GROUPS ACTING ON THE 4-SPHERE[1]

BY

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Introduction

Montgomery and Zippin have shown [16, p. 260] that if G is a compact connected topological group acting effectively as a transformation group on 3-dimensional Euclidean space E^3, then G is either T^1, the circle group, or $SO(3)$, the group of all proper rotations of E^3. In each case the action of G is (topologically) equivalent to the standard action of G as a group of linear transformations. In [7] Jacoby has classified all actions of T^1 on the 3-sphere S^3 with no stationary points. Again all such actions are equivalent to linear actions. In Section 2 we show by an elementary analysis that all actions of the torus group T^2 $(= T^1 \times T^1)$ on S^3 are equivalent to linear actions. This completes the study of compact connected transformation groups on S^3. All such groups are Lie groups, and their actions are equivalent to linear actions.

In this paper we study compact connected transformation groups on the 4-sphere S^4. Our main theorem is

THEOREM B. *Let G be a compact connected topological group acting effectively on S^4 such that there is an orbit of dimension $\geqq 2$. Then G is a Lie group, and the action of G is equivalent to a linear action.*

Our method is quite straightforward. In all cases under consideration the orbit space S^4/G is either a closed 2-cell or a closed arc. The group G must be a compact connected Lie group of dimension $\leqq 6$. We use our detailed knowledge of such groups plus our knowledge of the orbit space to obtain an explicit description of all possible actions. It is easily checked that each such action is equivalent to a linear action.

Montgomery and Zippin have given an action of T^1 on S^4 which cannot be equivalent to a differentiable action [15]. Thus the assumption on the dimension of the orbit in Theorem B is not superfluous. The problem of actions of T^1 on S^4 seems to be quite difficult even when one assumes differentiabilty.

In [9] Montgomery and Samelson give a complete discussion of groups acting transitively on spheres. In this paper we consider only the non-transitive case.

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1. Preliminaries

We shall follow the notation of Montgomery and Zippin's book [16] with the following exceptions: If G acts on M, we denote the image of the point $x \in M$ under the group element g by $g.x$, and we denote the space of orbits by M/G.

If the compact connected topological group G acts effectively on an n-manifold with an orbit of dimension $\geqq (n-2)$, then G is a Lie group [2]. Henceforth G will denote a compact connected Lie group.

An orbit $G(x)$ is *principal* if there is a neighborhood N of x such that for every $y \in N$, G_y is conjugate to G_x. It is known that if G acts on the n-sphere S^n, then all principal orbits are equivalent, and the set of points which lie on principal orbits is an open set whose complement has dimension $\leqq (n-2)$ [10; 12; 14]. Thus if G acts effectively on S^n, it must act effectively on each principal orbit. If G acts effectively on the k-dimensional coset space G/H, then dimension $G \leqq k(k+1)/2$ [16, p. 243]. Thus if G acts on S^4 such that the highest dimension of any orbit is 3 (resp. 2), then dimension $G \leqq 6$ (resp. 3). From the classification of compact connected Lie groups (see e.g. [18, p. 282]) it follows that every compact connected Lie group of dimension $\leqq 6$ can be represented as a factor group G/H, where

$$G = G_1 \times \cdots \times G_m$$

is a direct product, each factor G_i is either T^1 or Q_1, the group of unit quaternions, and H is a finite subgroup of the center of G.

We say that the action of G on M is *almost effective* if the subgroup $G_0 = \{g \in G \mid g.x = x \text{ for every } x \in M\}$ is finite.

The following results describe the orbit space in the cases under consideration.

1.1. *Let G act on S^n with an $(n-1)$-dimensional orbit. Let X be the set of points on $(n-1)$-dimensional orbits, and let $Y = S^n - X$. Then S^n/G is a closed arc, and Y/G consists of precisely two points, the end points of S^n/G. Furthermore all orbits of X are equivalent.*

Proof. S^n/G is compact and simply connected [13, Corollary 2 of Theorem 2]. Thus by [17], S^n/G is an arc. The other conclusions of 1.1 also follow from [17].

