Illinois Journal of Mathematics
Volume 49, Number 4, Winter 2005, Pages 1133–1143
S 0019-2082

MONOTONICITY RESULTS FOR THE PRINCIPAL EIGENVALUE OF THE GENERALIZED ROBIN PROBLEM

TIZIANA GIORGI AND ROBERT G. SMITS

ABSTRACT. We study domain monotonicity of the principal eigenvalue $\lambda_1^\Omega(\alpha)$ corresponding to $\Delta u = \lambda(\alpha)\, u$ in Ω, $\frac{\partial u}{\partial \nu} = \alpha\, u$ on $\partial\Omega$, with $\Omega \subset \mathcal{R}^n$ a $C^{0,1}$ bounded domain, and α a fixed real. We show that contrary to intuition domain monotonicity might hold if one of the two domains is a ball.

1. Introduction

We are interested in studying the domain monotonicity properties of the principal eigenvalue, $\lambda_1^\Omega(\alpha)$, of the following eigenvalue problem:

$$\begin{cases} \Delta u = \lambda(\alpha)\, u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \alpha\, u & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where throughout the regime $-\infty < \alpha < \infty$ and ν denotes the external normal vector.

Following [7], we say that a bounded domain $\Omega \subset \mathcal{R}^n, n \geq 2$, and its boundary are of class $C^{k,\beta}$, $0 \leq \beta \leq 1$, if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one-to-one mapping Ψ of B onto $D \subset \mathcal{R}^n$ such that:

(i) $\Psi(B \cap \Omega) \subset \mathcal{R}^n_+$;
(ii) $\Psi(B \cap \partial\Omega) \subset \partial\mathcal{R}^n_+$;
(iii) $\Psi \in C^{k,\beta}(B), \Psi^{-1} \in C^{k,\beta}(D)$.

Unless otherwise stated, we will assume Ω to be a $C^{0,1}$ bounded domain.

For $\alpha < 0$ one has $\lambda_1^\Omega(\alpha) < 0$, and, via the Rayleigh principle,

$$\lambda_1^\Omega(\alpha) = \sup_{u \in H^1(\Omega;\mathcal{R})} \frac{\alpha \int_{\partial\Omega} u^2\, d\sigma_x - \int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2\, dx}. \tag{2}$$

Received March 22, 2005; received in final form December 5, 2005.
2000 *Mathematics Subject Classification.* Primary 35P15. Secondary 35J25.
The first author was supported by the National Science Foundation-funded ADVANCE Institutional Transformation Program at NMSU, fund # NSF0123690. The second author was partially supported by a grant of the National Security Agency, grant # H98230-05-1-0060.

It is well-known that $\lambda_1^\Omega(\alpha)$ is simple, and that an eigenfunction $\phi_1^\Omega(\alpha; x)$ can be chosen with a single sign and normalized by letting $\int_\Omega \left(\phi_1^\Omega\right)^2 dx = 1$. In addition, as $\alpha \to -\infty$, $\lambda_1^\Omega(\alpha)$ converges to the principal eigenvalue, $\lambda_D^\Omega < 0$, for the Dirichlet problem:

$$\begin{cases} \Delta u = \lambda\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

For $\alpha = 0$, we are in the case of the Neumann problem, and $\lambda_1^\Omega(0) = 0$ is achieved by taking u identically equal to a constant.

For $\alpha > 0$, one continues to have a single-signed eigenfunction corresponding to a principal eigenvalue, which verifies the variational formulation (2); see [9], [6].

Traditionally, the study of $\lambda_1^\Omega(\alpha)$ for α positive is separated from the one for α negative, since they arise in different application contexts, and for example the limits of $\lambda_1^\Omega(\alpha)$ as $\alpha \to \infty$ or $\alpha \to -\infty$ depend differently on the geometry of the domain Ω. When $\alpha < 0$, the value of $-\lambda_1^\Omega(\alpha)$ corresponds to the fundamental frequency for an elastically supported membrane [8], [11], and is also seen as the exponential decay rate for heat in a non-perfectly insulated region. When $\alpha > 0$, the eigenvalue $\lambda_1^\Omega(\alpha)$ describes a growth rate for reaction-diffusion models with nonlinear boundary sources [9]. (For applications related to superconductivity see also [6].)

