HOMOTOPY INVARIANTS OF CHAIN COMPLEXES[1]

BY

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1. Introduction

It is well known that the homology of a free abelian chain complex constitutes a complete set of homotopy invariants. That is to say, two complexes are of the same homotopy type if and only if their homology groups are isomorphic.

If we consider instead complexes which are modules over some ring Λ, even under the restriction that they be projective as Λ-modules, the homology module remains a homotopy invariant but no longer determines the homotopy type of the complex.

We propose to supply the deficit in the following case. We shall suppose that Λ is a finite-dimensional algebra over a field K with the property that the classes of projective and injective left Λ-modules coincide. We shall call such an algebra a *weak-Frobenius* algebra; among these are included Frobenius algebras and in particular group-algebras of finite groups. The complexes we consider are finitely generated left Λ-modules whose derivations are Λ-endomorphisms. We shall for brevity consider only the ungraded case, but the graded case may be treated by the same methods.

The information required, in addition to the homology module, to determine the homotopy type of a complex X, consists of the boundary module BX and the extension class [2] of the canonical filtration $0 \subset BX \subset ZX \subset X$ of X. It is hardly surprising that these should supply sufficient information. What is perhaps more remarkable is the fact that, properly regarded, they are homotopy invariants of X.

The techniques and results of [2] are used below without specific reference, and the notation of that paper is employed without redefinition.

2. Homotopy and the ideal of projectives

Let Λ be a finite-dimensional algebra over a field K, and denote by $\mathcal{K}$ the category of Λ-projective chain complexes and by $\mathfrak{N}$ the ideal of nullhomotopic maps [1]. Then $\mathfrak{N}$ is generated by the null-complexes X^x, where X is a Λ-projective module and X^x is $X \oplus X$ with the derivation $d(x, y) = (0, x)$. If $\mathfrak{L}$ is the category of left Λ-modules and $\mathfrak{P}$ the ideal in $\mathfrak{L}$ generated by the projectives, then the boundary functor $B: \mathcal{K} \to \mathfrak{L}$ takes $\mathfrak{N}$ into $\mathfrak{P}$. Thus the composition $\mathcal{K} \to \mathfrak{L} \to \mathfrak{L}/\mathfrak{P}$ vanishes on $\mathfrak{N}$ and is a homotopy invariant. This is expressed for our present purposes in the following lemma.

LEMMA 2.1. *If $f, g: X \to Y$ are homotopic, then $Bf \equiv Bg$* mod $\mathfrak{P}$. *If f is a homotopy equivalence, then Bf is an equivalence modulo $\mathfrak{P}$.*

Received September 12, 1960.

[1] This work was supported by a National Science Foundation grant.

We shall introduce the notation $[A_n, \cdots, A_0]$ for the graded module A with homogeneous components A_q, $0 \leqq q \leqq n$ and 0 in other degrees, and $[f_n, \cdots, f_0]:[A_n, \cdots, A_0] \to [A'_n, \cdots, A'_0]$ for the map homogeneous of degree 0 having the value f_q on A_q.

Now if X is a chain complex, the *canonical filtration* of X, i.e., that given by $0 \subset BX \subset ZX \subset X$, has associated graded module $[BX, HX, BX]$. If $f:X \to Y$ is a chain map, then $[Bf, Hf, Bf]:[BX, HX, BX] \to [BY, HY, BY]$. The extension class of the canonical filtration of X, which we shall denote by $\hat{e}X$, is an element of $\mathfrak{E}[BX, HX, BX]$. In order to show the homotopy invariance of $\hat{e}X$ we shall have to supply, for a homotopy equivalence $f:X \to Y$, a way of comparing $\mathfrak{E}[BX, HX, BX]$ and $\mathfrak{E}[BY, HY, BY]$ depending only on the homotopy class of f and taking $\hat{e}X$ into $\hat{e}Y$.

We shall of course use Lemma 2.1 for this purpose. In preparation for this we begin by investigating the functorial character of $\mathfrak{E}$.

