TORSION-FREE AND MIXED ABELIAN GROUPS

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I. Introduction

In the initial sections of this paper we classify arbitrary torsion-free abelian groups in a manner similar to that used to classify the subgroups of the rationals. An existence theorem is presented to complete the picture, and these results are applied to give new examples of indecomposable groups of any finite rank.

The remaining sections are concerned with certain countable mixed groups of torsion-free rank 1; they are essentially classified by the invariants which came up in Kaplansky and Mackey's solution [6] of the analogous problem for modules over complete discrete valuation rings. The proof of the present classification theorem depends heavily on the author's adaptation [10] of their work to modules over (not necessarily complete) discrete valuation rings. Again an existence theorem shows the set of invariants is complete. Although these invariants are clumsy, they are used to solve cancellation, square-root, and isomorphic refinement problems.

II. Basic definitions and notation

All groups considered are abelian and are written additively. If G is a group, the set of all elements in G of finite order forms a subgroup T, the *torsion subgroup* of G. G is *torsion* if $T = G$; G is *torsion-free* if $T = 0$. An arbitrary group may contain elements of infinite order and elements of finite order. Since most work on abelian groups has been done on torsion groups and torsion-free groups, a general group is called *mixed* to distinguish it from these particular cases.

Let p be a prime integer, x an element of G. x is *divisible by* p^n in case there is a $y \in G$ such that $p^n y = x$. x has *p-height* n, denoted $h_p(x) = n$, if n is maximal with the property that x is divisible by p^n; if there is no such n, x has *infinite p-height*.

Let I denote the rational integers, Q the rational numbers. If G is a group, $Q \otimes_I G$ is a vector space over Q. The *rank* of G is the dimension of $Q \otimes G$. (There are other notions of rank in abelian group theory; the one defined above is sometimes called the "torsion-free rank" of G. No confusion should arise from our abbreviation since no other kind of rank will be used.) Closely allied to the concept of rank is that of independence: A set $\{x_i\}$ of elements of G is *independent* in case $\sum m_i x_i = 0$, $m_i \in I$, implies $m_i = 0$ for all i. In particular, each element in an independent set must have infinite order. A *basis* is a maximal independent subset; all bases have the same cardinality which is equal to the rank.

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If p is a prime in I, we may form I_p, the ring of quotients of I with respect to (p). This is the ring of *p-adic fractions* consisting of all elements of Q whose denominators are prime to p. Observe that p is the unique (up to associates) prime in the ring I_p. We must also consider (unitary) modules over I_p. The definitions given above for groups may be applied to I_p-modules, with the following simplification. Since I_p has only one prime, we speak of *the height* of an element x, $h(x)$, instead of the various p-heights.

For a complete discussion of these ideas, as well as the concept of *Ulm invariants*, the reader is referred to [4, 5, 8].

III. Torsion-free groups

THEOREM 1. *Let G and G' be torsion-free abelian groups of finite rank s. Then G is isomorphic to G' if and only if there are bases $x_1, \cdots, x_s$ in G, $x_1', \cdots, x_s'$ in G' such that $h_p(\sum m_i x_i) = h_p(\sum m_i x_i')$ for all primes p and all integers m_i.*

Proof. We define an isomorphism f as follows. First, set $f(x_i) = x_i'$ for all i. Suppose $y \neq 0$, $y \in G$. Then, since the x's form a basis, $my = \sum m_i x_i$ for some nonzero integer m and integers m_i. We may assume that $m = p^k$ for some prime p, since G modulo the subgroup generated by the x's is torsion, and is thus the direct sum of primary groups. Hence $h_p(\sum m_i x_i) \geqq k$; therefore, there is a unique (G' is torsion-free) element $y' \in G'$ such that $p^k y' = \sum m_i x_i'$. Set $f(y) = y'$. It is a simple matter to verify that f is a well-defined isomorphism.

Let P denote the set of primes in I, and let I^s denote the cartesian product of s copies of I. An ordered basis $x_1, \cdots, x_s$ induces a function $g: P \times I^s \rightarrow$ nonnegative integers and ∞ by $g(p, m_1, \cdots, m_s) = h_p(\sum m_i x_i)$. Theorem 1 is unsatisfactory as stated since the "invariant" it mentions depends on the choice of ordered basis of G. Suppose $y_1, \cdots, y_s$ is another basis of G, which induces a function f. There is a rational nonsingular $s \times s$ matrix $B = (b_{ij})$ such that $y_i = \sum b_{ij} x_j$. If n is the product of the denominators of the b_{ij}, then $ny_i = \sum (nb_{ij})x_j$, where now all coefficients are integers. Hence

$$\begin{aligned} f(p, nm_1, \cdots, nm_s) &= g(p, \textstyle\sum m_i nb_{i1}, \cdots, \sum m_i nb_{is}) \\ &= g(p, [m_1, \cdots, m_s]nB), \end{aligned}$$

where we consider $[m_1, \cdots, m_s]$ as a row matrix. It is easy to check that this relation is an equivalence relation, and that any two ordered bases of G determine the same equivalence class of functions. Thus this equivalence class is an invariant of G.

