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ON TORSION-FREE GROUPS IN O-MINIMAL STRUCTURES

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ABSTRACT. We consider groups definable in the structure $\mathbb{R}_{an}$ and certain o-minimal expansions of it. We prove: If $\mathbb{G} = \langle G, * \rangle$ is a definable abelian torsion-free group, then $\mathbb{G}$ is definably isomorphic to a direct sum of $\langle \mathbb{R}, + \rangle^k$ and $\langle \mathbb{R}^{>0}, \cdot \rangle^m$, for some $k, m \geqslant 0$. Futhermore, this isomorphism is definable in the structure $\langle \mathbb{R}, +, \cdot, \mathbb{G} \rangle$. In particular, if $\mathbb{G}$ is semialgebraic, then the isomorphism is semialgebraic.

We show how to use the above result to give an "o-minimal proof" to the classical Chevalley theorem for abelian algebraic groups over algebraically closed fields of characteristic zero.

We also prove: Let $\mathcal{M}$ be an arbitrary o-minimal expansion of a real closed field R and $\mathbb{G}$ a definable group of dimension n. The group $\mathbb{G}$ is torsion-free if and only if $\mathbb{G}$, as a definable group-manifold, is definably diffeomorphic to R^n.

1. Introduction

Throughout this paper we fix an o-minimal expansion $\mathcal{M}$ of a real closed field $R = \langle R, +, \cdot, 0, 1, < \rangle$. By "definable" we always mean definable in $\mathcal{M}$.

It is well-known that every abelian connected real Lie group is Lie isomorphic to a direct sum of copies of $\mathbb{R}_a$ and the circle group S^1 (see, for example, [2]). Here and everywhere below for a real closed field R we will denote by R_a its additive group $\langle R, +, 0 \rangle$, and by R_m the multiplicative group of positive elements $\langle R^{>0}, \cdot, 1 \rangle$.

From a model-theoretical point of view it is natural to ask whether or not this kind of decomposition holds in the category of groups definable in the o-minimal structure $\mathcal{M}$.

In general, the answer to the above question is negative. There are at least two obstacles. First, in the polynomial bounded case, the multiplicative group R_m is not definably isomorphic to the additive group of R_a, and one should at least allow also copies of the multiplicative group.

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Secondly, there are examples of two-dimensional definable groups that are not direct sums of one-dimensional definable subgroups. In [26] A. W. Strzebonski gave examples of abelian semialgebraic groups that are extensions of S^1 and are not definably isomorphic to direct sums of one-dimensional definable subgroups.

It seems that the presence of definably compact factors is essential in Strzebonski's example and in all other known examples, and the following question is open.

Question 1. Let G be an abelian torsion-free definable group. Is G definably isomorphic to a direct sum of definable copies of R_a and R_m?

In fact, the above question consists of two sub-questions.

Question 1A. Let G be an abelian torsion-free definable group. Is G a direct sum of definable one-dimensional subgroups?

Question 1B. Is every definable one-dimensional torsion-free group definably isomorphic to R_a or R_m?

It is not hard to show (see Claim 2.10 below) that in Question 1A it is sufficient to consider only definable groups of dimension 2. Also, since by Edmundo's Theorem every definably compact definable group has torsion points (see [8] or [1]), Question 1A is just a restatement of a question asked by the first author and C. Steinhorn in [16].

Question 1B was originally asked by the second author and C. Miller, and in [12] they gave a positive answer to it in the polynomially bounded case.

We believe that Question 1B also has a positive answer in the polynomially bounded case, and we state it here as a conjecture.

Conjecture 1. If $\mathcal{M}$ is polynomially bounded, then every abelian torsion-free definable group is a direct sum of definable one-dimensional subgroups.

We prove here Conjecture 1 for any o-minimal expansion of a real closed field in which every definable function in one variable has a definable Puiseux-like expansion at $+\infty$ (see Theorem 4.10). In the case of the real field this means that given a definable $f : (d, +\infty) \to \mathbb{R}$, there are $r_k > r_{k-1} > \cdots > r_1 > 0$ in $\mathbb{R}$, and $c_1, \ldots, c_k \in \mathbb{R}$ such that the function $f(x) - [c_1 x^{r_1} + \cdots + c_k x^{r_k}]$ is ultimately bounded.

In the structures below, every definable function in one variable has a definable Puiseux-like expansion at $+\infty$:

(1) $\mathbb{R}_{an}^{\mathbb{R}}$, the expansion of $\mathbb{R}_{an}$ by the functions $\{x \mapsto x^r : r \in \mathbb{R}\}$ on the positive real line (see C. Miller [11]). Definable functions in elementarily equivalent structures also have Puiseux-like expansions.

(2) $\mathbb{R}_{\mathcal{G}}$, the expansion of $\mathbb{R}_{an}$ by multisummable functions (see L. v.d. Dries and P. Speissegger [6, Theorem A]).

(For other examples, see L. v.d. Dries and P. Speissegger [5, Remark 2, on p. 4420].)

Thus, in particular, Conjecture 1 holds for semi-algebraic groups, i.e., groups definable in real closed fields. (It is possible that this last fact could be also deduced from the work of A. Pillay and E. Hrushovski [9].) In the Appendix we explain how one can use this fact to prove the classical Chevalley theorem for abelian groups. Namely, every abelian algebraic group over an algebraically closed field of characteristic zero has a linear algebraic subgroup $H \subseteq G$, such that G/H is an abelian variety.

We would like to remark that Conjecture 1 was also considered by M. Edmundo implicitly in [7], and we use here some of his results.

We thank Chris Miller for his useful suggestions.

1.1. The structure of the paper. In Section 2 we state known properties of torsion-free definable groups that we will need in this paper.

In Section 3 we consider definable abelian group extensions and give a criterion for splitting definable group extensions in terms of growth rates of definable global sections.

In Section 4 we consider the polynomially bounded case. Since in this case every one-dimensional torsion-free group is definably isomorphic to R_a or R_m, only extensions involving these groups need to be considered. We first present Edmundo's result that every definable extension of a definable ordered group by R_m definably splits. It reduces Conjecture 1 to definable extensions by R_a, which we prove for o-minimal structures in which every definable function has a Puiseux-like expansion at $+\infty$. We then use the Expansion Theorem of L. v.d. Dries and C. Miller to show that the conjecture holds in structures elementarily equivalent to $\mathbb{R}_{an}^{\mathbb{R}}$, and in reducts of these. This implies that any abelian torsion-free group definable in these structures is definably isomorphic to $R_m^k \times R_a^l$ (see Corollary 4.10).

As a corollary to the above we show (see Theorem 4.13) that for every definable groups $H \subseteq G$ in such a structure, if G/H is not a definably compact space, then G contains a 1-dimensional torsion-free definable group H_1 such that $H_1 \cap H = \{e\}$.

Let H be a definable subgroup of a definable group G and $\pi : G \to G/H$ the natural projection. In Section 5 we consider the existence of definable continuous and smooth sections of π. As in the classical case of Lie groups, we show existence of a definable continuous section in the case of definably contractible group H. We also consider the case when H is a normal subgroup and G/H is torsion-free. In this case we prove the existence of a definable smooth global section. It implies that every torsion-free definable group is diffeomorphic to the affine space R^n (see Corollary 5.8).

On notation. Frequently we will use $\oplus$ to denote the group operation. We will use then $\ominus x$ to denote the group inverse of x.

2. Preliminaries

If G is a definable group, then, by [17], for any $p \geqslant 0$ the group G has a structure of a definable C^p group manifold. This structure is unique in the following sense.