3. Functorial character of $\mathfrak{E}$

The extension set $\mathfrak{E}A$ of a module A graded by nonnegative degrees is a functor with respect to isomorphisms of such modules; if $\phi:A \approx B$ is such a map, then $\mathfrak{E}\phi:\mathfrak{E}A \approx \mathfrak{E}B$. We shall extend the definition of $\mathfrak{E}\phi$ to all maps $\phi:A \to B$ homogeneous of degree 0 and such that $\phi_q:A_q \approx B_q$ for $q > 0$. If $z \in \mathfrak{Z}A$, then $\phi z(\Omega\phi)^{-1}$ is well defined since $z \in \mathrm{Hom}_{<0}(\Omega A, A; K)$. But

$$\begin{aligned}
\square[\phi z(\Omega\phi)^{-1}] &= \phi(\square z)(\Omega^2\phi)^{-1} && \text{by [2, (2.6)]}\\
&= -\phi z(\Omega\phi)^{-1}(\Omega\phi)\Omega z(\Omega^2\phi)^{-1} && \text{by [2, (3.1)]}\\
&= -\phi z(\Omega\phi)^{-1}\Omega[\phi z(\Omega\phi)^{-1}],
\end{aligned}$$

so that $\phi z(\Omega\phi)^{-1} \in \mathfrak{Z}B$. If also $f \in \mathfrak{A}A$ so that $f = 1 + g, g \in \mathrm{Hom}_{<0}(A, A; K)$, then $1 + \phi g\phi^{-1}$ is well defined and is in $\mathfrak{A}B$. But

$$\begin{aligned}
(1 + \phi g\phi^{-1}) * \phi z(\Omega\phi)^{-1}&\\
&= [\phi z(\Omega\phi)^{-1} + \phi g z(\Omega\phi)^{-1} - \phi\square g(\Omega\phi)^{-1}]\Omega(1 + \phi g\phi^{-1})^{-1}\\
&= \phi(z + gz - \square g)(\Omega\phi)^{-1}\Omega(1 + \phi g\phi^{-1})^{-1}\\
&= \phi(f * z)(\Omega\phi)^{-1} \qquad \text{(see [2, Lemma 3.3])},
\end{aligned}$$

so that $z \to \phi z(\Omega\phi)^{-1}$ preserves orbits and defines $\mathfrak{E}\phi:\mathfrak{E}A \to \mathfrak{E}B$. Clearly $\mathfrak{E}$ is a covariant functor on maps of this type.

The meaning of $\mathfrak{E}\phi$ is elucidated by the following construction. We suppose (without loss of generality) that $\phi:A \to B$ is the identity in positive degrees, so that $A_q = B_q$, $q > 0$, and that $\boldsymbol{X} = (0 \to X \xrightarrow{\xi} X \xrightarrow{\xi''} A \to 0)$ is an extension of A. We define $\boldsymbol{A}$ to be the "constant" filtered module which we may indicate as $0 \subset A_0 \subset A_0 \cdots$, and $\boldsymbol{B}$ analogously in terms of B_0. We define also the filtration-preserving maps $\boldsymbol{\zeta}:\boldsymbol{A} \to \boldsymbol{X}$ by $\boldsymbol{\zeta} = (\zeta, \zeta'')$ where $\zeta_q = (\xi^q)_0:A_0 \to X_q$ and $\boldsymbol{\psi} = (\psi, \psi''):\boldsymbol{A} \to \boldsymbol{B}$ by $\psi_q = \phi_0$.