1.2. *Let G act on S^n with the highest dimension of any orbit $(n-2)$. Let there also be an orbit of dimension $< (n-2)$. Let X denote the set of points on $(n-2)$-dimensional orbits, and let $Y = S^n - X$. Then S^n/G is a closed 2-cell with Y/G as simple closed curve boundary. All orbits of X are equivalent.*

Proof. This result was proved by Montgomery, Samelson, and Yang [11] for the case in which there is a stationary point. The same proof goes through if there exists any orbit of dimension $< (n-2)$. Well known actions of

T^1 on S^3 show that the assumption of an orbit of lower dimension is not superfluous.

Throughout this paper if G acts on S^n, we shall denote by X the set of points which lie on principal orbits, and by Y the complement of X.

1.3. *Let G be a semisimple group acting on S^n $(n > 2)$ with the higest dimension of any orbit $(n - 2)$. Then there exists an orbit of dimension less than $(n - 2)$.*

Proof. If all orbits are of dimension $(n - 2)$, then S^n/G is a simply connected 2-manifold, thus a 2-sphere [2; 13]. If G/H is a coset space of G, then $\pi_1(G/H)$ is finite; thus each orbit $G(x)$ has finite fundamental group. Thus for every x the cohomology group $H^1(G(x); Q)$ is trivial (Q = rationals). Let p denote the projection $S^n \to S^n/G$. By the Beagle-Vietoris mapping theorem (see [1, Exposé VII])

$$p^*:H^2(S^n/G; Q) \to H^2(S^n; Q)$$

is an injection. This gives a contradiction.

Let G act on a space M with projection $p:M \to M/G$. A cross section for the action of G is a map $s:M/G \to M$ such that $p \circ s$ is the identity map of M/G. The following result will be used to prove the equivalence of different actions.

1.4. *Let G act in two ways on a compact space M. Let $(M/G)_1$ and $(M/G)_2$ denote the corresponding orbit spaces. If $x \in M$, we denote the stability groups of x with respect to the two actions by $(G_x)_1$ and $(G_x)_2$. Assume that the two actions admit cross sections s_1 and s_2 and that there is a homeomorphism h of $(M/G)_1$ onto $(M/G)_2$ such that $(G_{s_1(y)})_1 = (G_{s_2 \circ h(y)})_2$ for every $y \in (M/G)_1$. Then the two actions are equivalent.*

Proof. Let f_i $(i = 1, 2)$ denote the map from $(M/G)_i \times G$ onto M given by $f_i(y, g) = g.s_i(y)$. There is a natural action of G on $(M/G)_i \times G$ given by $g_1.(y, g) = (y, g_1 g)$, and f_i is equivariant with respect to this action. Let M_i denote the decomposition space of $(M/G)_i \times G$ whose points are the inverses $f_i^{-1}(x)$, and let f_i' be the induced map of M_i onto M. The action of G on $(M/G)_i \times G$ induces an action of G on M_i and f_i' is an equivariant homeomorphism with respect to this action. Consider the map

$$\alpha: (M/G)_1 \times G \to (M/G)_2 \times G$$

given by $(y, g) \to (h(y), g)$. The conditions given on s_1, s_2, and h imply that α induces an equivariant homeomorphism α' of M_1 onto M_2. The map $f_2' \alpha' f_1'^{-1}$ is an equivariant homeomorphism of M with respect to the two actions.

The following result is well known to specialists but has not, to the author's knowledge, appeared explicitly in the literature.

1.5. *Let G act on a locally compact space M, and assume that all orbits of G are equivalent. Let $x \in M$, and let $N = \{y \in M \mid G_y = G_x\}$. Then N is a locally trivial principal fiber bundle (see* [3, *Exposé* VI] *for definitions) with group $N(G_x)/G_x$ ($N(G_x)$ = the normalizer of G_x). M is an associated fiber bundle with N as principal bundle and G/G_x as fiber.*

The local triviality follows from Gleason's theorem [6]. The other properties are readily checked.