FACT 2.1 ([17]). *Let G, H be definable groups and $f : G \to H$ a definable group homomorphism. Then f is C^p with respect to the definable C^p group manifold structures on G and H.*

If H is a definable subgroup of a definable group G, then the set G/H of the left cosets of H can also be equipped with a structure of a definable manifold so that the canonical action of G on G/H is C^p. (See [14, Theorem 2.11].) We will always view G and G/H as definable C^1-manifolds, and all references to topological and differentiable structures will be with respect to these manifold structures.

2.1. Some facts about torsion-free definable groups.

In this section we list some basic facts about torsion-free definable groups.

FACT 2.2 ([25]). *A definable group G is torsion-free if and only if the o-minimal Euler characteristic of G is $+1$ or -1.*

By definable choice, if H is a definable normal subgroup of a definable group G, then there is a definable group K and a definable homomorphism $f : G \to K$ whose kernel is H. Thus we can always consider G/H as a definable group.

COROLLARY 2.3. *Let G be a torsion-free definable group. If H is a normal definable subgroup of G, then G/H is also torsion-free.*

COROLLARY 2.4. *If G is a torsion-free definable group, then G is definably connected.*

Recall that a definable group G is *definably compact* if for every definable function $f : (a, b) \to G$ the limit $\lim_{x\to b^-} f(x)$ exists in G.

FACT 2.5 ([16, Theorem 1.2]). *If a definable group G is not definably compact, then it has a definable one-dimensional torsion-free subgroup.*

Using induction on dimension of G we obtain the following.

FACT 2.6. *If G is an abelian definable group, then there are definable subgroups $G_0 < G_1 < \cdots < G_n < G$ such that G/G_n is definably compact and $(G_{i+1}/G_i) = 1$ is a torsion-free one-dimensional group for $i = 0, \ldots, n-1$.*

Since, by Edmundo's Theorem, every definably compact group has a torsion point, for torsion-free groups the above fact can be restated as follows.

Fact 2.7. *If $G = \langle G, \cdot, e\rangle$ is an abelian torsion-free definable group of dimension n, then there are definable subgroups $\{e\} = G_0 < G_1 < \cdots < G_n = G$ such that $\dim(G_{i+1}/G_i) = 1$ for $i = 0, \ldots, n-1$.*

Recall that a definable ordered group is a definable group H together with a definable order relation on H such that H with this ordering is an ordered group. It is not hard to see that every definable one-dimensional ordered group is definably isomorphic to a definable ordered group on R with continuous group operations whose ordering is the ordering of R.

Fact 2.8 ([26], [20]). *Let G be a torsion-free definable one-dimensional group. Then there is a definable order relation $<$ on G such that G with $<$ is a definable ordered group.*

Fact 2.9 ([19]). *Let H be a definable one-dimensional ordered group. Then H is abelian, divisible and has no proper nontrivial definable subgroups.*

2.2. Some preliminary results. We believe that all results in this section are well-known. However we could not find precise references, and decided to present them here with complete proofs.

Claim 2.10. *Assume that every abelian torsion-free definable group of dimension 2 is a direct sum of definable one-dimensional subgroups. Then every abelian torsion-free definable group is a direct sum of definable one-dimensional subgroups.*

Proof. Let G be an abelian torsion-free definable group. We proceed by induction on $\dim(G)$. There is nothing to prove if $\dim(G) \leqslant 2$.

Assume $\dim(G) > 2$. Since G is torsion-free, it has a one-dimensional definable subgroup H. Let $K = G/H$ and $\pi : G \to K$ be the projection map. Since $\dim(K) < \dim(G)$, by the induction hypothesis, K is a direct sum of definable one-dimensional subgroups $V_1, \ldots, V_k$. Let $U_i = \pi^{-1}(V_i)$. Each U_i has dimension 2, hence is a direct sum of H and a definable subgroup H_i. Obviously, G is a direct sum of $H, H_1, \ldots, H_k$. □

Claim 2.11. *If G is a torsion-free definable group, then G is solvable.*

Proof. Assume not, and let G be a torsion-free definable not solvable group of the smallest possible dimension. Since all one-dimensional groups are abelian by finite, $\dim(G) > 1$.

G must be definably simple. Indeed, if H is a proper nontrivial definable normal subgroup, then, by the induction hypothesis, both H and G/H must be solvable. This implies that G is solvable as well.

Every definably simple group definable in an o-minimal structure is elementarily equivalent to a simple centerless real Lie group [15, Theorem 5.1]. It is well-known that every simple centerless real Lie group has torsion elements. This is a contradiction. □

Fact 2.6 together with the previous claim yield:

Corollary 2.12. *If G is a torsion-free definable group, then G has a definable normal subgroup H such that* $\dim(G/H) = 1$.

3. Definable group extensions

Let H and K be groups. *An extension* of K by H is a group G, containing H, together with a homomorphism $\pi : G \to K$ such that the sequence $0 \to H \hookrightarrow G \xrightarrow{\pi} K \to 0$ is exact, i.e., π is a surjective homomorphism and H is the kernel of π. We will also call a surjective homomorphism $\pi : G \to K$ an extension of K when we do not need to mention H explicitly. An extension $\pi : G \to K$ is *an abelian extension* if G is an abelian group.

An extension $\pi : G \to K$ is called *a definable extension* if G, K and π are definable.

We say that a definable extension $\pi : G \to K$ *definably splits* if there is a definable homomorphism $h : K \to G$ such that $\pi \circ h = \mathrm{id}_K$.

Recall that for a surjective map $\pi : A \to B$ a global section of π is a map h from B to A such that $\pi \circ h = \mathrm{id}_B$. If $U \subseteq B$ and $h : U \to A$ satisfies $\pi \circ h = \mathrm{id}_U$, then h is called a section of π over U. Thus a definable extension $\pi : G \to K$ definably splits if and only if π has a definable global section that is also a group homomorphism.

The following is obvious.

Claim 3.1. *Let $\pi : G \to K$ be a definable abelian extension of K by H. The following are equivalent.*

1. *G is a direct sum of H and a definable subgroup H_1,*
2. *The extension $\pi : G \to K$ definably splits.*

Consider a torsion-free abelian definable group G of dimension 2. Let $H < G$ be a definable subgroup of H of dimension 1, $K = G/H$ and $\pi : G \to K$ the natural projection. It is easy to see that G is a direct sum of one-dimensional definable subgroups if and only if the projection $\pi : G \to K$ definably splits. Since, by Fact 2.8, both K and H are definable ordered groups, in order to answer Question 1A, we need to understand when a definable abelian extension of a one-dimensional ordered group by a one-dimensional ordered group splits. Below we give a criterion for such a splitting in terms of growth rates of definable sections. We will follow ideas from [12] and [16].

For the rest of this section we fix a definable group $G = \langle G, *, e\rangle$ and an ordered one-dimensional definable group $K = \langle K, \oplus, 0\rangle$. We do not assume that G is abelian.

3.1. Growth rates of sections and definable splitting. We will follow the approach from [12, Section 1.3-1.4] and will be brief.

For a function $f : K \to G$ and $x, y \in K$ we define $\Delta_y f(x)$ to be $f(x \oplus y) * f(x)^{-1}$. Obviously, if f is definable, then so is $\Delta_y f(x)$.