Finally let the direct sum $\boldsymbol{X} + \boldsymbol{B}$ have the decomposition $(\boldsymbol{i}, \boldsymbol{j}, \bar{\boldsymbol{\imath}}, \bar{\boldsymbol{\jmath}})$ where

$\boldsymbol{i} = (i, i''), \cdots, \boldsymbol{\bar{\jmath}} = (\bar{\jmath}, \bar{\jmath}'')$, and let

$$(3.1) \qquad 0 \to \boldsymbol{A} \xrightarrow{-\boldsymbol{i}\boldsymbol{\zeta} + \bar{\boldsymbol{\imath}}\boldsymbol{\psi}} \boldsymbol{X} + \boldsymbol{B} \xrightarrow{\boldsymbol{\varrho}} \boldsymbol{Y} \to 0$$

be exact. It is easy to see that $S''\boldsymbol{Y} = B$ and that $S''\boldsymbol{\varrho} = \phi\bar{\jmath}$ so that

$$\boldsymbol{Y} = (0 \to Y \xrightarrow{\eta} Y \xrightarrow{\eta''} B \to 0)$$

and $\boldsymbol{\varrho} = (\rho, \phi\bar{\jmath})$. Note that $\boldsymbol{\varrho i}: \boldsymbol{X} \to \boldsymbol{Y}$ has $S''(\boldsymbol{\varrho i}) = \phi$.

Proposition 3.2. *If ϕ, $\boldsymbol{X}$, $\boldsymbol{Y}$ are as above, then* $\mathfrak{e}\boldsymbol{Y} = (\mathfrak{S}\phi)\mathfrak{e}\boldsymbol{X}$.

For suppose that $s \in \Psi\boldsymbol{X}$ is a left-splitting of $\boldsymbol{X}$. Then

$$\rho(isj + \bar{\imath}\bar{\jmath})(-i\zeta + \bar{\imath}\psi) = \rho(-is\zeta + \bar{\imath}\psi) = 0$$

since $s\zeta = \zeta$, and there is thus a $t: Y \to Y$ such that $t\rho = \rho(isj + \bar{\imath}\bar{\jmath})$. But $t\eta\rho = t\rho(i\xi j + \bar{\imath}\bar{\jmath}) = (isj + \bar{\imath}\bar{\jmath})(i\xi j + \bar{\imath}\bar{\jmath})\rho = \rho$. Thus $t \in \Psi Y$. The conjugate left splitting t^* is given by $t_q^* = \rho_q is^*$ for $q > 0$, and $t_0^* = \rho_0 \bar{\imath}$, so that $\Box t^* = \rho i \Box s^*$. Thus

$$\begin{aligned}\delta t = \textstyle\sum_n \eta'' t^n \Box t^* &= \textstyle\sum_n \eta''\rho(is^n j + \bar{\imath}\bar{\jmath})i\Box s^* \\ &= \textstyle\sum_n \eta''(\rho i)s^n \Box s^* \\ &= \textstyle\sum_n \phi\xi'' s^n \Box s^* \\ &= \phi(\delta s)(\Omega\phi)^{-1}\end{aligned}$$

since $(\Omega\phi)^{-1}$ is the identity in positive degrees. The conclusion follows from the fact that $\delta s \in \mathfrak{e}\boldsymbol{X}$, $\delta t \in \mathfrak{e}\boldsymbol{Y}$.

Corollary 3.3. *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are filtered modules with associated graded modules $S''\boldsymbol{X}$, $S''\boldsymbol{Y}$, $\phi: S''\boldsymbol{X} \to S''\boldsymbol{Y}$ is an isomorphism in positive degrees, and* $(\mathfrak{S}\phi)\mathfrak{e}\boldsymbol{X} = \mathfrak{e}\boldsymbol{Y}$, *then there is a filtration-preserving map $\boldsymbol{\phi}: \boldsymbol{X} \to \boldsymbol{Y}$ such that $S''\boldsymbol{\phi} = \phi$.*

4. Invariance modulo $\mathfrak{P}^*$

The proof that $\hat{\mathfrak{e}}X$ is a homotopy invariant of a complex X depends centrally on the following proposition. We retain the notation of §3 and write $\mathfrak{P}^*$ for the ideal generated by injective modules.