As a corollary we obtain

1.6. *Let G and M be as in* 1.5, *and assume that M/G is paracompact and contractible. Then M is homeomorphic to $(M/G) \times (G/G_x)$. There is a cross section for M/G on which the isotropy group is constant.*

This follows from [3, Exposé VIII].

If G acts on S^n ($\subset E^{n+1}$) as a group of orthogonal transformations, we define the *suspension* of this action to be the linear action of G on S^{n+1} obtained as follows: Write E^{n+2} as $E^{n+1} \times E^1$, let G act by $g.(x, y) = (g.x, y)$, and restrict the action to the unit sphere S^{n+1} in E^{n+2}.

2. Actions of T^2 on S^3

We use (θ, ϕ) $(0 \leqq \theta, \phi \leqq 2\pi)$ as coordinates on T^2. Every closed 1-parameter subgroup of T^2 is of the form

$$G_{m,n} = \{(\theta, \phi) \mid m\theta + n\phi = 0 \ (2\pi)\}$$

where m and n are relatively prime integers. Let $R(\alpha)$ denote the 2×2 rotation matrix with angle α. Then any representation of T^2 with representation space E^4 is equivalent to one of the form

$$(\theta, \phi) \to R(m\theta + n\phi) \oplus R(p\theta + q\phi). \tag{2.1}$$

A necessary and sufficient condition that the representation be faithful (i.e., that the induced action be effective) is that $mq - np = \pm 1$.

Consider the action of T^2 on S^3 induced by (2.1), and assume that

$$mq - np = \pm 1.$$

Let I denote the closed unit interval which we identify with S^3/T^2, and define $h: I \to S^3$ by $h(t) = (t, 0, \sqrt{(1 - t^2)}, 0)$. Then h is a cross section, and $G_{h(0)} = G_{p,q}$, $G_{h(1)} = G_{m,n}$, and $G_{h(t)} = \{e\}$ if $0 < t < 1$.

Assume now that T^2 acts effectively on S^3. Since T^2 is abelian and the action is effective, it follows that if x lies on a principal orbit, then $G_x = \{e\}$. By 1.1 the orbit space is an arc; we identify it with I. Let $s: I \to S^3$ be a cross section. The existence of s follows from [13, Theorem 2]. By 1.1 and 1.6, X is homeomorphic to $(0, 1) \times T^2$. The space Y is the union of the two singular orbits. Each orbit of Y must be either a stationary point or a 1-sphere. From Alexander duality it follows that each orbit of Y is a 1-sphere.

It follows from [17] that the groups $G_{s(0)}$ and $G_{s(1)}$ are connected; thus each is a circle group, and we have $G_{s(0)} = G_{p,q}$, $G_{s(1)} = G_{m,n}$ for integers m, n, p, and q. Using the construction given in 1.4 we see that S^3 is homeomorphic to $T^2 \times I$ with $T^2 \times 0$ collapsed to $T^2/G_{p,q}$ and $T^2 \times 1$ collapsed to $T^2/G_{m,n}$. From consideration of the Mayer-Vietoris sequence for the triple $(s^{-1}([0, 1/2]), s^{-1}([1/2, 1]), s^{-1}(1/2))$ we see that the space described above has the homology groups of S^3 only if $mq - np = \pm 1$. It is an easy consequence of 1.4 that the action of T^2 is equivalent to a linear action of the form (2.1).

The group T^3 cannot act effectively on S^3; there would have to be a 3-dimensional orbit for such an action.

Combining this result with the much more difficult theorems of Montgomery-Zippin and Jacoby (see the Introduction) we obtain

THEOREM A. *Let G be a compact connected topological group acting effectively on S^3. Then G is either T^1, T^2, or $SO(3)$. In each case the action of G is equivalent to a linear action.*

3. Actions of T^2 on S^4

Let T^2 act effectively on S^4. Then, using the Lefschetz fixed-point theorem, it follows from [4] that there is a stationary point. By 1.2 the orbit space is a 2-cell. According to [8] there are precisely two stationary points and four classes of inequivalent orbits. Furthermore all isotropy groups are connected. Let D denote the closed unit disk in E^2, and let

$$A_+ \text{ (resp. } A_-) = \{(x_1, x_2) \in E^2 \mid x_1^2 + x_2^2 = 1, \quad x_1 > 0 \text{ (resp. } x_1 < 0)\}.$$