For any $f : K \to G$ we have

$$\begin{gathered}\Delta_{y\oplus z} f(x) = \Delta_y f(x \oplus z) * \Delta_z f(x),\\ \Delta_{\ominus y} x = [\Delta_y (x \ominus y)]^{-1}.\end{gathered} \tag{1}$$

If f is a definable function from $K \to G$, then it follows from (1) that the set $\{y \in K : \lim_{x\to+\infty} \Delta_y f(x) \in G\}$ is a subgroup of K. Since K is a definable one-dimensional ordered group, it has no proper definable nontrivial subgroup. Thus if for a definable $f : K \to G$ the limit $\lim_{x\to+\infty} \Delta_y f(x)$ exists for some $y \neq 0$, then the limit exists for all $y \in K$.

Assume now that $f : K \to G$ is a definable function such that the limit $\lim_{x\to+\infty} \Delta_y f(x)$ exists for all $y \in K$. It follows from (1) then that the function $y \mapsto \lim_{x\to+\infty} \Delta_y f(x)$ is a definable homomorphism from K to G.

Let $\pi : G \to K$ be a definable extension and $h : K \to G$ a definable global section of π. It is easy to see that $\Delta_y h(x) \in \pi^{-1}(y)$ for all $y, x \in K$.

Thus we have the following claim.

Claim 3.2. *Let $\pi : G \to K$ be a definable extension, and $h : K \to G$ a definable global section of π. Then the set $\{y \in K : \lim_{x\to\infty} \Delta_y h(x) \in G\}$ is a subgroup of K. It is either trivial or the whole K. In the latter case the function $y \mapsto \lim_{x\to+\infty} \Delta_y f(x)$ is a group homomorphism and also a global section of π; in particular, π definably splits.*

The following corollary to Claim 3.2 was first proved by M. Edmundo [7, Lemma 5.1].

Corollary 3.3. *If $\pi : G \to K$ is a definable extension of a definable one-dimensional ordered group K by a definably compact group H, then π definably splits.*

Proof. Let $f : K \to G$ be a definable global section and $d \neq 0 \in K$. Since $\Delta_d f(x) \in \pi^{-1}(d)$ for any $x \in K$, and the fiber $\pi^{-1}(d)$ is definably compact, the limit $\lim_{x\to\infty} f(x \oplus c) * f(x)^{-1}$ exists in G □

Let $\pi : G \to K$ be a definable extension of K by a group H, and $f : K \to G$ a definable global section of π. If $g : K \to H$ is any definable function, then $f(x) * g(x)$ is also a definable global section of π, and any definable global section can be obtained from f this way by choosing an appropriate g. Thus we have the following result.

COROLLARY 3.4. *Let $\pi : G \to K$ be a definable extension of a definable one-dimensional ordered group K by a definable group H. The following conditions are equivalent:*

(1) *π definably splits.*
(2) *There is a definable global section f of π and $c \neq 0 \in K$ such that the limit $\lim_{x\to\infty} f(x \oplus c) * f(x)^{-1}$ exists in G.*
(3) *For any definable global section f of π there is a definable function $g : K \to H$ and $c \neq 0 \in K$ such that the limit $\lim_{x\to\infty} f(x\oplus c) * g(x \oplus c) * g(x)^{-1} * f(x)^{-1}$ exists in G.*

3.2. Abelian torsion-free definable extensions. In this section we assume that G is an abelian torsion-free definable group of dimension 2, $H < G$ a definable subgroup of dimension 1, $K = G/H$ and $\pi : G \to K$ the natural projection. We have that both H and K are definable one-dimensional ordered groups.

Let f be a definable global section of π, $g : K \to H$ a definable function, and $c \neq 0 \in K$. Consider the limit

$$\lim_{x\to\infty} f(x \oplus c) * f(x)^{-1} * g(x \oplus c) * g(x)^{-1}.$$

Obviously, the above limit exists in G if and only if the limit

$$\lim_{x\to\infty} f(x \oplus c) * f(x)^{-1} * f(c)^{-1} * g(x \oplus c) * g(x)^{-1}$$

exists. Since f is a global section of π, both $f(x \oplus c) * f(x)^{-1} * f(c)^{-1}$ and $f(x \oplus c) * f(x)^{-1} * f(c)^{-1} * g(x \oplus c) * g(x)^{-1}$ are in H for any $x \in K$. Also, since H and K are one-dimensional ordered groups, for any definable function $h : K \to H$ the limit $\lim_{x\to+\infty} h(x)$ exists in H if and only if $h(x)$ is ultimately bounded, i.e., there is $C \in H$ such that $C^{-1} < h(x) < C$ for all sufficiently large $x \in K$. Thus we can restate Corollary 3.4 in the following way.

CLAIM 3.5. *Let $\pi : G \to K$ be a definable abelian extension of a definable one-dimensional ordered group K by a definable one-dimensional ordered group H, and $f : K \to G$ a definable global section of π. The following conditions are equivalent:*

(1) *The extension $\pi : G \to K$ definably splits.*
(2) *There is a definable function $h : K \to H$ such that for some $c \neq 0 \in K$ the function*

$$[f(x \oplus c) * f(x)^{-1} * f(c)^{-1}] * \Delta_c h(x)$$

is ultimately bounded (as a function from K to H).

Remark. The criterion given in Claim 3.5 has also a geometric interpretation. In [16] it was shown how to associate to every definable curve $\gamma : [0,+\infty) \to G$ with no endpoint in G a definable, torsion-free one-dimensional subgroup $H_\gamma \subseteq G$. (H_γ is the set of all limit points of $\gamma(s) * \gamma(t)^{-1}$, as s and t tend to $+\infty$.) In order for a definable abelian extension G of a definable one-dimensional ordered group K by a definable one-dimensional ordered subgroup H to split, it is necessary to find a definable curve γ so that H_γ is different from H. If $f : K \to G$ and $h : K \to H$ satisfy clause (2) in Claim 3.5, then for the curve $\gamma(t) = f(t) * h(t)$ the associated group H_γ will be different from H.

4. The polynomially bounded case

Recall that $\mathcal{M}$ is polynomially bounded if for every definable function $f : R \to R$ there is $n \in \mathbb{N}$ such that $|f(x)| < x^n$ for all sufficiently large x.

The following was proved in [12].

Fact 4.1. *If $\mathcal{M}$ is polynomially bounded, then every definable one-dimensional ordered group is definably isomorphic to R_a or R_m.*

Thus, by Fact 2.7, in the polynomially bounded case, every abelian torsion-free definable group of dimension 2 is a definable abelian extension of R_a or R_m by R_a or R_m.

4.1. Abelian extensions by R_m. If $\mathcal{M}$ is polynomially bounded, then the triviality of definable abelian extensions of any definable one-dimensional ordered group by R_m follows from [7, Lemma 5.2]. We present its proof here for the sake of completeness.

The following lemma follows from [12, Proposition 3.2].

Lemma 4.2. *Assume $\mathcal{M}$ is polynomially bounded. Let $H = \langle H, *, e\rangle$ be a definable group definably isomorphic to R_m. If $f(x,y) : R \times R \to H$ is a definable function, then there is $C \in R$ such that $\lim_{x\to+\infty} f(x,y_1) * f(x,y_2)^{-1}$ exists in H for all $y_1, y_2 > C$.*

Theorem 4.3. *Assume $\mathcal{M}$ is polynomially bounded. Let H be a definable group definably isomorphic to R_m and $\pi : G \to K$ a definable abelian extension of a definable one-dimensional ordered group K by H. Then π definably splits.*

Proof. We will denote the group operation of G by $*$, and the group operation of K by $\oplus$. Let $f : K \to G$ be a definable global section.