Proposition 4.1. *Suppose that ϕ, $\phi^{\#}: A \to B$ are homogeneous of degree 0 and that $\phi_q = \phi_q^{\#}: A_q \approx B_q$, $q > 0$. If further $\phi_0 = \phi_0^{\#} \bmod \mathfrak{P}^*$, then $\mathfrak{S}\phi = \mathfrak{S}\phi^{\#}$.*

Once more we may suppose that $\phi_q = \phi_q^{\#} = 1: A_q$, $q > 0$. We use the characterization of $\mathfrak{S}\phi$ given in 3.2, and define $\boldsymbol{Y}^{\#}$ analogously to $\boldsymbol{Y}$, by the sequence

$$0 \to \boldsymbol{A} \xrightarrow{-\boldsymbol{i}\boldsymbol{\zeta} + \bar{\boldsymbol{\imath}}\boldsymbol{\psi}^{\#}} \boldsymbol{X} + \boldsymbol{B} \xrightarrow{\boldsymbol{\varrho}^{\#}} \boldsymbol{Y}^{\#} \to 0.$$

We need only show that $\boldsymbol{Y}$ and $\boldsymbol{Y}^{\#}$ are equivalent extensions of B.

Since $\phi_0^\# - \phi_0 \in \mathfrak{P}^*$, there are maps $\theta_q : X_q \to B_0$ such that $\theta_0 = \phi_0$, $\theta_{q+1}\xi_q = \theta_q$. These constitute a map $\boldsymbol{\theta}: \boldsymbol{X} \to \boldsymbol{B}$ such that $\boldsymbol{\theta\zeta} = \boldsymbol{\psi}^\# - \boldsymbol{\psi}$. Now

$$\boldsymbol{\rho}^\#(1 + \bar{\boldsymbol{i}}\boldsymbol{\theta j})(-\boldsymbol{i\zeta} + \bar{\boldsymbol{i}}\boldsymbol{\psi}) = \boldsymbol{\rho}^\#(-\boldsymbol{i\zeta} + \boldsymbol{i\psi}^\#) = 0$$

so that there is a $\boldsymbol{\mu}: \boldsymbol{Y} \to \boldsymbol{Y}^\#$ such that $\boldsymbol{\mu\rho} = \boldsymbol{\rho}^\#(1 + \bar{\boldsymbol{i}}\boldsymbol{\theta j})$; this is of course the required equivalence.

COROLLARY 4.2. *If $\phi: A \to B$, $\phi_q: A_q \approx B_q$ for $q > 0$, and ϕ_0 is an equivalence modulo $\mathfrak{P}^*$, then $\mathfrak{E}\phi$ is bijective.*

We shall also need the duals of these results. These follow from 4.1, 4.2 by the general duality theory of categories. We shall want however to make explicit the following notation. Suppose now that A, B are modules graded by degrees $\leqq n$, that $\phi: A \to B$ is homogeneous of degree 0, and that $\phi_q: A_q \approx B_q$, $q < n$. Then $\mathfrak{E}^*\phi: \mathfrak{E}B \to \mathfrak{E}A$, defined by $z \to \phi^{-1}z\Omega\phi$, is the dual of $\mathfrak{E}$.

PROPOSITION 4.1*. *If A, B are graded by degrees $\leqq n$, ϕ, $\phi^\#: A \to B$ are homogeneous of degree* 0 *and $\phi_q = \phi_q^\#: A_q \approx B_q$ for $q < n$, and if $\phi_n^\# \equiv \phi_n$* mod $\mathfrak{P}$, *then $\mathfrak{E}^*\phi^\# = \mathfrak{E}^*\phi$.*

COROLLARY 4.2*. *If further ϕ_n is an equivalence modulo $\mathfrak{P}$, then $\mathfrak{E}^*\phi$ is bijective.*