We denote by p the projection $S^4 \to S^4/T^2$. We may identify the orbit space S^4/T^2 with D in such a way that $(0, 1)$ and $(0, -1)$ correspond to the stationary points and $G_x = G_{m,n}$ for $p(x) \in A_+$, $G_x = G_{p,q}$ for $p(x) \in A_-$, where $G_{m,n}$ and $G_{p,q}$ are circle subgroups of T^2.

We wish to show that there are local cross sections at the points of A_+ and A_-. Let $y' = p(y) \in A_+$, and let K be a slice at y (see [13] for definitions). We may assume that $p(K)$ is a 2-cell which meets A_+ in an arc. The stability group G_y acts freely on $K \cap X$, and the orbit space $(K \cap X)/G_y = p(K \cap X)$ is contractible. Thus by 1.6 there is a cross section for $p(K \cap X)$ in $K \cap X$. Each point of $K \cap Y$ is stationary under G_y. Thus the cross section can be extended to a cross section for $p(K)$ in K.

3.1. *There exists a global cross section for S^4/T^2.*

Proof. We follow the method of [11]. Write $D - \{(0, 1), (0, -1)\}$ as the union of a countable number of 2-cells which intersect nicely with each other and with A_+ and A_-. The method given above shows that we can construct cross sections for 2-cells which intersect A_+ or A_-. The global cross section is constructed step by step. One checks that there is no trouble in "patching together" cross sections over neighboring 2-cells. Since (0, 1) and (0, −1) are stationary points, the cross section defined on $D - \{(0, 1), (0, -1)\}$ extends to a cross section of D.

Let $\gamma \subset D$ be the line segment $\{(t, 0) \mid -1 \leqq t \leqq 1\}$. Using the cross section for $D - \{(0, 1), (0, -1)\}$, we see that S^4 − (two points) is homeomorphic to the product of $p^{-1}(\gamma)$ with the open interval. Thus $p^{-1}(\gamma)$ has the homology groups of S^3. But $p^{-1}(\gamma)$ is homeomorphic to $T^2 \times I$ with $T^2 \times 0$ collapsed to $T^2/G_{p,q}$ and $T^2 \times 1$ collapsed to $T^2/G_{m,n}$. It was shown in Section 2 that a necessary condition for this space to have the homology groups of S^3 is that $mq - np = \pm 1$. It now follows from 1.4 that the action of T^2 on S^4 is equivalent to the suspension of a linear action of T^2 on S^3.

4. Some properties of $SO(3)$

The group $SO(3)$ has the following conjugacy classes of subgroups (see [19] for a discussion of the finite subgroups of $SO(3)$): the groups S_p of all rotations about the axis determined by a point $p \in S^2$; the groups N_p of all $g \in SO(3)$ such that $g.p = \pm p$ for a point $p \in S^2$; the cyclic groups Z_n; the dihedral groups D_n of order $2n$; the groups H_T, H_C, and H_I, of all rotational symmetries of the tetrahedron, cube, and icosahedron respectively.

The following property of the dihedral groups will be used.

4.1. *If $n > 2$, then D_n is contained in precisely one group of the form N_p. The dihedral group D_2 is contained in three distinct groups of the form N_p.*

The coset spaces $SO(3)/S_p$ and $SO(3)/N_p$ are homeomorphic to the 2-sphere S^2 and the projective plane P^2 respectively. If V is a finite subgroup of $SO(3)$, then $SO(3)/V$ is an orientable 3-manifold with $H_2(SO(3)/V) = 0$. Using the covering map $\pi: Q_1 \to SO(3)$ we can calculate the fundamental group $(= \pi^{-1}(V))$ of $SO(3)/V$ and thus the first homology group. We list the results (see [5] for a typical computation):

$$\begin{aligned} &H_1(SO(3)/Z_n) = Z_{2n}, && H_1(SO(3)/D_{2n}) = Z_2 \oplus Z_2, \\ (4.2)\quad &H_1(SO(3)/D_{2n+1}) = Z_4, && H_1(SO(3)/H_T) = Z_3, \\ &H_1(SO(3)/H_C) = Z_2, && H_1(SO(3)/H_I) = 0. \end{aligned}$$