Since $f(x \oplus y) * f(x)^{-1} * f(y)^{-1} \in H$ for all $x, y \in K$, by Lemma 4.2, for all sufficiently large $y_1, y_2 \in K$ the limit

$$\lim_{x\to+\infty} \big(f(x \oplus y_1) * f(x)^{-1} * f(y_1)^{-1} * f(x) * f(y_2) * f(x \oplus y_2)^{-1}\big)$$

exists in H. Therefore, for all sufficiently large $y_1, y_2 \in K$, the limit $\lim_{x\to+\infty}(f(x\oplus y_1) * f(x\oplus y_2)^{-1})$ exists in G. Hence, if $z \neq 0 \in K$, then, for all sufficiently large $y \in K$, the limit $\lim_{x\to+\infty}(f(x\oplus y\oplus z) * f(x\oplus y)^{-1})$ exists in G, and so does $\lim_{x\to+\infty}(f(x\oplus z) * f(x)^{-1})$. Theorem 4.3 now follows from Corollary 3.4. □

Thus in order to establish Conjecture 1, it remains to prove that in the polynomially bounded case every definable abelian extension of a definable one-dimensional ordered group by a group definably isomorphic to the additive group R_a definably splits.

4.2. On abelian extensions by R_a. Let $G = \langle G, *, e\rangle$ be a definable abelian extension of a definable one-dimensional ordered group $K = \langle K, \oplus, 0\rangle$ by a group H definably isomorphic to R_a. We fix a definable global section f of this extension. In contrast to the case of R_m it is not true anymore that for any $c \neq 0 \in K$ the limit $\lim_{x\to+\infty} f(x\oplus c) * f(x)^{-1}$ always exists in G. (An elementary example is $G = R_a \times R_a$ with the global section $x \mapsto (x, x^2)$.) According to Claim 3.5, this extension splits if and only if there is a definable function $h : K \to H$ such that

$$f(x \oplus c) * f(x)^{-1} * f(c)^{-1} * \Delta_c h(x)$$

is ultimately bounded for some $c \neq 0 \in K$. Taking the group inverse of the above expression we obtain that the extension splits if and only if there is a definable function $h : K \to H$ such that

$$f(x \oplus c)^{-1} * f(x) * f(c) * \Delta_c h(x)^{-1}$$

is ultimately bounded for some $c \neq 0 \in K$.

It is not hard to see that for a given definable $h : K \to H$ the function

$$f(x \oplus y) * f(x)^{-1} * f(y)^{-1} * \Delta_z h(x)^{-1}$$

is ultimately bounded, as a function of x, for some $y = z \neq 0 \in K$ if and only if it is ultimately bounded for all nonzero $y, z \in K$ if and only if it is ultimately bounded for some nonzero $y, z \in K$.

Since in the polynomially bounded case every definable one-dimensional ordered group is definably isomorphic to R_a or R_m, we have the following claim.

Claim 4.4. *Assume $\mathcal{M}$ is polynomially bounded. Suppose that for any definable function $g : R \to R$ there are definable functions $h_a, h_m : R \to R$ such that both $g(x) - [h_a(x + 1) - h_a(x)]$ and $g(x) - [h_m(2x) - h_m(x)]$ are ultimately bounded. Then every definable abelian extension of a definable one-dimensional ordered group by the additive group R_a splits, and Conjecture 1 holds for $\mathcal{M}$.*

In the next section we will show that in many interesting cases, e.g., in the semi-algebraic case, the assumptions of the above claim hold.

4.3. Power expansions at infinity. We recall some notions and results from [11].

A definable power function is a definable group endomorphism of R_m. The set of all power functions is an ordered field, with addition given by pointwise multiplication, multiplication given by composition, and the positive elements are precisely the strictly increasing functions. If f is a definable power function, then f is differentiable with $f'(x) = f'(1)f(x)/x$.

The map $f \mapsto f'(1)$ is an embedding of the ordered field of definable power functions into R. We will denote by Λ the image of this map, called the field of definable exponents. If $r \in \Lambda$, then the definable power function f with $f'(1) = r$ will be denoted by x^r, and instead of $f(a)$ we will write a^r. Note that in general Λ is not a definable subset of R. The set Λ contains $\mathbb{Q}$, and for $q \in \mathbb{Q}$ the function x^q coincides with the standard one. By Λ_{fin} we will denote the set of all $r \in \Lambda$ such that $-n < r < n$ for some $n \in \mathbb{N}$.

Let $f(x)$ be a definable function from an unbounded interval $(d, +\infty)$ into R. We say that f has *a definable Puiseux-like expansion at* $+\infty$ if there are $r_k > r_{k-1} > \cdots > r_1 > 0$ in Λ_{fin}, and $c_1, \ldots, c_k \in R$, such that the function $f(x) - [c_1 x^{r_1} + \cdots + c_k x^{r_k}]$ is ultimately bounded.

It follows from [4] that in $\mathbb{R}^{\mathbb{R}}_{an}$ and elementarily equivalent structures definable functions have definable Puiseux-like expansions at $+\infty$:

Fact 4.5 ([4, Expansion Theorem]). *Let $\mathbb{R}^{\mathbb{R}}_{an}$ be the expansion of $\mathbb{R}_{an}$ by all power functions $x \mapsto x^r, r \in \mathbb{R}$. Then every function $f : \mathbb{R} \to \mathbb{R}$ definable in $\mathbb{R}^{\mathbb{R}}_{an}$ has an $\mathbb{R}^{\mathbb{R}}_{an}$-definable Puiseux-like expansion at $+\infty$, and the same is true for all structures elementarily equivalent to $\mathbb{R}^{\mathbb{R}}_{an}$.*

Claim 4.6. *Let $\mathcal{M} = \mathbb{R}^{\mathbb{R}}_{an}$, and $\mathcal{M}_0$ a reduct of $\mathcal{M}$ to a language containing the language of ordered fields. Then every $\mathcal{M}_0$-definable function $f : \mathbb{R} \to \mathbb{R}$ has an $\mathcal{M}_0$-definable Puiseux-like expansion at $+\infty$. The same is true when $\mathcal{M}$ is elementarily equivalent to $\mathbb{R}^{\mathbb{R}}_{an}$.*

Proof. Let $\mathcal{M}_1$ be a structure elementarily equivalent to $\mathcal{M}_0$. Passing to an elementary extension, if necessary, we can assume that $\mathcal{M}$ is saturated. Hence it has an expansion $\mathcal{N}$ elementarily equivalent to $\mathbb{R}^{\mathbb{R}}_{an}$.

Let $f : R \to R$ be an $\mathcal{M}_1$-definable function. By Fact 4.5, it has an $\mathcal{N}$-definable Puiseux-like expansion at $+\infty$. Let $x^{r_1}, \ldots, x^{r_k}$ be $\mathcal{N}$-definable power functions, with $r_k > r_{k-1} > \cdots > r_1 > 0$, and $c_1, \ldots, c_k \in R \setminus \{0\}$, such that the function $f(x) - [c_1 x^{r_1} + \ldots c_k x^{r_k}]$ is ultimately bounded. We need to show that $x^{r_1}, \ldots, x^{r_k}$ are $\mathcal{M}_1$-definable.