A classical result in the literature tells us that the Dirichlet eigenvalue λ_D^Ω satisfies a so-called *domain monotonicity property*, namely

$$\text{if} \quad \Omega_1 \subseteq \Omega_2 \quad \text{then} \quad \lambda_D^{\Omega_1} - \lambda_D^{\Omega_2} \leq 0. \tag{4}$$

On the other hand, for our principal eigenvalue $\lambda_1^\Omega(\alpha)$ any kind of domain monotonicity property fails in general for any value of α and even for convex domains, as intuition suggests and as the following constructions show.

We start by noticing that

$$\lim_{\alpha \to 0} \frac{\lambda_1^\Omega(\alpha)}{\alpha} = \frac{|\partial\Omega|}{|\Omega|}. \tag{5}$$

The above limit is a consequence of Theorem 2.1 in [6], where the limit from the right is obtained, and the following two inequalities. The first one, which holds for $\alpha < 0$, is due to Sperb [13]:

$$\lambda_1^\Omega(\alpha) \leq \frac{1}{\frac{1}{\mu_2} + \frac{|\Omega|}{\alpha\, |\partial\Omega|}}, \qquad \alpha < 0, \tag{6}$$

where $\mu_2 < 0$ denotes the second eigenvalue of the Neumann problem for the positive Laplacian. (Note that we quote the Sperb result rewritten in our notation, where α is negative.) The second one is trivially obtained by taking in (2) as test-function $u \equiv 1$:

$$\lambda_1^{\Omega}(\alpha) \geq \alpha \frac{|\partial\Omega|}{|\Omega|}, \quad \text{for any } \alpha \neq 0. \tag{7}$$

For $\alpha > 0$ fixed, inequality (7) shows that one can find a sequence of regions $B(\mathbf{0},1) \subset \Omega_n \subset B(\mathbf{0},2)$ with $|\partial\Omega_n| \to \infty$, so that eventually $\lambda_1^{\Omega_n}(\alpha) > \lambda_1^{B(\mathbf{0},1)}(\alpha) > \lambda_1^{B(\mathbf{0},2)}(\alpha) > 0$, and thus in general monotonicity with respect to domains fails for any positive α.

For $\alpha < 0$ fixed, due to the scaling property we can find an $\epsilon > 0$ for which $\lambda_D^{B(\mathbf{0},1+\epsilon)} < \lambda_1^{B(\mathbf{0},1)}(\alpha) < \lambda_1^{B(\mathbf{0},2)}(\alpha) < 0$, and following Dancer and Daners [5] we can construct Ω_n with $|\partial\Omega_n| \to \infty$ and $B(\mathbf{0},1) \subset \Omega_n \subset B(\mathbf{0},1+\epsilon)$, so that

$$\lim_{n\to\infty} \lambda_1^{\Omega_n}(\alpha) = \lambda_D^{B(\mathbf{0},1+\epsilon)}.$$

This implies that for fixed $\alpha < 0$ one eventually has $\lambda_1^{\Omega_n}(\alpha) < \lambda_1^{B(\mathbf{0},1)}(\alpha) < \lambda_1^{B(\mathbf{0},2)}(\alpha) < 0$. Hence in general monotonicity with respect to domains fails for both positive and negative α. On the other hand, balls are smooth enough to expect monotonicity for sub-domains for all $\alpha \neq 0$, as well as for certain classes of containing regions. However, while one might hope that balls can be replaced by simple objects such as convex sets, this is not the case as our next construction shows.

By decomposing a convex polygon P, containing a ball B, into triangles with a vertex at the center of B, we can see that

$$|\partial P|/|P| \leq |\partial B|/|B|,$$

with equality if and only if all faces of P are tangent to B. We take a square S that circumscribes a ball B, and consider a polygon C obtained by cutting from S a corner in a way so that $B \subset C \subset S$, and with the new side of C not touching B, so that

$$|\partial C|/|C| < |\partial B|/|B| = |\partial S|/|S|. \tag{8}$$

For $\alpha < 0$, the inequalities (8), (6) and the limit (5) then imply that $\lambda_1^C(\alpha) - \lambda_1^S(\alpha)$ is positive for $\alpha < 0$ small. On the other hand, since domain monotonicity holds for the Dirichlet problem, for α large negative $\lambda_1^C(\alpha) - \lambda_1^S(\alpha)$ is negative.