The representation of weight two of $SO(3)$, which defines an action of $SO(3)$ on S^4, can be described as follows: The representation space $\mathfrak{M}$ is the space of all 3×3 symmetric matrices of trace zero, and $SO(3)$ acts on $\mathfrak{M}$ by $g(m) = gmg^{-1}$ (conjugation of the matrix m by the matrix g). Using trace $(m_1 m_2)$ as inner product in $\mathfrak{M}$ we identify S^4 with the set of all m of norm 1. This inner product is invariant under $SO(3)$.

We now construct a cross section for this action. Let S^1 be the set of all diagonal matrices of S^4. Since every symmetric matrix can be orthogonally diagonalized, it follows that every orbit of $SO(3)$ on S^4 intersects S^1. If $x \in S^1$ has distinct eigenvalues, then $G_x = D_2$, the group of all diagonal matrices of $SO(3)$. If x has two equal eigenvalues, then G_x is one of the three groups N_{e_1}, N_{e_2}, and N_{e_3} (e_1, e_2, and e_3 are the standard basis vectors of E^3). There are precisely six points on S^1 with two equal eigenvalues. One checks that there are precisely six arcs on S^1 which cross-section the orbit space $S^4/SO(3)$. We obtain

4.3. *Let $SO(3)$ act on S^4 by the representation of weight two. Let $i, j \in \{1, 2, 3\}$, $i \neq j$. There is a cross-sectioning arc cd for the orbit space $S^4/SO(3)$ such that $G_c = N_{e_i}$, $G_d = N_{e_j}$, and $G_x = D_2$ on $(cd - (c \cup d))$.*

5. Actions of $SO(3)$ on S^4 with 3-dimensional orbits

Let $SO(3)$ act on S^4 with at least one 3-dimensional orbit. By 1.1, S^4/G is a closed arc with Y/G as end points, and all orbits of X are equivalent. It follows from 1.6 that X is homeomorphic to the product of $G(x)$, a typical orbit of X, with an open interval. Each orbit of Y must be homeomorphic to either S^2, P^2, or a point.

5.1. *Y is homeomorphic to $P^2 \cup P^2$.*

Proof. If there is a stationary point p, then letting $p = p_\infty$ we get an induced action of $SO(3)$ on E^4. But then, by [16, p. 252], the other exceptional orbit must be a stationary point. Suppose there are stationary points. Then X is homeomorphic to $E^4 - p_\infty$ which is simply connected; thus $G(x)$ must be simply connected. But $SO(3)$ has no simply connected 3-dimensional coset spaces. It follows that there are no stationary points. From the Alexander duality theorem we see that $H^2(Y) \approx H_1(G(x))$ which is finite. Thus Y cannot have any S^2 orbits. It follows that both orbits of Y are homeomorphic to projective planes.

5.2. *There is a cross-sectioning arc cd for $S^4/SO(3)$ such that G_x is constant on $(cd - (c \cup d))$.*

Proof. By 1.6 there is a cross section s for $X/SO(3)$ on which G_x is constant. By Alexander duality we have

$$H_1(X) \approx H_1(SO(3)/G_x) \approx H^2(Y) \approx Z_2 \oplus Z_2 .$$

From 4.2 we see that G_x is a dihedral group D_{2n}.

Let f be a mapping of the open unit interval I^0 into a complete metric space M. We say that a point x of M belongs to the *cluster set* of f at 0 (resp. 1) if there is a sequence $t_i \in I^0$ such that $t_i \to 0$ (resp. 1) and $f(t_i) \to x$. The cluster set of f at each end point is connected. The map f can be extended to a continuous map of I into M if the cluster set at each end point reduces to a point.