THEOREM 1'. *Let G and G' be torsion-free abelian groups of finite rank s. Then G is isomorphic to G' if and only if they have the same equivalence class of height functions.*

Several comments may be made at this point. Theorem 1 may be proved for torsion-free modules of any rank over principal ideal domains, the proof being identical to the one given above. Also, if the groups have rank 1, Theorem 1′ is precisely the usual classification of the additive subgroups of the rationals by means of "characteristics" or "Steinitz numbers."

There is yet another formulation of this theorem. Instead of considering ordered bases, one may consider the free subgroups they generate. The height functions now become functions of only two variables, $f(p, a)$, where p is a prime, and a varies over a free subgroup of rank s; indeed, this is the approach of Campbell [1]. (I remark that Campbell's paper and my own were done independently.) It is now obvious that the invariants really tell how the free subgroups of rank s are situated within G. However, I prefer the functions of several variables (i.e., ordered bases instead of free subgroups) since the equivalence relation is less cumbersome, thus permitting an application (Theorem 3).

In order to prove the existence theorem, we first consider the simpler case of modules over I_p.

Let $f: I_p^s \to$ nonnegative integers and ∞. Abbreviate the argument $(r_1, \cdots, r_s)$ of f by r.

LEMMA. *Suppose the function f above satisfies:*

(i) $f(pr) = f(r) + 1$, *where* $\infty + 1 = \infty$;

(ii) $f(ur) = f(r)$, *where u is a unit in I_p*;

(iii) $f(r) > f(r')$ *implies* $f(r + r') = f(r')$.

Then there exists a reduced torsion-free I_p-module M of rank s containing an ordered basis $x_1, \cdots, x_s$ such that $h(\sum r_i x_i) = f(r_1, \cdots, r_s)$, for all $r_i \in I_p$.

Proof. Let V be an s-dimensional vector space over Q with ordered basis $x_1, \cdots, x_s$. Let $M = [y \in V \mid y = (1/p^k) \sum r_i x_i$, where $k \leqq f(r_1, \cdots, r_s)]$. Note that M is the *set* consisting of precisely these elements, and is not, a priori, the I_p-module they generate. In particular, $(1/p^k) \sum r_i x_i \notin M$ if $k > f(r_1, \cdots, r_s)$.

We shall now prove that M is an I_p-module. Suppose y and y' are nonzero elements of M. Then $y = (1/p^k) \sum r_i x_i$ and $y' = (1/p^{k'}) \sum r_i' x_i$, where $k \leqq f(r)$, $k' \leqq f(r')$, and $k \leqq k'$. Also $y + y' = (1/p^k) \sum (r_i + p^{k-k'} r_i') x_i$.

Case 1. $f(r) \leqq f(p^{k-k'} r')$. Then $k \leqq f(r) \leqq f(r + p^{k-k'} r')$, by (iii). Hence $y + y' \in M$.

Case 2. $f(r) > f(p^{k-k'} r')$. By (iii), $f(r + p^{k-k'} r') = f(p^{k-k'} r')$. Suppose $k > f(p^{k-k'} r') = f(r') + k - k'$, by (i). Then $0 > f(r') - k'$, contradicting $k' \leqq f(r')$. Again $k \leqq f(r + p^{k-k'} r')$ and $y + y' \in M$.

Let $y \in M$, and let $r' \neq 0$ be in I_p. Then $r' = up^n$, where u is a unit in I_p. Hence $r'y = (1/p^k) \sum r_i u p^n x_i$. But $k \leqq f(r) = f(ur) \leqq f(up^n r)$ by (ii) and (i). Hence $r'y \in M$.

Thus M is an I_p-module. By our earlier remark, $h(\sum r_i x_i) = f(r_1, \cdots, r_s)$.

THEOREM 2. *Let $f\colon P \times I^s \to$ nonnegative integers and ∞ satisfy the following conditions*:

(i) *$f(p, pm) = f(p, m) + 1$, where $\infty + 1 = \infty$;*
(ii) *$f(p, nm) = f(p, m)$ where $(n, p) = 1$;*
(iii) *$f(p, m) > f(p, m')$ implies $f(p, m + m') = f(p, m')$.*
(m denotes an s-tuple of integers.) Then there exists a torsion-free abelian group G of rank s containing a basis $x_1, \cdots, x_s$ with $h_p(\sum m_i x_i) = f(p, m_1, \cdots, m_s)$ for all p and m_i.