As was noted in [13], since $\lim_{t\to+\infty} f(t)/t^{r_k} = c_k \neq 0$, for any $x > 0$, $\lim_{y\to+\infty} f(xy)/f(y) = x^{r_k}$. Hence x^{r_k} is $\mathcal{M}$-definable. Considering the function $f(x) - c_k x^{r_k}$, we obtain that $x^{r_{k-1}}$ is $\mathcal{M}$-definable, etc. □

Our goal is to show that the premise of Claim 4.4 holds in the case when every definable function has a definable Puiseux-like expansion at $+\infty$.

CLAIM 4.7. *Let r be a positive element of Λ_{fin} and n be the largest nonnegative integer with $r > n$. Then $(x+1)^r$ has a definable Puiseux-like expansion at $+\infty$. More precisely,*

$$(x+1)^r - \left[rx^r + \binom{r}{1}x^{r-1} + \cdots + \binom{r}{n}x^{r-n}\right]$$

is ultimately bounded. (Here, as usual, $\binom{r}{i} = r(r-1)\cdots(r-i+1)/i!$.)

Proof. Since r is positive and $r \leqslant n+1$,we have $0 \leqslant x^r \leqslant x^{n+1}$ for all $x > 1$. Hence $0 \leqslant x^r/x^{n+1} \leqslant 1$ for all $x > 1$. Since the $(n+1)$-th derivative of x^{n+1} is 1, by L'Hôpital's rule, the $(n+1)$-th derivative of x^r is ultimately bounded. The claim now follows from Taylor's formula. (See [3, p. 114] for L'Hôpital's rule and Taylor's formula.) □

LEMMA 4.8. *Let f be a definable function from an unbounded interval $(d, +\infty)$ into R. Assume f has a definable Puiseux-like expansion at $+\infty$. Then the following hold.*

(1) *There is a definable $h_a(x)$ such that $f(x) - [h_a(x+1) - h_a(x)]$ is ultimately bounded.*
(2) *There is a definable $h_m(x)$ such that $f(x) - [h_m(2x) - h_m(x)]$ is ultimately bounded.*

Proof. (1) Choose the minimal subset $S \subseteq \Lambda_{fin}$ such that:

(a) S consists of positive elements;
(b) if $r \in S$ and $r - 1 > 0$, then $r - 1 \in S$;
(c) there are $r_k > r_{k-1} > \cdots > r_1$ in S, such that $f(x) - [c_1 x^{r_1} + \ldots c_k x^{r_k}]$ is ultimately bounded for some $c_1, \ldots, c_k \in R$.

We will denote this set by $S(f)$. We will proceed by induction on the size of $S(f)$.

If $S(f)$ is empty, then f is bounded and we can take $h_a \equiv 0$.

Assume $S(f)$ is non-empty, and let $r_k > r_{k-1} > \cdots > r_1$ and $c_1, \ldots, c_k$ be as in (c). Since $S(f)$ is minimal satisfying (a)-(c), r_k is the largest element of $S(f)$ and c_k is not zero.

Consider the function $g(x) = \frac{c_k}{r_k+1} x^{r_k+1}$. By Claim 4.7, the function $u(x) = f(x) - [g(x+1) - g(x)]$ has a definable Puiseux-like expansion at $+\infty$, and it is not hard to see that $S(u) \subsetneqq S(f)$. By the induction hypothesis, there is a definable $h_0(x)$ such that $u(x) - [h_0(x+1) - h_0(x)]$ is ultimately bounded. We can take $h_a(x) = h_0(x) + u(x)$.

(2) Let $r_k > r_{k-1} > \cdots > r_1 > 0$ be in Λ_{fin}, and $c_1, \ldots, c_k \in R$, such that $f(x) - [c_1 x^{r_1} + \ldots c_k x^{r_k}]$ is ultimately bounded. It is easy to check that we can take $h_m(x) = b_1 x^{r_1} + \cdots + b_k x^{r_k}$, where $b_i = c_i/(2^{r_i} - 1)$. □

THEOREM 4.9. *Assume that every definable function $f : R \to R$ has a definable Puiseux-like expansion at $+\infty$. Then every abelian torsion-free definable group is a direct sum of definable one-dimensional subgroups, each definably isomorphic to R_a or R_m.*

Proof. If every definable function has a definable Puiseux-like expansion at $+\infty$, then the structure is polynomially bounded. Thus Theorem 4.9 follows from Claim 4.4 and Lemma 4.8. □

COROLLARY 4.10. *Assume $\mathcal{M}$ is elementarily equivalent to $\mathbb{R}^{\mathbb{R}}_{an}$. Then every abelian torsion-free $\mathcal{M}$-definable group $(G, *)$ is definably isomorphic to $R^k_m \times R^l_a$, and this isomorphism is definable in the structure $\langle R, +, \cdot, (G, *)\rangle$.*

COROLLARY 4.11. *Every abelian, torsion-free, semilagebraic group is a direct sum of one-dimensional semialgebraic subgroups.*

The following is an immediate corollary to Fact 2.6 and Theorem 4.9.

COROLLARY 4.12. *Assume that every definable function $f : R \to R$ has a definable Puiseux-like expansion at $+\infty$. If G is a definable abelian group, then there is a definable torsion-free subgroup $H \subseteq G$, such that H a direct sum of definable 1-dimensional torsion-free subgroups and G/H is definably compact.*

We finish this section with a corollary on definable homogeneous spaces.

THEOREM 4.13. *Assume that every definable function $f : R \to R$ has a definable Puiseux-like expansion at $+\infty$. Let G be a definable group, $H \subseteq G$ a definable subgroup and assume that G/H is not a definably compact space. Then G has a 1-dimensional torsion-free definable subgroup whose intersection with H is trivial.*

Proof. Let us assume first that H is a normal subgroup. By Fact 2.5, there is a 1-dimensional torsion-free subgroup A_1 of G/H. The preimage of A_1 under the quotient map is a definable subgroup H_1 of G, containing H, whose dimension equals $\dim(H) + 1$, and H_1/H is not definably compact.

Now take an element g in H_1 which is not in H but has an infinite order. Then g is contained in an infinite abelian subgroup A_2 (e.g., the center of the centralizer of g), and by dimension considerations we have $A_2 \cdot H = H_1$.

By Corollary 4.12, we have 1-dimensional, torsion-free definable subgroups $B_1, \dots, B_k$ of A_2 such that the quotient $A_2/(B_1 \oplus \cdots \oplus B_k)$ is definably compact. At least one of the B_i's is not contained in H, for otherwise H_1/H would be definably compact.

We now return to the general case (i.e., we do not assume that H is normal in G). Assume towards a contradiction that all definable 1-dimensional, torsion-free subgroups of G are contained in H. Then every conjugate of H will also contain all definable 1-dimensional torsion-free subgroups. Consider now the intersection, N, of all conjugates of H. This is a definable normal subgroup of G which contains all 1-dimensional torsion-free subgroup of G. By the lemma, G/N must be definably compact, contradicting the fact that G/H was not definably compact. □

4.4. Groups definable in $\mathbb{R}_{an}$. In the case when $\mathcal{M} = \mathbb{R}_{an}$ we have a stronger result.

THEOREM 4.14. *Let G be a connected abelian group, definable in $\mathbb{R}_{an}$. Then G is a direct sum of $\mathbb{R}_{an}$-definable, one-dimensional groups.*

Proof. By the analytic cell-decomposition in $\mathbb{R}_{an}$, G can be equipped, definably, with the structure of a real analytic group. As such, it is a direct sum of real analytic 1-dimensional subgroups. Namely,

$$G = A_1 \oplus \cdots \oplus A_k \oplus B_1 \oplus \cdots \oplus B_l,$$

where the A_i's are Lie isomorphic to the circle group S^1, and the B_i's are Lie isomorphic to $\langle \mathbb{R}, + \rangle$.