If $SO(3)$ acts transitively on P^2 there is a one-one correspondence between stability groups G_z and points $z \in P^2$. By 4.1 if $n > 1$, there is only one subgroup of the form N_p which contains D_{2n}. There are exactly three subgroups of this form which contain D_2.

For any point y in the cluster set of s at either end point we must have $G_y \subset D_{2n}$, and G_y must be of the form N_p. Thus the cluster set of s at each end point is finite and connected, hence a point. Thus s can be extended to a cross section for $S^4/SO(3)$. This cross section will be an arc which we denote by cd.

5.3. $G_c \neq G_d$.

Proof. Let M be the space $I \times (SO(3)/D_{2n})$ with $0 \times (SO(3)/D_{2n})$ collapsed to $SO(3)/G_c$ and $1 \times (SO(3)/D_{2n})$ collapsed to $SO(3)/G_d$. As in 1.4 we see that M is homeomorphic to S^4. We show that if $G_c = G_d$, then $H_2(M) \neq 0$, which gives a contradiction. Let p denote the natural projection of M onto I. Let $A = p^{-1}([0, 1/2])$ and $B = p^{-1}([1/2, 1])$. Consider the Mayer-Vietoris sequence for (M, A, B):

$$H_2(M) \to H_1(A \cap B) \xrightarrow{i_{1*} \oplus i_{2*}} H_1(A) \oplus H_1(B).$$

Here i_1 is the injection $(A \cap B) \subset A$, and i_2 is the injection $(A \cap B) \subset B$. Now

$$H_1(A \cap B) \approx H_1(SO(3)/D_{2n}) \approx Z_2 \oplus Z_2,$$

and

$$H_1(A) \approx H_1(B) \approx H_1(P^2) \approx Z_2.$$

Thus i_{1*} and i_{2*} have nontrivial kernels. If $G_c = G_d$, then clearly kernel i_{1*} = kernel i_{2*}; this implies that $i_{1*} \oplus i_{2*}$ has a nontrivial kernel and hence that $H_2(M) \neq 0$.

5.4. *Let $SO(3)$ act as a transformation group on S^4 such that there exists a 3-dimensional orbit. Then the action is equivalent to the action of $SO(3)$ on S^4 induced by the representation of weight two.*

Proof. Since $D_{2n} \subset G_c$, $D_{2n} \subset G_d$, and $G_c \neq G_d$, it follows from 4.1 that $n = 1$, hence that $G_x = D_2$ on $(cd - (c \cup d))$. By translating our cross section we may assume that on $(cd - (c \cup d))$, G_x is the group of diagonal matrices of $SO(3)$. (Remember that D_2 is defined only to within conjugacy.) Hence G_c and G_d must each be one of the groups N_{e_1}, N_{e_2}, or N_{e_3}. Recalling 4.2 we apply 1.4 to prove 5.4.

6. Actions of $SO(3)$ on S^4 with 2-dimensional orbits

Let $SO(3)$ act on the 4-sphere with the highest dimension of any orbit two. Then by 1.2, $S^4/SO(3)$ is a closed 2-cell with $Y/SO(3)$ as simple closed curve boundary, and all orbits of X are equivalent. It is easily seen that the orbits of X must be orientable; hence each orbit of X is a 2-sphere. All orbits of Y must be stationary points since $SO(3)$ has no 1-dimensional coset spaces. Thus Y is a simple closed curve.

6.1. *There exists a cross section s for $S^4/SO(3)$ such that G_x is constant on $s(X/SO(3))$.*

Proof. By 1.6 there exists a cross section for $X/SO(3)$ on which G_x is constant. Since every orbit of Y is a stationary point, this cross section can be extended to a cross section for $S^4/SO(3)$.

We now apply 1.4 to obtain

6.2. *Let $SO(3)$ act as a transformation group on S^4 with the highest dimension of any orbit two. Then the action is equivalent to the double suspension of the standard action of $SO(3)$ on S^2.*

7. Actions of Q_1 on S^4

Let $\pi: Q_1 \to SO(3)$ be the standard two-to-one covering map, and let K = kernel π. Our knowledge of the subgroups of $SO(3)$ enables us to classify all subgroups of Q_1. The following result is easily obtained.