Proof. Let V be an s-dimensional vector space over Q with ordered basis $x_1, \cdots, x_s$. For each prime p, define a new function $g_p: I_p^s \rightarrow$ nonnegative integers and ∞ as follows. Let $r_1, \cdots, r_s$ be elements in I_p, and let n be the product of their denominators. Set $g_p(r_1, \cdots, r_s) = f(p, nr_1, \cdots, nr_s)$. By the lemma, let G_p be the I_p-module with basis $x_1, \cdots, x_s$ having g_p as its height function. Repeat this construction for all p using the same x's in the same order. Set $G = \bigcap_p G_p$. G is a torsion-free group of rank s containing $x_1, \cdots, x_s$. Further, in G_p, considered as a group, $h_q(\sum m_i x_i) = \infty$ for all primes $q \neq p$, while $h_p(\sum m_i x_i) = f(p, m)$ by (ii), where the m_i are integers. Hence the p-height of $\sum m_i x_i$ in G must be precisely $f(p, m)$.

A group is *decomposable* in case it can be written as a direct sum of two of its proper subgroups. Theorems 1 and 2 immediately give the following result.

COROLLARY. *A torsion-free group G of finite rank is decomposable if and only if G contains a basis $x_1, \cdots, x_s, y_1, \cdots, y_t$ such that*

$$h_p(\sum m_i x_i + \sum m_j y_j) = \min \{h_p(\sum m_i x_i), h_p(\sum m_j y_j)\}$$

for all primes p and all integers m_i and m_j.

These results are now used to produce new examples of indecomposable groups of any finite rank.

THEOREM 3. *There exist indecomposable groups of any finite rank.*

Proof. For simplicity of notation, we shall only give an example of an indecomposable group of rank 2, but the generalization to any finite rank is straightforward from the construction. The basic idea is to define a height function f such that given any basis and a partition of it into ordered disjoint subsets, there will be some linear combination of those elements whose height will not obey the "min rule" for some prime p.

Let $p \leftrightarrow (b_{ij})$ be a one-one correspondence between the primes p and the 2×2 nonsingular matrices (b_{ij}) over Q. Let n_p be the product of the denominators of the b_{ij}, and let k_p be the maximal power of p dividing $n_p \sum_i b_{ij}$. Define a function f: $P \times I \times I \rightarrow$ nonnegative integers and ∞ as follows. Given p, set

$$f(p, n_p b_{11}, n_p b_{12}) = f(p, n_p b_{21}, n_p b_{22}) = k_p,$$

and set

$$f(p, \sum n_p b_{i1}, \sum n_p b_{i2}) = k_p + 1.$$

For fixed p, f may be extended over all pairs of integers so that (i), (ii), and (iii) of the existence theorem are satisfied. (Generators and relations seem to offer the quickest way to see this.) It should be emphasized that the b's in the above construction are the entries of the matrix corresponding to the prime p. Since we have defined f at each prime p, we have constructed a function satisfying the conditions of Theorem 2. Let G be the group determined by f, with elements x_1 and x_2 such that $h_p(\sum m_i x_i) = f(p, m_1, m_2)$. Suppose G is decomposable. By the corollary, there is a basis x_1' and x_2' of G such that $h_p(\sum m_i x_i') = \min_i h_p(m_i x_i)$ for all primes p and all integers m_i. But there exists a matrix (b_{ij}) such that

$$f'(p, nm_1, nm_2) = f(p, \sum nm_i b_{i1}, \sum nm_i b_{i2})$$

(where f' is the height function determined by x_1' and x_2', and n is the product of the denominators of the b_{ij}). In particular, $f'(p, n, 0) = f'(p, 0, n) = k_p$ while $f'(p, n, n) = k_p + 1$, for the prime p corresponding to (b_{ij}). This means $h_p(nx_1') = h_p(nx_2')$, but $h_p(nx_1' + nx_2') > h_p(nx_i')$, and so the min rule is not obeyed. This contradiction shows that G is indecomposable.

IV. A structure theorem for mixed groups

A *KM group* is a countable abelian group of rank 1 such that for any prime p, $h_p(x)$ is infinite if and only if x has finite order prime to p. In a moment we shall characterize a KM group in terms of certain modules associated to it.

LEMMA 1. *Let G be a group, $x \in G$, and let p be a prime with $(m, p) = 1$, m an integer. Then $h_p(x) = h_p(mx)$.*

Proof. This is immediate from the existence of integers a and b such that $amx + bpx = x$.

