that the range of T has at most dimension 3. Now let $X_1, \cdots, X_{n-1}$ and $Z_1, \cdots, Z_{m-1}$ be matrices such that

$$V_1 = \langle X, X_1, \cdots, X_{n-1} \rangle \quad \text{and} \quad V_2 = \langle X, Z_1, \cdots, Z_{m-1} \rangle$$

are of dimension n and m respectively, consist of rank 1 matrices, and moreover satisfy

$$\dim (V_1 + V_2) = \dim \langle X, X_1, \cdots, X_{n-1}, Z_1, \cdots, Z_{m-1} \rangle = n + m - 1.$$

Then $T(V_1 + V_2) \subseteq \langle G_1, G_2, G_3 \rangle$, so by Lemma 1 we conclude that

$$n + m - 1 \leqq 3, \qquad n + m \leqq 4.$$

But we are dealing with the case $n + m \geqq 5$, and the proof of the lemma is complete.

Lemma 5 tells us that T preserves rank 1, and we may apply a theorem in [1; p. 1218] that asserts that

$$T(X) = AXB$$

or

$$T(X) = AX'B \quad \text{in case } m = n,$$

for all X in $M_{m,n}$ where $A \in M_{m,m}$, $B \in M_{n,n}$, and $\det A \det B \neq 0$. Thus

$$P_r(AXB) = S_r\, C_r(X) \tag{11}$$

or

$$P_r(AX'B) = S_r\, C_r(X). \tag{12}$$

Now choose X_0 so that $AX_0B = \sum_{i=1,\ j=1}^{m\quad n} E_{ij}$. Then

$$\rho(X_0) = 1, \qquad C_r(X_0) = 0,$$

but

$$P_r(AX_0B) = r! \sum_{\omega \epsilon Q_{m,r}, \tau \epsilon Q_{n,r}} E_{\omega,\tau} \neq 0.$$

This contradiction completes the proof for $m + n \geqq 5$.

If $m = n = r = 2$, then $S_r = \alpha \epsilon F$, $\alpha \neq 0$, and

$$\det(\Delta T(X)) = \operatorname{per}(T(X)) = \alpha \det X.$$

Hence if $\rho(X) = 1$, then $\det X = 0$, and $\det(\Delta T(X)) = 0$. Since ΔT is nonsingular, $\rho(\Delta T(X)) = 1$. Another application of [1; p. 1218] yields the result.

COROLLARY. *There is no linear map T on $M_{n,n}$ $(n > 2)$ into itself such that for all $X \epsilon M_{n,n}$*, per $T(X) = \det X$.

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