Now, each of the A_i's is a compact real analytic submanifold of G, and therefore it is definable in $\mathbb{R}_{an}$.

On the other hand, by Fact 2.6, G contains a definable torsion-free subgroup such that G/H is compact. By Corollary 4.10, H is a direct sum of 1-dimensional groups. Since every compact group must have a torsion point, the intersection of H with the product of the A_i's is trivial. It is now easy to see that G is a direct sum of H and the A_i's. □

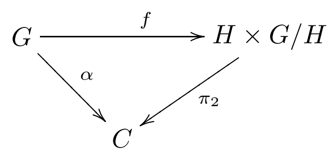
REMARK. The first part of the above argument works in any o-minimal expansion of $\mathbb{R}_{an}$, with an analytic cell decomposition. Namely, every definable group in such a structure is a real analytic Lie group, and the A_i's above are definable in it. However, we do not know if this carries over to elementarily equivalent structures.

5. On the existence of continuous and smooth global sections

We continue to work in an o-minimal expansion $\mathcal{M}$ of a real closed field R.

Given a definable group G with a definable subgroup H, one can ask if there is a definable continuous or smooth global section of π, where $\pi : G \to G/H$

is the natural projection. (By "smooth" we will always mean C^1 with respect to R.) Obviously, the existence of a continuous (smooth) definable global section of π is equivalent to the existence of a definable continuous (smooth) bijection $f : G \to H \times G/H$ such that the diagram



is commutative. (Here π_2 is the projection map from $H \times G/H$ onto G/H.)

Of course it is not true that such a definable continuous global section exists for all definable $H < G$. A well-known example is $G = SU(2, \mathbb{C})$ with H consisting of all the diagonal matrices. In this case there is no continuous global section of $\pi : G \to G/H$. ($SU(2, \mathbb{C})$ is homeomorphic to the 3-sphere S^3, H is homoeomorphic to the 1-sphere S^1, $SU(2, \mathbb{C})/H$ is homeomorphic to the 2-sphere S^2, and S^3 is not homeomorphic to $S^1 \times S^2$: The 3-sphere S^3 is simply connected, but $S^1 \times S^2$ is not since it has a nontrivial 1-cycle corresponding to S^1.)

If H is a closed subgroup of a real Lie group G, then a continuous global section of the projection $\pi : G \to G/H$ exists if H or G/H is contractible (see, for example, [24, p. 56]).

We prove an analogous result in the category of definable groups and maps. First we consider the case of a definably contractible definable subgroup H of a definable group G, and we prove the existence of a definable continuous global section.

We do not consider the general case when G/H is definably contractible, even though one should be able to find a definable continuous global section in this case as well. Instead we consider the case when H is a normal definable torsion-free subgroup, and in this case we obtain the existence of a definable *smooth* global section. It implies, in particular, that every definable torsion-free group is definably diffeomorphic to R^n.

5.1. Extensions by definably contractible groups.

THEOREM 5.1. *Let H be a definably contractible definable subgroup of a definable group G and π the projection map from G onto the set G/H of the left cosets of H. Then there is a definable continuous global section $\alpha : G/H \to G$ of π.*

Proof. We will need the following claim.

CLAIM 5.2. *Let H be a definable subgroup of a definable group G and π the projection map onto G/H. Then there is a finite cover $U_1, \ldots, U_k$ of G/H*

by definable open sets $U_1, \dots, U_k$ such that for each $i = 1, \dots, k$ there is a definable continuous section $\varphi : U_i \to G$ of π over U_i.[1]

Proof. By definable choice, there is a definable global section $\sigma : G/H \to G$ of π. Using o-minimality, we can find a definable open dense $U \subseteq G/H$ such that σ is continuous on U. By [14, Claim 2.12], finitely many translates of U cover G/H. Let $g_1, \dots, g_k \in G$ be such that $G/H = g_1U \cup \cdots \cup g_kU$.

For $i = 1, \dots, k$ we denote by U_i the set g_iU, and by σ_i the function $x \mapsto g_i\sigma(g_i^{-1}x)$. Obviously, each σ_i is a continuous section of π over U_i. □

Let k be the minimal positive integer such that G/H can be covered by k definable open sets U_i with definable continuous sections $\varphi_i : U_i \to G$ of π over U_i.

If $k = 1$, then there is a continuous definable global section and we are done.

We will assume that $k > 1$ and derive a contradiction. Let U_i, $i = 1, \dots, k$, be definable open subsets covering G/H and φ_i definable continuous sections of π over U_i.

By [3, Chapter 6, Lemma 3.6], there are definable open $V_1 \subseteq U_1$, $V_2 \subseteq U_2$ such that $\mathrm{cl}(V_i) \subseteq U_i$, $i = 1, 2$, and $V_1 \cup V_2 \cup U_3 \cdots \cup U_k = G/H$. To reduce k, and obtain a contradiction, we will construct a definable continuous section of π over $V = V_1 \cup V_2$.

Let $\varphi_{1,2}$ be the function $x \mapsto \varphi_1(x)^{-1}\varphi_2(x)$ on the set $U_1 \cap U_2$. Clearly, $\varphi_{1,2} : U_1 \cap U_2 \to H$.

Let $\psi : V \to G$ be a definable section of π over V. For $i = 1, 2$ we will denote by ψ_i the function $x \mapsto \varphi_i(x)^{-1}\psi(x)$ on V_i. Clearly, $\psi_i : V_i \to H$ and

$$\psi_1(x) = \varphi_{1,2}(x)\psi_2(x) \text{ on } V_1 \cap V_2 \tag{2}$$

The converse is also true: For any two definable functions $\psi_i : V_i \to H$, $i = 1, 2$, satisfying (2), we can define a section $\psi : V \to G$ of π over V by setting $\psi(x) = \varphi_i(x)\psi_i(x)$ on each V_i. This section ψ is continuous on V if and only if both ψ_i are continuous.

Thus we need to find definable continuous functions $\psi_i : V_i \to H$, $i = 1, 2$, such that $\psi_1(x) = \varphi_{1,2}(x)\psi_2(x)$ on $V_1 \cap V_2$. We take $\psi_2(x) \equiv e$ on V_2, where e is the identity element of H, and show that $\varphi_{1,2} \restriction V_1 \cap V_2$ has a definable continuous extension to V_1. Since the closure of $V_1 \cap V_2$ is contained in $U_1 \cap U_2$, $\varphi_{1,2}$ is continuous on $\mathrm{cl}(V_1 \cap V_2)$, and, since G is definably contractible, by [3, Chapter 8, Corollary 3.10], $\varphi_{1,2} \restriction \mathrm{cl}(V_1 \cap V_2)$ has a continuous extension to U_1. □

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[1]The conclusion of the claim states that $\pi : G \to G/H$ is a definable principal fiber bundle with base G/H and bundle group H. In the rest of the proof of the theorem we show that every definable principal fiber bundle with contractible bundle group is trivial.

5.2. Extensions of torsion-free groups. We will need the following technical claim. Although it should be well-known, we could not find a good reference for it.