LEMMA 2. *Let G be a group, $x \in G$, and p a prime. If $h(1 \otimes x) \geqq k$ in $I_p \otimes G$, then $h_p(x) \geqq k$ in G.*

Proof. $1 \otimes x \in p^k(I_p \otimes G) = I_p \otimes p^k G$. Hence $1 \otimes x = \sum r_i \otimes g_i$, $g_i \in p^k G$. Let m be the product of the denominators of the r_i. Then $(m, p) = 1$ and $1 \otimes mx = 1 \otimes \sum (mr_i)g_i$. But $1 \otimes t = 0$ implies $nt = 0$, where $(n, p) = 1$. Hence $nmx = \sum (nmr_i)g_i$ which implies $h_p(nmx) \geqq k$. But $(nm, p) = 1$, and so our result follows from Lemma 1.

COROLLARY. *Let G be a countable group of rank 1. G is a KM group if and only if each I_p-module $I_p \otimes G$ has no elements of infinite height, for all primes p.*

LEMMA 3. *Let M be an I_p-module of rank 1 with no elements of infinite height. Let S be a finitely generated submodule, $z \notin S$. Then the coset $z + S$ contains an element of maximal height.*

Proof. This is Lemma 3.8 in [10].[1]

[1] Lemma 3 is false if rank $M > 1$ [10, Example 5.9].

LEMMA 4. *Let S be a finitely generated subgroup of the KM group G, $x \notin S$ and $px \in S$ for some prime p. Then the coset $x + S$ contains an element of maximal p-height.*

Proof. First we show that $1 \otimes x \notin I_p \otimes S$. Let x^* denote $x + S$.

$$x^* \in \ker\,(G/S \to I_p \otimes G/S)$$

if and only if x^* has finite order prime to p. However x^* has order p, and so it is not in this kernel. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & G/S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_p \otimes S & \longrightarrow & I_p \otimes G & \xrightarrow{f} & I_p \otimes (G/S) & \longrightarrow & 0, \end{array}$$

where the downward maps are $y \to 1 \otimes y$. Then $f(1 \otimes x) = 1 \otimes x^*$ which we have just seen is nonzero. But $f(1 \otimes x) = 1 \otimes x + I_p \otimes S$. Therefore $1 \otimes x \notin I_p \otimes S$. By the above Corollary, $I_p \otimes G$ has no elements of infinite height. Hence Lemma 3 implies the coset $1 \otimes x + I_p \otimes S$ contains an element of maximal height, and so there are only finitely many different heights occurring in it. By Lemma 2, there are only finitely many distinct p-heights occurring in the coset $x + S$, and so there is an element of maximal p-height in it.

LEMMA 5. *Let G and G' be KM groups. Let S and S' be finitely generated subgroups of G and G' respectively, and let f be a height-preserving isomorphism of S onto S'. Let $x \in G$ with $px \in S$ for some prime p. Then f can be extended to a height-preserving isomorphism between $\{x, S\}$ and a suitable subgroup of G' containing S'.*

Proof. By Lemma 4, we may assume x has maximal p-height in $x + S$. Precisely as in [6], one may find an element $x' \in G'$ such that $x' \notin S'$, $px' \in S'$, x' has maximal p-height in the coset $x' + S'$, and $h_p(x) = h_p(x')$. In order to complete the proof, one need only verify that if $(m, p) = 1$, then

$$h_q(mx + s) = h_q(mx' + f(s))$$

for all primes q and all $s \in S$. If $q = p$, then the fact that x and x' are elements of maximal p-height in their cosets modulo S, respectively S', yields the desired result. If $q \neq p$, then, by Lemma 1,

$$h_q(mx + s) = h_q(pmx + ps) = h_q(pmx' + pf(s)) = h_q(mx' + f(s))$$

since f is height-preserving. The lemma now follows.

THEOREM 4. *Let G and G' be KM groups. G is isomorphic to G' if and only if there exist elements of infinite order $x \in G$ and $x' \in G'$ such that $h_p(mx) = h_p(mx')$ for all integers m and primes p, and G and G' have isomorphic torsion subgroups.*

Proof. Let A be the subgroup generated by x, A' the subgroup generated by x', and let $f: A \to A'$ be defined by $f(x) = x'$. f is height-preserving, by the choice of x and x'. This isomorphism is now extended stepwise to an isomorphism of G and G' by Lemma 5. To ensure catching all of G and G', we take fixed lists of elements of each and alternate between adjoining an element of G and an element of G'. Since the elements of G and G' have finite order modulo A and A' respectively, we can suppose that at each step we are adjoining an element x such that px lies in the preceding subgroup. This is precisely the situation of Lemma 5.