For $\alpha > 0$, since C has a face which is not tangent to B and has corners, the work of Lacey et al. [9] implies that

$$\lambda_1^C(\alpha) - \lambda_1^B(\alpha) > 0$$

for large α. At the same time, the inequality (8) with (5) shows that for $\alpha > 0$ small we have

$$\lambda_1^C(\alpha) - \lambda_1^B(\alpha) < 0.$$

Therefore

$$\lambda_1^C(\alpha) - \lambda_1^B(\alpha)$$

switches sign.

In the positive direction, in this work we show, among other results, that the domain monotonicity property of the Dirichlet problem can be partially recovered if Ω_2 is a ball. In particular, we prove the following results.

Let $B \subset \mathcal{R}^n, n \geq 2$, *be a ball and* $\Omega \subset B$. *If* $\alpha < 0$ *then* $\lambda_1^{\Omega}(\alpha) \leq \lambda_1^{B}(\alpha) < 0$, *while if* $\alpha > 0$ *we know* $\lambda_1^{\Omega}(\alpha) \geq \lambda_1^{B}(\alpha) > 0$. (See Theorem 1 and Theorem 2.5 in [6].)

Let $\alpha < 0$. *If* $\Omega \subset \mathcal{R}^n, n \geq 2$, *is a convex domain that contains a ball* B, *then*

$$\lambda_1^{B}(\alpha) \leq \lambda_1^{\Omega}(\alpha) < 0.$$

(See Theorem 2 and Corollary 3.)

Let $\alpha < 0$ *and* $\Omega_1 \subset B \subset \Omega_2$, *where* $B \subset \mathcal{R}^n, n \geq 2$ *is a ball and* Ω_1, Ω_2 *are convex domains, we have that* $\lambda_1^{\Omega_1}(\alpha) \leq \lambda_1^{\Omega_2}(\alpha) < 0$. (See Corollary 4.)

Remark 1. In general, Corollary 4 is not true if a suitable ball separating the two convex domains does not exist, as evidenced in the discussion above. A similar effect regarding the Neumann heat kernel is presented in [3], where Bass and Burdzy show that one can find convex domains $\Omega_1 \subset \Omega_2$, points $x, y \in \Omega_1$, and a time t for which $P_N^{\Omega_1}(t, x, y) \leq P_N^{\Omega_2}(t, x, y)$. In words, this says that contrary to expectation it is more likely that a Brownian motion with reflection will move from x to y within the larger domain Ω_2.

Remark 2. For $\alpha > 0$ a weaker version of Corollary 3 holds, provided in our Theorem 2.7 in [6], namely:

Let $\alpha > 0$. *If* $P \subset \mathcal{R}^n, n \geq 2$, *is a convex polyhedron that circumscribes a ball* B, *then*

$$\lambda_1^{\Omega}(\alpha) \geq \lambda_1^{B}(\alpha) > 0.$$

The set C in (8) is a convex polyhedron, so our previous construction implies that for $\alpha > 0$ the hypothesis of circumscribing can not be relaxed in general.

We summarize part of our results in Figure 1, where we show the relation between the eigenvalues $\lambda_1^{\Omega}(\alpha)$ for the square $[-1, 1] \times [-1, 1]$, the ball it circumscribes, $B(\mathbf{0}, 1)$, and the ball $B(\mathbf{0}, 2)$. Our work implies that a similar picture holds for every dimension and every convex polyhedron and corresponding balls.

2. Domain monotonicity results

Theorem 1. *Let* $B \subset \mathcal{R}^n, n \geq 2$, *denote a ball. If* $\Omega \subset B$ *is a* $C^{0,1}$ *domain, then for* $\alpha < 0$ *one has* $\lambda_1^{\Omega}(\alpha) \leq \lambda_1^{B}(\alpha) < 0$, *while for* $\alpha > 0$ *one has* $\lambda_1^{\Omega}(\alpha) \geq \lambda_1^{B}(\alpha) > 0$.