Claim 5.3. *Let S be a definable smooth manifold definably homeomorphic to an open interval I in R. Then S is definably diffeomorphic to I.*

Proof. Let $\varphi : I \to S$ be a definable homeomorphism. By o-minimality, φ is smooth outside of finitely many points $a_1, \dots, a_n \in I$. Since S is a definable smooth manifold, for every $k = 1, \dots, n$, we can choose a definable open in S subset J_k, containing $\varphi(a_i)$, such that J_k is definably diffeomorphic to an open interval J'_k via a definable map ψ_k, and the intervals $J_1, \dots, J_n$ are pairwise disjoint. Working in these charts we can assume $J_k = J'_k$ and $\psi_k = \mathrm{id}$.

For each k we pick a definable open interval J^0_k containing $\varphi(a_k)$ such that J_k contains the closure of J^0_k. Let $I^0_k = \varphi^{-1}(J^0_k)$. To prove the claim it is sufficient to find for each k a definable bijection $\varphi_k : I^0_k \to J^0_k$ such that the function

$$x \mapsto \begin{cases} \varphi_k(x) & \text{if } x \in I^0_k, \\ \varphi(x) & \text{if } x \in I_k \setminus I^0_k, \end{cases}$$

is C^1. The proof of the existence of such functions is elementary and is left to the reader. □

Corollary 5.4. *If G is a definable one-dimensional ordered group, then G is definably isomorphic to an ordered group G_1 on R whose group operation is smooth as a function from $R \times R$ to R.*

Proof. By the previous claim G is diffeomorphic to R as a smooth manifold. □

Lemma 5.5. *Let $K = \langle K, \oplus \rangle$ be an abelian torsion-free definable one-dimensional group. Then for any definable extension $\alpha : G \to K$ of K there is a definable smooth global section $h : K \to G$ of α.*

Proof. We can assume that the universe of K is R, the neutral element of K is 0, and $\oplus$ is smooth on $R \times R$.

We will denote the group operation of G by $*$ and the identity element of G by e.

By definable choice, there is a definable global section $g : R \to G$ of α. Since g is piece-wise smooth, there is $r \in R$ such that g is smooth on $(r, +\infty)$. Fix $a > r$ and consider the map $g_1(x) : x \mapsto g(a \oplus x) * g(a)^{-1}$. It is easy to see that g_1 is also a global section of α, $g_1(0) = e$, and $g_1(x)$ is smooth on $[0, +\infty)$, i.e., it has a definable smooth extension on an open interval containing $[0, +\infty)$.

Let $h(x)$ be the map

$$x \mapsto \begin{cases} g_1(x) & \text{if } x \geqslant 0, \\ g_1(\ominus x)^{-1} & \text{if } x < 0. \end{cases}$$

Clearly, $h(x)$ is a global section of α, $h(0) = e$, h is continuous on R and smooth on $R \setminus \{0\}$. The function $h(x)$ has also one-sided derivatives at 0, and we are left to show that $h'(0^-) = h'(0^+)$, i.e., we need to show that the differential at zero of $g_1(\ominus x)^{-1}$ equals the differential at zero of $g_1(x)$. Since $g_1(0) = e$, this follows from the Chain Rule and the fact that for any definable group $\langle H, \cdot, e\rangle$ the differential of the map $x \mapsto x^{-1}$ at e is $-\mathrm{Id}$ [2]. □

THEOREM 5.6. *Let K be a torsion-free definable group. Then for any definable extension $\alpha : G \to K$ of K there is a definable smooth global section of α.*

Proof. We proceed by induction on $\dim(K)$. The one-dimensional case is covered in Claim 5.5. Thus we can assume that $\dim(K) > 1$.

Let $\alpha : G \to K$ be a definable extension of K by H. By Corollary 2.12, there is a definable normal subgroup K_1 of K such that the dimension of $K_0 = K/K_1$ is one. Let $G_1 = \alpha^{-1}(K_1)$ and β be the restriction of α to G_1. Then $\beta : G_1 \to K_1$ is a definable extension of K_1, and by the induction hypothesis, it has a definable smooth global section $f : K_1 \to G_1$.

Let $\pi : K \to K_0$ be the projection map and $\gamma = \pi \circ \alpha$. Then $\gamma : G \to K_0$ is a definable extension of K_0. Since $\dim(K_0) = 1$ it has a smooth definable global section $g : K_0 \to G$.

Consider the function $h : K \to K$ defined as $h : x \mapsto \alpha(g(\pi(x)))$. It is not hard to see that h is a definable smooth global section of π, and $xh(x)^{-1} \in K_1$.

The function $x \mapsto g(\pi(x)) \cdot f(xh(x)^{-1})$ is a definable smooth global section of α. □

COROLLARY 5.7. *Let G be a definable torsion-free group of dimension n. Then G is definably diffeomorphic to R^n.*

Proof. We proceed by induction on $\dim(G)$. If G has dimension 1, then it is a definable one-dimensional ordered group and is definably diffeomorphic to R.

Assume $\dim(G) = n + 1$. Then, by Corollary 2.12, G has a definable normal subgroup H with $\dim(H) = n$. By the induction hypothesis, both H and G/H are definably diffeomorphic to R^n and R, respectively. Now we can apply Theorem 5.6. □

[2] Since $e \cdot x = x \cdot e = x$, we have that the differential of the $x \cdot y$ at (e, e) is $dx + dy$. Now apply the implicit function theorem.

COROLLARY 5.8. *Let G be a definable group. The following conditions are equivalent.*

(1) *G is torsion-free.*
(2) *G is definably diffeomorphic to R^n.*
(3) *The Euler characteristic of G is $+1$ or -1.*

Appendix A. A theorem of Chevalley

We will now show how the results of Section 4 yield "a semialgebraic proof" to the classical theorem of Chevalley for abelian algebraic groups over fields of characteristic zero (see [21]).

We believe that the method suggested below could be applied in other cases as well to translate semi-algebraic information into the algebraic category. (Note that usually one deduces *real algebraic* information from the semi-algebraic one but this is not our point of view here.) We begin with some preliminaries.

Let K be an algebraically closed field of characteristic zero. We fix a maximal real closed subfield $R \subseteq K$, a square root of -1, which we denote by i, and identify K with R^2 in an obvious way: $z = a + bi$, for $a, b \in R$. We assume that R is ω^+-saturated.

Every subset of R^{2n} can now be identified with a subset of K^n via the map $(a_1, \dots, a_n, b_1, \dots b_n) \mapsto (a_1 + ib_1, \dots, a_n + ib_n)$. For $(\bar{a}, \bar{b}) = (a_1, \dots, a_n, b_1, \dots, b_n) \in R^{2n}$ and $A \subseteq R$, we write $\dim_R(\bar{a}, \bar{b})/A$ for its semialgebraic dimension (i.e., $\operatorname{tr.deg}(\mathbb{Q}(\bar{a}, \bar{b})/\mathbb{Q}(A))$) and $\dim_K(a_1 + ib_1, \dots, a_n + ib_n)/A$ for its algebraic dimension (i.e., $\operatorname{tr.deg}(\mathbb{Q}(a_1 + ib_1, \dots, a_n + ib_n)/\mathbb{Q}(A))$.

For a semi-algebraic set $S \subseteq R^{2n}$, we write $\dim_R(S)$ for its semi-algebraic dimension. If S happens to be also an algebraic (or constructible) subset of K^n, we write $\dim_K S$ for its dimension in the sense of the algebraically closed field K. Notice that since R is assumed to be sufficiently saturated, if $S \subseteq R^{2n}$ is semialgebraic and definable over $A \subseteq R$, then

$$\dim_R(S) = \max\{\dim_R(\bar{a}, \bar{b})/A : (\bar{a}, \bar{b}) \in S\}$$

and if $S \subseteq K^n$ is constructible and definable over A, then

$$\dim_K(S) = \max\{\dim_K(a_1 + ib_1, \dots, a_n + ib_n)/A : \bar{a} + i\bar{b} \in S\}.$$

Given $S \subseteq R^{2n}$, we write $Z_K(S)$ for the Zariski closure of S inside K^n. For example, if $S \subseteq R^2$ is any semialgebraic infinite set, then $Z_K(S) = K$.