COROLLARY (Cancellation Theorem). *Let T be a countable torsion group with p-primary components T_p, and suppose that the Ulm invariants of each T_p are finite. Let G and G' be KM groups. If $T \oplus G \approx T \oplus G'$, then $G \approx G'$.*

Proof. Since the groups are isomorphic, there is an $x \in T \oplus G$ and an $x' \in T \oplus G'$, each of infinite order, such that $h_p(mx) = h_p(mx')$ for all primes p and all integers m. Now $x = t + g$ and $x' = t' + g'$, $t, t' \in T$, $g \in G$, and $g' \in G'$. Since t and t' have finite order, we may assume each is zero. Hence g and g' satisfy the height equation. By Ulm's Theorem, we may cancel T from either side to obtain that the torsion subgroups of G and G' are isomorphic. Therefore G and G' are isomorphic, by Theorem 4.

One may object to our formulation of Theorem 4 for the same reason as he objected to Theorem 1; the "invariants" given are not really invariants of the group G since the collection of heights $h_p(mx)$ depends on a choice of element x. Any $x \in G$ of infinite order determines a function $f: P \times I \to$ nonnegative integers and ∞ by $f(p, m) = h_p(mx)$. (∞ is a value only when $m = 0$, since G is a KM group.) By Lemma 1, this function is completely determined if we know $f(p, p^k)$ for all primes p and integers $k \geqq 0$. In other words, each x of infinite order determines a family of sequences of integers, one sequence for each prime p. In examining modules over complete discrete valuation rings (in which rings there is a unique prime), Kaplansky and Mackey saw that two modules are isomorphic if and only if they have isomorphic torsion submodules and equivalent height sequences. Our theorem is thus the true analogue of their theorem; we have one sequence for each prime. To understand this collection of heights even better, recall the situation in torsion-free groups of rank 1. There each nonzero element determines a characteristic, i.e., its p-height for each prime p. Thus each element determines a "horizontal" collection of numbers; in modules, each element determines a "vertical" collection of numbers. We have seen that in mixed groups, each element of infinite order determines a "two-dimensional" array of numbers; call such an array the *Ulm tower of x*, and denote it Ux. The problem of dependence on choice of element arose in both the module and torsion-free group cases; the way to solve it is via an appropriate equivalence relation. We proceed to do this here.

Let $\{\alpha_k\}$ and $\{\beta_k\}$ be strictly increasing sequences of nonnegative integers.

$\{\alpha_k\} \sim \{\beta_k\}$ in case there are nonnegative integers m and n such that $\alpha_{k+m} = \beta_{k+n}$ for all k. Let A and B be families of such sequences: A consists of sequences $\{\alpha_k^p\}$; B of sequences $\{\beta_k^p\}$. $A \sim B$ in case $\{\alpha_k^p\} = \{\beta_k^p\}$ for almost all p, and $\{\alpha_k^p\} \sim \{\beta_k^p\}$ for the others.

It is easy to check that we have defined an equivalence relation, and that if x and y are elements of infinite order in a KM group, then they determine equivalent Ulm towers. Hence the structure theorem can be restated as follows:

THEOREM 4′. *Let G and G' be KM groups. $G \approx G'$ if and only if they have isomorphic torsion subgroups and the same equivalence class of Ulm towers.*

V. An existence theorem and applications

A strictly increasing sequence $\{\alpha_k\}$ of nonnegative integers has a *gap at k* in case $\alpha_{k+1} > \alpha_k + 1$.

LEMMA (Kaplansky). *Let $\{\alpha_k\} = \{h_p(p^k x)\}$, where $x \in G$. $\{\alpha_k\}$ has a gap at k implies the $\alpha_k{}^{\text{th}}$ Ulm invariant of the p-primary component of the torsion subgroup of G is nonzero.*

Proof. See [5, page 58].

Motivated by this lemma, we define a notion of compatibility. Let $\{\alpha_n\}$ be a strictly increasing sequence of nonnegative integers, and let $\{n_i\}$ be its subsequence of gaps (this subsequence may be finite); let T be a primary abelian group. $\{\alpha_n\}$ and T are *compatible* if the $\alpha_{n_i}{}^{\text{th}}$ Ulm invariant of T is not zero for each i.

THEOREM 5. *For each prime p, let there be given a strictly increasing sequence of nonnegative integers $\{\alpha_n^p\}$ and a countable p-primary group T_p with no elements of infinite height such that $\{\alpha_n^p\}$ and T_p are compatible. Then there exists a KM group G with torsion subgroup $\sum T_p$ and which contains an element x with $h_p(p^n x) = \alpha_n^p$, all p and n.*

Proof. We divide the proof into two steps: The first step constructs certain "building blocks" whose invariants depend on only one prime; the second step puts the building blocks together to form the desired group.

Our building block shall have torsion subgroup T_p and shall contain an element x such that $h_q(q^n x) = n$ if $q \neq p$, $= \alpha_n^p$ if $q = p$. In this step we omit the superscript p on the α's.

Let H have generators $x, x_0, x_1, \cdots$ and relations $p^{\alpha_n} x_n - p^n x$, i.e., F is free on the x's, S is the subgroup generated by the relations, and $H = F/S$. Clearly H is a countable group of rank 1; let x^* be the image of x in H.