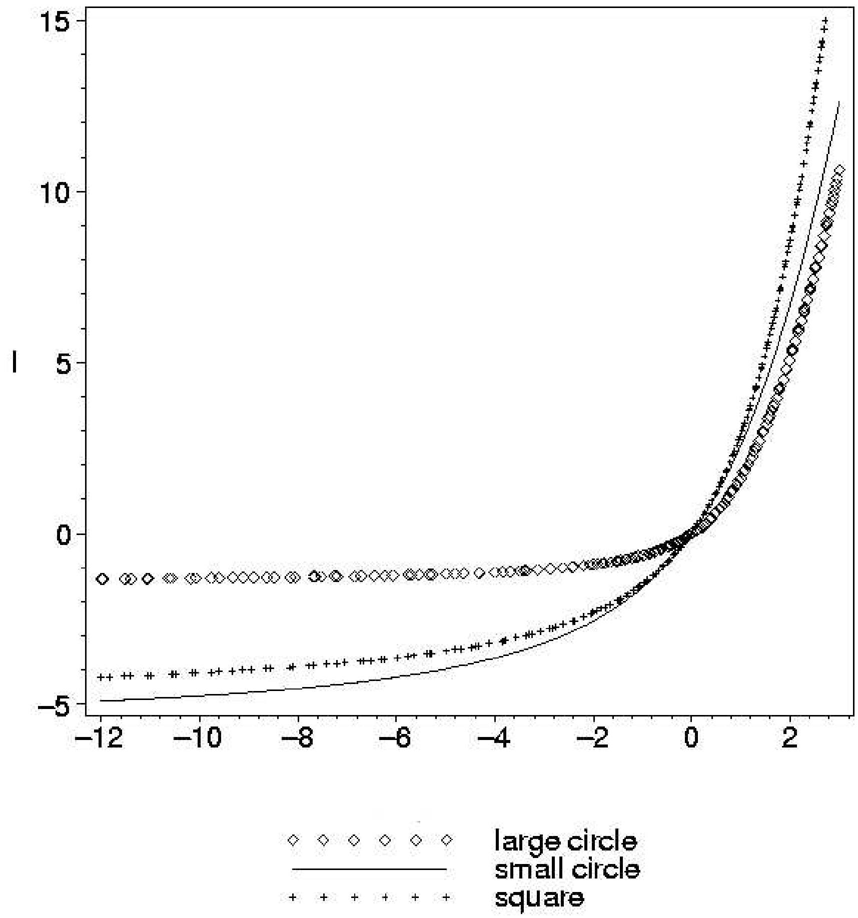

FIGURE 1. Graphs $y = \lambda_1^\Omega(\alpha)$: the larger ball contains the square, which circumscribes the smaller ball

Proof. By scaling and translation, we can assume $B = B(\mathbf{0}, 1)$. It is well-known that the eigenfunction $\phi_1^B(\alpha; x)$ is radially symmetric and single signed, so that it can be taken positive, and considered as a function of its radial component $r = |x|$, say $\phi(r) := \phi_1^B(\alpha; x)$.

Case $\alpha < 0$: Since we have $\Delta\phi_1^B = \lambda_1^B(\alpha)\,\phi_1^B < 0$ that is ϕ_1^B superharmonic, we know that ϕ is decreasing as a function of r. Moreover, using the fact that ϕ_1^B is log-concave [12], we deduce that

$$\frac{\partial}{\partial r} \log \phi(r) = \frac{\phi'(r)}{\phi(r)} < 0$$

is decreasing, and hence has its minimum in B at $r = 1$, where it equals $\alpha < 0$.

For $x \in \partial\Omega$, we define

$$\alpha^*(x) = \frac{\frac{\partial \phi_1^B}{\partial \nu}(\alpha; x)}{\phi_1^B(\alpha; x)},$$

where $\nu = \nu(x)$ is the normal at x to $\partial\Omega$. Since

$$\nabla \phi_1^B(\alpha; x) = \phi'(r)\, \frac{x}{r},$$

$\phi'(r) < 0$, and

$$\left| \frac{x}{r} \cdot \nu \right| \le 1,$$

we have that

$$\frac{\partial \phi_1^B}{\partial \nu}(\alpha; x) = \phi'(r)\, \frac{x}{r} \cdot \nu \ge \phi'(r);$$

from $\phi > 0$ we conclude

$$\alpha^*(x) \ge \frac{\phi'(r)}{\phi(r)} \ge \frac{\phi'(1)}{\phi(1)} = \alpha.$$