7.1. *The only (closed) subgroups of Q_1 which do not contain K are finite cyclic groups of odd order.*

As a consequence we have

7.2. *Let Q_1 act effectively on a space M. Then there must exist $x \in M$ such that $G_x = e$ or $G_x = Z_n$ for odd n. In particular there must be a 3-dimensional orbit.*

Let Q_1 act effectively on S^4. Let $x \in X$. Then by 7.2, $G_x = \{e\}$ or $G_x = Z_{2n+1}$. Thus, as in Section 5,

$$H_1(X) \approx H_1(Q_1/G_x) \approx Z_{2n+1} \text{ or } 0.$$

By Alexander duality $H^2(Y)$ is isomorphic to Z_{2n+1} or 0. But each orbit of Y is homeomorphic to S^2, P^2, or a point. It follows that each orbit of Y is a stationary point, and hence that $G_x = e$. By 1.6 there is a cross section for X/Q_1. Since each orbit of Y is a stationary point, the cross section can be extended to a cross section for S^4/Q_1.

The group Q_1 acts transitively on S^3 $(= Q_1)$ by left multiplication such that each isotropy group is trivial. This is a linear action. Using 1.4 we obtain

7.3. *Let Q_1 act effectively on S^4. Then the action is equivalent to the suspension of the action of Q_1 on S^3 given by left multiplication.*

8. Actions of $SO(3) \times T^1$, $Q_1 \times T^1$, and $Q_1 \times Q_1$

Let $SO(3) \times T^1$ act effectively on S^4. We consider $SO(3)$ and T^1 as subgroups of $SO(3) \times T^1$ and denote by π_1 the projection $SO(3) \times T^1 \to SO(3)$.

8.1. *Let H be a 1-dimensional subgroup of $SO(3) \times T^1$ whose identity component H^* is not included in either $SO(3)$ or T^1. Then H is included in a maximal torus $S_p \times T^1$ of $SO(3) \times T^1$.*

Proof. The group H^* is abelian and connected and is thus included in a maximal torus of $SO(3) \times T^1$. Every such maximal torus is of the form $S_p \times T^1$. The image $\pi_1(H)$ must be included in N_p, otherwise it would be 3-dimensional. Hence $H \subset N_p \times T^1$. Using the multiplication table for $N_p \times T^1$, one sees that H must be included in $S_p \times T^1$.

8.2. *The action of $SO(3)$ on S^4 induced by the action of $SO(3) \times T^1$ is not equivalent to the action of $SO(3)$ on S^4 induced by the representation of weight two.*

Proof. Assume the actions are equivalent. Let x lie on a principal orbit. Then $SO(3)$ acts transitively on the $(SO(3) \times T^1)$ orbit of x. The isotropy group G_x $(= (SO(3) \times T^1)_x)$ is 1-dimensional and $G_x \cap SO(3) = D_2$, $G_x \cap T^1 = \{e\}$. (Any element in $G_x \cap T^1$ would act trivially on every principal orbit.) It follows from 8.1 that G_x is included in a maximal torus $S_p \times T^1$. But this gives a contradiction since $G_x \cap SO(3) = D_2$, which is not included in S_p.

By 5.4, 6.2, and 8.2 we may assume that the induced action of $SO(3)$ on S^4 is equivalent to the double suspension of the transitive action of $SO(3)$ on S^2. The orbit space for this action is a closed 2-cell; interior points correspond to 2-sphere orbits, and boundary points to stationary points. The action of $SO(3) \times T^1$ on S^4 induces an action of T^1 on the orbit space $S^4/SO(3)$. It is an easy consequence of [17] that every action of T^1 on a 2-cell is equivalent to a linear action. Furthermore the action of T^1 on the 2-cell $S^4/SO(3)$ must be effective. For if $g \in T^1$ $(g \neq e)$ left each S^2 orbit fixed, then g would be orientation-reversing on each S^2 orbit, and hence orientation-reversing on S^4, which is impossible.