(i) $h_p(p^n x^*) = \alpha_n$.

By our construction, $h_p(p^n x^*) \geqq \alpha_n$. Suppose this inequality were strict. For notational convenience, we shall denote p^k by $[k]$. In F

$$(1) \qquad [1 + \alpha_n]y - [n]x = \sum b_k([\alpha_k]x_k - [k]x).$$

Let $y = ax + \sum a_i x_i$. Then we have the equations

(2) $\quad [1 + \alpha_n]a_i = b_i [\alpha_i]$ and $[1 + \alpha_n]a - [n] = -\sum b_i[i]$.

Since the height of the left side of (1) is n,

$$h_p(\sum b_i([\alpha_i]x_i - [i]x) = \min h_p(b_i[\alpha_i]x_i) \quad \text{and} \quad h_p(\sum b_i[i]x) = n.$$

By (2), $h_p(b_i[\alpha_i]x_i) > n$. Hence $h_p(\sum b_i[i]x) = n$. Now

$$\sum b_i[i]x = \sum_{i \leqq n} b_i[i]x + \sum_{i>n} b_i[i]x,$$

and clearly the height of the second term $> n$. But if $i \leqq n$, $\alpha_i \leqq \alpha_n$ and so $b_i = a_i[1 + \alpha_n - \alpha_i]$. Hence $\sum_{i \leqq n} b_i[i]x = \sum_{i \leqq n} a_i[1 + \alpha_n - \alpha_i + i]x$. But $\alpha_n - \alpha_i \geqq n - i$ which implies $1 + \alpha_n - \alpha_i + i > n$. Hence $h_p(\sum_{i \leqq n} b_i[i]x) > n$, a contradiction. Hence $h_p(p^n x^*) = \alpha_n$.

(ii) *If $q \neq p$, $h_q(q^n x^*) = n$.*

If this height $> n$, we have in F, $q^{n+1}y - q^n x = \sum b_i([\alpha_i]x_i - [i]x)$. Again $y = ax + \sum a_i x_i$ and we obtain $q^{n+1}a - q^n = -\sum b_i[i]$ and $q^{n+1}a_i = b_i[\alpha_i]$. Since $q \neq p$, $b_i = c_i q^{n+1}$ for all i, c_i an integer. Therefore $\sum b_i([\alpha_i]x_i - [i]x) = \sum c_i q^{n+1}([\alpha_i]x_i - [i]x)$ which implies the q-height of the right side of the original equation $\geqq n + 1$, while the q-height of the left side $= n$ (x being a basis element of the free group F).

(iii) *The torsion subgroup of H is p-primary.*

Calculations similar to those in (ii) show that if q is a prime $\neq p$, $qy \in S$ implies $y \in S$.

(iv) *H has no elements of infinite p-height.*

By (i) and (ii), it suffices to look at an element z^* of finite order; by (iii) we may assume the order is p. Lifting to F, $pz = \sum b_i([\alpha_i]x_i - [i]x)$ and $z = ax + \sum a_i x_i$. Hence $pa_i = b_i[\alpha_i]$. Suppose also that $[k]y - z = \sum c_i([\alpha_i]x_i - [i]x)$, where $y = mx + \sum m_i x_i$. Then $[k]m_i - a_i = c_i[\alpha_i]$. For large k, $a_i = [k]m_i - c_i[\alpha_i] = [\alpha_i] d_i$, some integers d_i, and also $pa_i = b_i[\alpha_i]$. Hence $b_i \in (p)$ for all i and $z \in S$, since F is torsion-free.

Let L be the torsion subgroup of H. We have almost proved that H is a building block; we have not yet shown that L is isomorphic to T_p. Let C_i be a cyclic group of order $[\alpha_{n_i} + 1]$. It seems reasonable that $L = \sum C_i$, but the calculations appear tedious. Therefore we resort to another approach.

Lemma. *Let $\{\alpha_n\}$ be a strictly increasing sequence of nonnegative integers, and let $\{n_i\}$ be its subsequence of gaps. Let T_p be the direct sum of cyclic I_p-modules of order $([\alpha_{n_i} + 1])$. Then there exists a countable I_p-module M of rank 1 with no elements of infinite height whose torsion submodule is T_p and which contains an element x such that $h(p^n x) = \alpha_n$. Further, M is a direct summand of any other countable I_p-module of rank 1 with no elements of infinite height which contains an element whose heights give the sequence $\{\alpha_n\}$.*

Proof. See the corollary to Theorem 6.2 in [10].