We may combine these results for finitely graded modules as follows. If $[\phi_n, \cdots, \phi_0]: [A_n, \cdots, A_0] \to [B_n, \cdots, B_0]$, $\phi_q: A_q \approx B_q$, $0 < q < n$, and ϕ_n, ϕ_0 are equivalences modulo $\mathfrak{P}$, $\mathfrak{P}^*$, set

$$\mathfrak{E}^\bullet[\phi_n, \cdots, \phi_0] = \mathfrak{E}^*[\phi_n, 1, \cdots, 1]^{-1}\mathfrak{E}[1, \phi_{n-1}, \cdots, \phi_0]: \mathfrak{E}[A_n, \cdots, A_0] \to \mathfrak{E}[B_n, \cdots, B_0].$$

Then $\mathfrak{E}^\bullet$ is a covariant functor on such maps, and $\mathfrak{E}^\bullet[\phi_n, \cdots, \phi_0]$ is always bijective. In consequence of 4.1–4.2*, and of 3.3 and its dual, the functor $\mathfrak{E}^\bullet$ has the following property.

PROPOSITION 4.3. *Suppose $S''\boldsymbol{X} = [A_n, \cdots, A_0]$, $S''\boldsymbol{Y} = [B_n, \cdots, B_0]$, and $[\phi_n, \cdots, \phi_0]: S''\boldsymbol{X} \to S''\boldsymbol{Y}$ satisfies the conditions listed above. Then there is a filtration-preserving map $\boldsymbol{\phi}: \boldsymbol{X} \to \boldsymbol{Y}$ with $S''\boldsymbol{\phi} = [\phi_n, \cdots, \phi_0]$ if and only if $\mathfrak{E}^\bullet[\phi_n, \cdots, \phi_0]\mathfrak{e}\boldsymbol{X} = \mathfrak{e}\boldsymbol{Y}$.*

5. The homotopy invariants

If X and Y are chain complexes and $f: X \to Y$, then f is certainly filtration-preserving with respect to the canonical filtrations of X and Y; taking a slight liberty with notation we write

$$S''f = [Bf, Hf, Bf]: [BX, HX, BX] \to [BY, HY, BY]$$

for the associated homogeneous map.

A filtration-preserving map, on the other hand, is not necessarily a chain map. We have indeed the following easily verified criterion.

LEMMA 5.1. *If $f:X \to Y$ preserves the canonical filtrations and*

$$S''f = [B'f, Hf, Bf]:[BX, HX, BX] \to [BY, HY, BY],$$

then f is a chain map if and only if $B'f = Bf$.

We now introduce the assumption that Λ is a weak-Frobenius algebra, so that $\mathfrak{P} = \mathfrak{P}^*$ and consider Λ-projective chain complexes. Combining 2.1, 4.1, 4.1*, 4.3, and 5.1 we have our principal result.

THEOREM 5.2. *If $f:X \to Y$ is a homotopy equivalence, then*

$$\mathfrak{E}^{\bullet}S''f:\mathfrak{E}[BX, HX, BX] \to \mathfrak{E}[BY, HY, BY]$$

is a bijection, and $(\mathfrak{E}^{\bullet}S''f)\hat{\mathfrak{e}}X = \hat{\mathfrak{e}}Y$. If $g:X \to Y$ is homotopic to f, then $\mathfrak{E}^{\bullet}S''g = \mathfrak{E}^{\bullet}S''f$. Conversely, if $h:HX \approx HY$ and $b:BX \to BY$ is an equivalence modulo $\mathfrak{P}$ such that $\mathfrak{E}^{\bullet}[b, h, b]\hat{\mathfrak{e}}X = \hat{\mathfrak{e}}Y$, then there is a homotopy equivalence $f:X \to Y$ such that $Bf = b$, $Hf = h$.

In other words the triple $(HX, BX, \hat{\mathfrak{e}}X)$, where HX is a Λ-module, BX is a module "modulo $\mathfrak{P}$", and $\hat{\mathfrak{e}}X \in \mathfrak{E}[BX, HX, BX]$, is a homotopy invariant of the complex X, and two complexes have equivalent invariants if and only if they are of the same homotopy type.

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