The function $\phi_1^B(\alpha; x)$ is a classical solution to

$$\begin{cases} \Delta \phi_1^B(\alpha; x) = \lambda_1^B(\alpha)\, \phi_1^B(\alpha; x) & \text{in } \Omega, \\ \frac{\partial \phi_1^B(\alpha; x)}{\partial \nu} = \alpha^*(x)\, \phi_1^B(\alpha; x) & \text{on } \partial\Omega, \end{cases}$$

and since Ω is a $C^{0,1}$ domain, classical results say that $\lambda_1^B(\alpha)$ is the principal eigenvalue of the above eigenvalue problem, and it is given by the Rayleigh quotient. Therefore, we have:

$$\begin{aligned} \lambda_1^B(\alpha) &= \frac{\int_{\partial\Omega} \alpha^*(x)\, \left(\phi_1^B(\alpha; x)\right)^2\, d\sigma_x - \int_\Omega |\nabla \phi_1^B(\alpha; x)|^2\, dx}{\int_\Omega \left(\phi_1^B(\alpha; x)\right)^2\, dx} \\ &= \sup_{u \in H^1(\Omega; \mathcal{R})} \frac{\int_{\partial\Omega} \alpha^*(x)\, u^2(x)\, d\sigma_x - \int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega u^2(x)\, dx} \\ &\ge \sup_{u \in H^1(\Omega; \mathcal{R})} \frac{\alpha \int_{\partial\Omega} u^2(x)\, d\sigma_x - \int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega u^2(x)\, dx} = \lambda_1^\Omega(\alpha). \end{aligned}$$

Case $\alpha > 0$: In this case, we have $\Delta \phi_1^B = \lambda_1^B(\alpha)\, \phi_1^B > 0$, that is, ϕ_1^B is subharmonic, and ϕ is an increasing function of r. We set $\eta = \frac{n}{2} - 1$ and

$\sqrt{\lambda} = \sqrt{\lambda_1^B(\alpha)}$. Then, using the explicit representation in terms of modified Bessel functions of the eigenfunction of a circle, we have that

$$\phi(r) = \frac{I_\eta\left(\sqrt{\lambda}\, r\right)}{\left(\sqrt{\lambda}\, r\right)^\eta},$$

and the recursive relation $\frac{d}{dr}\left(r^{-\eta} I_\eta(r)\right) = r^{-\eta} I_{\eta+1}(r)$ (see [10, p. 110]) implies that $(\phi'(r)/\phi(r))' > 0$ (see equation (15), p. 242, in [1]).

For $x \in \partial\Omega$, and $\alpha^*(x)$ defined as before, since $\phi'(r) > 0$, we now conclude that

$$\alpha^*(x) \leq \frac{\phi'(r)}{\phi(r)} \leq \frac{\phi'(1)}{\phi(1)} = \alpha,$$

and proceeding as in the previous case we obtain $\lambda_1^B(\alpha) \leq \lambda_1^\Omega(\alpha)$.

Remark 3. The sketch of the proof of Theorem 1 for $\alpha > 0$ is given for the benefit of the reader, and was obtained by the authors in [6].

We have seen in the introduction that in general domain monotonicity fails if we allow the larger domain to have arbitrary geometry, and even fails for general convex domains. We do have a positive result for general larger convex domains if the smaller domain is a ball.

Theorem 2. *Let $\alpha < 0$ and let $B \subset \mathcal{R}^n, n \geq 2$, denote a ball. If P is a convex polyhedron and $B \subset P$, then*

$$\lambda_1^B(\alpha) \leq \lambda_1^P(\alpha) < 0.$$

Proof. From scaling and up to translations, we can assume $B \equiv B(\mathbf{0}, 1)$ and take as eigenfunction the radially symmetric function

$$\phi_1^{B(\mathbf{0},1)}(\alpha; x) = \frac{J_{\frac{n}{2}-1}\left(\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\quad |x|\right)}{\left(\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\quad |x|\right)^{\frac{n}{2}-1}},$$

where J_η denotes the Bessel function of order η.