Let us return to the building blocks. H is contained in $I_p \otimes H$, and the torsion submodule of $I_p \otimes H$ is still L. By the lemma, $L = V \oplus \sum C_i$, and V is a module direct summand of $I_p \otimes H$; a fortiori, V is a group direct summand of H. Hence $H = V \oplus H'$, where H' has torsion $\sum C_i$. By compatibility, $\sum C_i$ is a direct summand of T_p, i.e., $T_p = B \oplus \sum C_i$. Hence $H' \oplus B$ is the building block we are seeking.

Now that the first part of the construction is over, we abandon all previous notation and start afresh. For each prime p, let G_p be a building block with torsion T_p and which contains an element x_p with $h_q(q^n x_p) = n$ if $q \neq p$, $= \alpha_n^p$ if $q = p$. Set $G = (\sum G_p)/S$, where S is generated by the elements $x_p - x_2$, for all primes p. Clearly G is a countable group of rank 1. Let x^* denote the common image of x_p in G.

(i) $h_p(p^i x^*) = \alpha_i^p$.

Clearly this height $\geqq \alpha_i^p$. Set $\alpha_i^p = \alpha$. Suppose $p^{1+\alpha} y^* = p^i x^*$. Lifting this equation to $\sum G_q$, $p^{1+\alpha} y - p^i x_p = \sum a_q(x_q - x_2)$, where $y = \sum y_q$. Looking at each coordinate gives $p^{1+\alpha} y_q = a_q x_q$ if $q \neq p$, $q \neq 2$. Hence $a_q \in (p^{1+\alpha})$, by the height condition on x_q. Further $p^{1+\alpha} y_2 = -\sum a_q x_2$, and so $\sum a_q \in (p^{1+\alpha})$. Hence $a_p \in (p^{1+\alpha})$. But $p^{1+\alpha} y_p = p^i x_p + a_p x_p$ which implies $h_p(p^i x_p) \geqq 1 + \alpha > \alpha$, a contradiction. Hence $h_p(p^i x^*) = \alpha_i^p$. A similar argument is necessary (and easy) for the case $p = 2$.

(ii) *The torsion subgroup of G is $\sum T_p$.*

Clearly S is a torsion-free subgroup of $\sum G_p$. We claim it is a pure subgroup. Suppose $p^i z = \sum a_q(x_q - x_2)$, p a prime. Now $z = \sum z_q$. Therefore $p^i z_q = a_q x_q$ if $q \neq 2$. Hence $a_q \in (p^i)$ for $q \neq p$. Further $p^i z_2 = (\sum a_q) x_2$ implies that $\sum a_q \in (p^i)$. Hence $a_p \in (p^i)$, and so, for all q, $a_q = p^i b_q$, for integers b_q. Therefore $p^i(\sum b_q(x_q - x_2)) = \sum a_q(x_q - x_2)$, and S is pure.

Let $\pi: \sum G_p \to (\sum G_p)/S$ be the natural map. Since S is torsion-free, $\pi(\sum T_p) \approx \sum T_p$. Suppose $nz^* = 0$, i.e., $nz \in S$. By purity, there exists an $s \in S$ such that $nz = ns$. Therefore $z = s + t$, $t \in T = \sum T_p$, since T is the torsion subgroup of $\sum G_p$. Hence $z^* = t^* \in \pi(T)$, so that $\pi(T)$ is the torsion subgroup of G. This completes the proof of Theorem 5.

Let us now return to Ulm towers $\{\alpha_n^p\}$. We introduce a partial order among Ulm towers by $\{\alpha_n^p\} \leqq \{\beta_n^p\}$ in case $\alpha_n^p \leqq \beta_n^p$ for all p and n. Let us call an equivalence class of Ulm towers a *castle*. This partial order on towers does not in general induce a partial order on castles, as we shall presently see.

THEOREM 6. *Let G and H be KM groups. G is almost isomorphic to a subgroup of H if and only if there exist elements of infinite order $x \in G$ and $y \in H$ with $Ux \leqq Uy$.*

Proof of necessity. We may assume G is a subgroup of H by enlarging the torsion subgroup of H. The result is now trivial.

Proof of sufficiency. The inductive construction of Theorem 4 may be repeated here, with the difference that we do not alternate between the fixed lists of generators.

COROLLARY.[2] *There exist KM groups G and H, each almost isomorphic to a subgroup of the other, and yet G and H are not almost isomorphic.*

Proof. Let G have torsion subgroup $T = \sum_{i=1}^{\infty} C(2^i)$ ($C(2^i)$ is the cyclic group of order 2^i), and let G contain an element x with $h_p(p^n x) = n$ if $p \neq 2$, $h_2(2^n x) = 2n$; let H have torsion subgroup T and contain an element y with $h_p(p^n y) = n$ if $p \neq 2$, $h_2(2^n y) = 2n + 1$. Ux and Uy are not equivalent, so that G and H are not almost isomorphic. On the other hand, $Ux \leqq Uy$, and $Uy \leqq U2x$ so that Theorem 6 implies each of G and H is almost isomorphic to a subgroup of the other.