The following lemma will be needed in order to transfer results from the semi-algebraic context into the algebraic one:

LEMMA A.1. *Let K be an algebraically closed field, R a maximal real closed subfield of K.*

(1) *Let $S \subseteq R^{2n}$ be a semi-algebraic set, $Z_K(S) \subseteq K^n$. Then*

$$\frac{1}{2}\dim_R(S) \leqslant \dim_K(Z_K(S)) \leqslant \dim_R(S).$$

(2) *Let V be an algebraic variety over K, and let $S \subseteq V(K)$ be a semi-algebraic subset of the K-points of V (namely, a subset of $V(K)$ which is definable in $\langle R, +, \cdot\rangle$). Then*

$$\frac{1}{2}\dim_R(S) \leqslant \dim_K(Z_K(S)) \leqslant \dim_R(S).$$

Proof. First notice that this lower bound is really optimal. Indeed, consider a semi-algebraic subset of R^2 of R-dimension 2. Then, $Z_K(S)$ is equal to K and thus has K-dimension 1.

(1) Consider now a semi-algebraic $S \subseteq R^{2n}$, $\dim_R S = k$, and assume that it is definable over $A \subseteq R$. Let $(\bar{a}, \bar{b})$ be a generic element in S over A. Let F be the field generated by A. Then the transcendence degree of $F(a_1, \dots, a_n, b_1, \dots, b_n)$ over F equals k.

Consider now S as a subset of K^n. Its elements are $\{(x_1+iy_1, \dots, x_n+iy_n) : (\bar{x}, \bar{y}) \in S\}$. Since the field $F(a_1+ib_1, \dots, a_n+ib_n)$ is contained in the algebraic closure of $F(a_1, \dots, a_n, b_1, \dots, b_n)$, we have $\dim_K(a_1 + ib_1, \dots, a_n + ib_n)/F \leqslant k$.

Let $L \subseteq K^n$ be the smallest Zariski closed subset of K^n which contains $(a_1 + ib_1, \dots, a_n + ib_n)$, and is defined over A (sometimes called the locus of $(a_1 + ib_1, \dots, a_n + ib_n)$ over A). We have $\dim_K L = \dim_K(a_1 + ib_1, \dots, a_n + ib_n/A) \leqslant k$.

We therefore showed that every element of S whose R-dimension is k, is contained in a Zariski closed subset of K^n whose K-dimension is not greater than k.

By the compactness theorem, we can find finitely many Zariski closed subsets of K^n, defined over A, each of K-dimension not greater than k, which cover S, up to a subset of S of R-dimension less than k. Applying induction on $\dim_R(S)$ we obtain a Zariski closed subset of K^n which contains S and its K-dimension is not greater than k.

As for the other direction, we point out that if $V \subseteq K^n$ is an algebraic (or even constructible) set, then we have $\dim_K V = 2\dim_R V$. Since $S \subseteq Z_K(S)$, we must clearly have $2\dim_K(Z_K(S)) \geqslant \dim_R(S)$.

(2) follows from (1), by working in the charts which make up V. □

Let us now consider the consequences of the above observation for algebraic groups. We denote by K_a the group $\langle K, +\rangle$ and by K_m the group $\langle K^*, \times\rangle$.

LEMMA A.2. *Let G be an algebraic group over K, and let H be a semi-algebraic, definably connected subgroup of G such that $\dim_R H = 1$. Then:*

(i) *$Z_K(H)$ is an algebraic abelian subgroup of G whose algebraic dimension is* 1.
(ii) *If H is (semi-algebraically) isomorphic to R_a, then $Z_K(H)$ is (algebraically) isomorphic to K_a.*
(iii) *If H is (semi-algebraically) isomorphic to R_m, then $Z_K(H)$ is algebraically isomorphic to K_m.*

Proof. (i) Follows immediately from Lemma A.1.

(ii) Assume that f is a semi-algebraic isomorphism between R_a or R_m and H. The graph of f, call it S_f, is a semi-algebraic abelian subgroup of the algebraic group G_1, where $G_1 = K_a \times G$ or $G_1 = K_m \times G$. We have $\dim_R(S_f) = 1$. Therefore, by the above lemma, its Zariski closure in G_1, denoted by S_1, is an abelian subgroup of G_1 whose K-dimension is 1 as well. Furthermore, S_1 must be a connected algebraic group, since otherwise S_f will be contained in S_1^0, its algebraic connected component. Let H_1 be the projection of S_1 onto G. H_1 is an abelian connected subgroup of G containing H, such that $\dim_K H_1 = 1$. It is easy to see that $H_1 = Z_K(H)$.

Assume first that $\mathrm{dom}(f) = R_a$. The intersection of S_1 with $K_a \times \{e\}$ is trivial and its projection on K must equal K. The intersection of S_1 with $\{0\} \times H_1$ is finite (by dimension considerations), hence S_1 is a surjective, finite-to-one homomorphism from H_1 onto K_a. Since K_a has no torsion, and H_1 is connected, S_1 is an isomorphism between H_1 and K_a.

Assume now that $\mathrm{dom}(f) = R_m$. The intersection of S_1 with $K_m \times \{e\}$ is a finite group F. We may replace K_m by K_m/F, which is again isomorphic, as an algebraic group, to K_m. As before, we now have an algebraic, surjective, finite-to-one homomorphism from H_1 onto K_m. It follows that H_1 is isomorphic to K_m. □

Before turning to Chevalley's Theorem, we need one more lemma.

LEMMA A.3. *Let X be an algebraic variety over K. If X is definably compact (in its o-minimal group topology), then X is a complete variety. In particular, if G is an algebraic group which is definably compact, then G is an abelian variety.*

Proof. The following is easy to verify:

If X is a definably compact set in an o-minimal structure and Y is any other definable set, then the projection map $\pi : X \times Y \to Y$ is a closed map (with respect to the o-minimal topology).

Consider now any algebraic variety. It follows from quantifier elimination that if $X \subseteq V$ is a constructible set (i.e., definable in $\langle K, +, \cdot \rangle$) and X is relatively closed in V, in the o-minimal topology, then X is Zariski closed in V.

The above two observations imply that if X is an algebraic variety which is definably compact, then it is a complete variety. □

THEOREM A.4. *Let G be an abelian algebraic group over an algebraically closed field K of characteristic zero. Then there is an algebraic subgroup H which is isomorphic to a linear group over K, such that G/H is an abelian variety.*

Proof. We may assume that G is a connected algebraic group. (Applying the theorem to the connected component of G will yield the general result.)

Let L be the algebraic subgroup of G generated by all algebraic subgroups of G which are isomorphic to K_a or K_m. We claim that G/L is definably compact, and thus, by Lemma A.3, is an abelian variety. Indeed, if not, then by Theorem 4.13 there is a 1-dimensional semi-algebraic subgroup of G which is semi-algebraically isomorphic to R_a or R_m and has trivial intersection with L. Applying Lemma A.2 to this subgroup we obtain an algebraic copy of K_a or K_m which is not contained in L.

Since G is abelian, it is not hard to see that L is linear (it is only here that we need to use the commutativity of G). □

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