As remarked in the proof of Theorem 1, the eigenfunction $\phi_1^{B(\mathbf{0},1)}$ is radially symmetric everywhere, and strictly positive and radially decreasing on $B(\mathbf{0}, 1)$. The eigenvalue $\lambda_1^{B(\mathbf{0},1)}(\alpha)$ satisfies the implicit equation

$$\frac{-\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\, J_{\frac{n}{2}}\left(\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\right)}{J_{\frac{n}{2}-1}\left(\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\right)} = \alpha \tag{9}$$

(see [10]), which for $n = 1$ and $(-1, 1)$ reduces to

$$-\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\,\tan\left(\sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}\right) = \alpha.$$

Moreover, $\phi_1^{B(\mathbf{0},1)}$ is analytic on all of $\mathcal{R}^n$ and verifies

$$\Delta\phi_1^{B(\mathbf{0},1)}(\alpha;x) = \lambda_1^{B(\mathbf{0},1)}(\alpha)\,\phi_1^{B(\mathbf{0},1)}(\alpha;x) \quad \text{in } \mathcal{R}^n. \tag{10}$$

To prove our theorem, we look at the domain $\widetilde{P} = P \cap \{\phi_1^{B(\mathbf{0},1)}(\alpha;x) > 0\}$. Due to convexity of the domains and the fact that the eigenfunction is radially decreasing up to the values when is zero, there are three possible distinct cases, which we consider separately.

Case 1: If $\partial\widetilde{P} \cap \partial P \neq \emptyset$ and $\partial\widetilde{P} = \partial P$, that is, if $\{\phi_1^{B(\mathbf{0},1)}(\alpha;x) > 0\} \supset P$, then $\widetilde{P} = P$, and we can define

$$\alpha^*(x) := \frac{\frac{\partial\phi_1^{B(\mathbf{0},1)}}{\partial\nu}(\alpha;x)}{\phi_1^{B(\mathbf{0},1)}(\alpha;x)} < 0 \text{ for a.e. } x \in \partial P,$$

where $\nu = \nu(x)$ is the normal at x to ∂P. Via integration by parts we get

$$\lambda_1^{B(\mathbf{0},1)}(\alpha) = \frac{\int_{\partial P}\alpha^*(x)\,\left(\phi_1^{B(\mathbf{0},1)}(\alpha;x)\right)^2 d\sigma_x - \int_P |\nabla\phi_1^{B(\mathbf{0},1)}(\alpha;x)|^2\,dx}{\int_P\left(\phi_1^{B(\mathbf{0},1)}(\alpha;x)\right)^2\,dx}.$$

If we can show that on ∂P one has $\alpha^*(x) \leq \alpha < 0$, then the above will yield

$$\begin{aligned}\lambda_1^{B(\mathbf{0},1)}(\alpha) &\leq \frac{\int_{\partial P}\alpha\,\left(\phi_1^{B(\mathbf{0},1)}(\alpha;x)\right)^2 d\sigma_x - \int_P |\nabla\phi_1^{B(\mathbf{0},1)}(\alpha;x)|^2\,dx}{\int_P\left(\phi_1^{B(\mathbf{0},1)}(\alpha;x)\right)^2\,dx}\\ &\leq \sup_{u\in H^1(P;\mathcal{R})}\frac{\alpha\int_{\partial P}u^2(x)\,d\sigma_x - \int_P|\nabla u(x)|^2\,dx}{\int_P u^2(x)\,dx} = \lambda_1^P(\alpha),\end{aligned}$$

which is the desired inequality.

Recalling that we are working on a convex domain where $\phi_1^{B(\mathbf{0},1)}(\alpha;x) > 0$, we can use that $\phi_1^{B(\mathbf{0},1)}(\alpha;x)$ is radially decreasing in P. For any $x \in \partial P$, we consider the hyperplane T_x tangent to P at x, and we define by $\widetilde{T_x}$ the closest hyperplane parallel to T_x and tangent to $B(\mathbf{0},1)$. We then consider the intersection point, $\tilde{x}$, between $\widetilde{T_x}$ and the ray from the center of the ball through x (note that $|\tilde{x}| \geq 1$). Set $\eta = \frac{n}{2} - 1$, $\sqrt{\lambda} = \sqrt{-\lambda_1^{B(\mathbf{0},1)}(\alpha)}$, and $\phi(r) = \phi_1^{B(\mathbf{0},1)}(\alpha;x)$, so that the eigenfunction is simply

$$\phi(r) = \frac{J_\eta\left(\sqrt{\lambda}\,r\right)}{\left(\sqrt{\lambda}\,r\right)^\eta}. \tag{11}$$