We remark that if each of G and H is almost isomorphic to a *pure* subgroup of the other, then G and H are almost isomorphic.

The example above shows that the partial order on Ulm towers induces a relation among castles which may fail to be antisymmetric. On the other hand, if we do have a collection C of castles which is partially ordered under this relation, we shall call C *unrelated.* Thus a set C of castles is unrelated in case the following condition holds: Let c and c' be castles in C. If there are Ulm towers α and β in c, α' and β' in c' such that $\alpha \leqq \alpha'$ and $\beta' \leqq \beta$, then $c = c'$.

THEOREM 7. *Let $G = \sum_{i=1}^{s} G_i$, each G_i a KM group. Suppose the collection C of all castles arising from elements in G is unrelated. Then any two decompositions of G into groups of rank 1 have isomorphic refinements.*

Proof. Let $[Ux]$ denote the castle of an element $x \in G$. We have the following arithmetic in C.

I. If x and y are dependent, $[Ux] = [Uy]$.

II. $[U(x + y)] \geqq [Ux]$.

III. If t has finite order and $x \in G$, then $[Ut] \geqq [Ux]$.

For any castle $c \in C$, let $G(c)$ denote all elements of G whose castle $\geqq c$, and let $G'(c)$ denote the subgroup of G generated by all elements whose castle $> c$. Since C is unrelated, these sets are well-defined; further, $G(c)$ is a subgroup which contains the torsion subgroup of G.

If $c \in C$, let (for notational convenience) $G_1, \cdots, G_k$ be those summands whose castle $\geqq c$. We claim $G(c) = \sum_{j=1}^{k} G_j$ plus the torsion subgroups of the other summands. Clearly $G(c)$ contains this subgroup; we must show it contains no more. Let $x \in G$ have infinite order, and choose $x_i \in G_i$ of infinite

[2] This example was inspired by correspondence with Ti Yen.

order. By I, we may assume $x = \sum_{i=1}^{s} a_i x_i$. Hence $[Ux] \leqq [Ux_i]$ for all i. But $c \leqq [Ux]$. Thus $c \leqq [Ux_i]$ for $i > k$, a contradiction. In a similar manner, one may see that $G'(c)$ consists of those summands whose castle $> c$ plus the torsion subgroups of the other summands. Hence the number of summands with castle $= c$ is the rank of $G(c)/G'(c)$, which is an invariant of G. Thus, at any rate, the castles occurring in any decomposition of G into groups of rank 1 are uniquely determined. Suppose $\sum H_i$ is another decomposition of G into groups of rank 1. We may now assume that the castle of G_i = the castle of H_i for all i. Hence we may find elements of infinite order $x_i \in G_i$, $y_i \in H_i$ which have identical Ulm towers. By Theorem 5, there exist groups K_i of rank 1 such that $G_i \approx K_i \oplus T_i$ and $H_i \approx K_i \oplus T'_i$ and such that all the Ulm invariants of the torsion subgroup of K_i are finite. Hence $G = \sum K_i \oplus \sum T_i \approx \sum K_i \oplus \sum T'_i$. If T is the torsion subgroup of $\sum K_i$, then $T \oplus \sum T_i \approx T \oplus \sum T'_i$. Since there are only finitely many K_i's, all the Ulm invariants of T are finite, and so Ulm's Theorem allows us to cancel T and conclude $\sum T_i \approx \sum T'_i$. Since these groups are countable with no elements of infinite height, they are the direct sum of cyclic groups. But it is well known that any two decompositions of such groups have isomorphic refinements. This completes the proof of the theorem.

Suppose $G = \sum G_i$, each G_i a KM group; suppose further that the set of castles of the G_i's is unrelated, i.e., it is not the case that two distinct G_i's are each almost isomorphic to a subgroup of each other. I conjecture that G satisfies the hypotheses of Theorem 7, but I have been unable to verify this.

THEOREM 8. *Let G and H be KM groups such that $G \oplus G \approx H \oplus H$. Then $G \approx H$.*

Proof. Let $x \in G$ have infinite order. Then there exists an element $(y, z) \in H \oplus H$ which has the same Ulm tower as $(x, 0)$. We may assume $(y, z) = (aw, bw)$, where a and b are nonzero integers. Hence

$$h_p(p^n x) = \min \{h_p(p^n aw), h_p(p^n bw)\}.$$

This second tower is equivalent to the Ulm tower of w. Hence G and H have the same castle. Let T be the torsion subgroup of G, and let V be the torsion subgroup of H. Then $T \oplus T \approx V \oplus V$. By Ulm's Theorem, $T \approx V$. Hence $G \approx H$, by Theorem 4.

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