It follows that each principal orbit is homeomorphic to $S^2 \times S^1$ and that the action of $SO(3) \times T^1$ on the orbit is given by letting $SO(3)$ act on the first factor and T^1 act on the second factor. The two singular orbits are a 2-sphere and a 1-sphere; T^1 acts trivially on the first, and $SO(3)$ acts trivially on the second.

Let G_1 and G_2 act orthogonally on the spheres S^n and S^m respectively. We define the *join* of these two actions as the action of $G_1 \times G_2$ on S^{n+m+1} obtained as follows: Write E^{n+m+2} as $E^{n+1} \times E^{m+1}$, let $G_1 \times G_2$ act by

$$(g_1, g_2).(x_1, x_2) = (g_1.x_1, g_2.x_2),$$

and restrict the action to the unit sphere S^{n+m+1} in E^{n+m+2}.

8.3. *The action of $SO(3) \times T^1$ on S^4 is equivalent to the join of the standard actions of $SO(3)$ on S^2 and T^1 on S^1.*

Proof. Using cross sections for the action of T^1 on the 2-cell $S^4/SO(3)$ and the action of $SO(3)$ on S^4 we obtain a cross section for the action of $SO(3) \times T^1$ on S^4. The cross sections may be chosen such that 1.4 applies to prove 8.3.

Let $Q_1 \times T^1$ act almost effectively on S^4. We may assume that Q_1 acts effectively, otherwise we would have an induced action of $SO(3) \times T^1$ on S^4. It follows from 7.3 that the principal orbits are 3-spheres and that the two singular orbits are stationary points. According to [9] the action of $Q_1 \times T^1$ on each S^3 orbit must be linear. The unitary group $U(2)$ is isomorphic to $Q_1 \times T^1$ modulo a subgroup of order two. The group $U(2)$ acts transitively on S^3 (considered as the unit sphere in a 2-dimensional complex vector space), and the only linear, transitive, almost effective action of $Q_1 \times T^1$ on S^3 is that induced by the action of $U(2)$. It follows that the action of $Q_1 \times T^1$ on S^4 is equivalent to the suspension of the linear action of $U(2)$ on S^3.

We show that $SO(3) \times SO(3)$ cannot act effectively on S^4. Assume such an action were given. Since $SO(3) \times T^1 \subset SO(3) \times SO(3)$, it follows that the action of the first factor is not equivalent to that induced by the representation of weight two. Thus $S^4/SO(3)$ is a 2-cell, and we get an induced action of $SO(3)$ on this 2-cell. This action must be trivial, since $SO(3)$ has no 1-dimensional coset spaces. Thus each principal orbit is a 2-sphere. This gives a contradiction since dimension $(SO(3) \times SO(3)) > 3$.

Let $Q_1 \times Q_1$ act almost effectively on S^4. Then at least one factor must act effectively or we would have an induced action of $SO(3) \times SO(3)$ on S^4. It follows from 7.3 that the principal orbits are 3-spheres and that the two singular orbits are stationary points. The rotation group $SO(4)$ is a quotient of $Q_1 \times Q_1$ by a subgroup of order two. It follows from [9, p. 461] that the action of $Q_1 \times Q_1$ on S^3 induces an action of $SO(4)$ on S^3 which is equivalent to the standard action of $SO(4)$ on S^3. It is immediate that the action given is equivalent to the suspension of the standard action of $SO(4)$ on S^3.

9. Elimination of the remaining cases

9.1. *T^3 cannot act effectively on S^4.*

Proof. Assume we are given such an action. Since T^3 is abelian, each principal orbit is homeomorphic to T^3. By [4] there is a stationary point. Hence there are precisely two stationary points. From 1.1 and 1.6 it follows that $S^4 - \{\text{two points}\}$ is homeomorphic to the product of an open interval with T^3, which is obviously false.

All of the remaining compact connected Lie groups of dimension $\leqq 6$ contain subgroups isomorphic to T^3 and hence cannot act effectively on S^4.

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