In this notation, using the log-concavity of the eigenfunction for a ball (as in Theorem 1), we know that $\frac{\phi'(r)}{\phi(r)}$ is radially decreasing. Therefore, if $\tilde{r} = |\tilde{x}|$ we have that

$$\alpha^*(x) = \frac{\frac{\partial \phi_1^{B(\mathbf{0},1)}}{\partial \nu}(\alpha; x)}{\phi_1^{B(\mathbf{0},1)}(\alpha; x)} \leq \frac{\phi'(\tilde{r})}{\phi(\tilde{r})} \frac{1}{\tilde{r}}.$$

This allows us to reduce the problem of showing $\alpha^*(x) \leq \alpha$ for $x \in \partial P$ to proving

$$\frac{\phi'(r)}{r\,\phi(r)} \leq \alpha \qquad \text{for } 1 \leq r < \text{first positive zero of } \phi(r).$$

Since at $r = 1$,

$$\frac{\phi'(r)}{r\,\phi(r)} = \alpha < 0,$$

we will have the wanted result if

$$\frac{\phi'(r)}{r\,\phi(r)}$$

is decreasing as a function of $r > 1$. If we consider in (11) the change of variables $s = \sqrt{\lambda} r$, the previous property is shown if we derive that

$$\frac{\psi'(s)}{s\,\psi(s)}$$

is decreasing for $0 < s < j_{\eta,1}$, where $\psi(s) = s^{-\eta} J_\eta(s)$ and $j_{\eta,1}$ is the first positive zero of $J_\eta(s)$.

The following recursive relation [10, p. 100] verified by the Bessel functions

$$\frac{d}{ds}\left(s^{-\eta}\, J_\eta(s)\right) = -s^{-\eta}\, J_{\eta+1}(s)$$

changes the problem to proving that

$$\frac{\psi'(s)}{s\,\psi(s)} = \frac{\frac{d}{ds}\left(s^{-\eta}\, J_\eta(s)\right)}{s\, s^{-\eta}\, J_\eta(s)} = -\frac{s^{-\eta}\, J_{\eta+1}(s)}{s\, s^{-\eta}\, J_\eta(s)} = -\frac{J_{\eta+1}(s)}{s\, J_\eta(s)}$$

is decreasing, or equivalently that $J_{\eta+1}(s)/(s\, J_\eta(s))$ is increasing for $0 < s < j_{\eta,1}$, where $j_{\eta,1}$ is the first positive zero of $J_\eta(s)$. But this last statement is exactly Lemma 2.4 in the second proof of the Payne-Pólya-Weinberger Conjecture by Ashbaugh and Benguria [2] (with appropriate changes in notation), hence the theorem follows.

Case 2: If $\partial \widetilde{P} \cap \partial P \neq \emptyset$ and $\partial \widetilde{P} \cap \partial P \neq \partial P$, then $\partial \widetilde{P} \cap \partial B(\mathbf{0},1) \neq \emptyset$ as well. On $\partial \widetilde{P} \cap \partial P$ the eigenfunction $\phi_1^{B(\mathbf{0},1)}$ is positive. Hence on this part of the boundary we can consider $\alpha^*(x)$, and show as in Case 1 that for $x \in \partial \widetilde{P} \cap \partial P$ we have $\alpha^*(x) \leq \alpha < 0$. Since on $\partial \widetilde{P} \cap \partial B$ the eigenfunction $\phi_1^{B(\mathbf{0},1)}$ is zero, we can define a function $v \in H^1(P; \mathcal{R})$ as $v(x) = \phi_1^{B(\mathbf{0},1)}$ for

$x \in \widetilde{P}$, and $v(x) = 0$ for $x \in P \setminus \widetilde{P}$. We then multiply (10) by $\phi_1^{B(\mathbf{0},1)}$, and integrate by parts, to obtain

$$\begin{aligned}
\lambda_1^{B(\mathbf{0},1)}(\alpha) &= \frac{\int_{\partial\widetilde{P}\cap\partial P} \alpha^*(x) \, \left(\phi_1^{B(\mathbf{0},1)}(\alpha;x)\right)^2 \, d\sigma_x - \int_{\widetilde{P}} |\nabla \phi_1^{B(\mathbf{0},1)}(\alpha;x)|^2 \, dx}{\int_{\widetilde{P}} \left(\phi_1^{B(\mathbf{0},1)}(\alpha;x)\right)^2 \, dx} \\
&\leq \frac{\alpha \int_{\partial\widetilde{P}\cap\partial P} v^2(x) \, d\sigma_x - \int_P |\nabla v(x)|^2 \, dx}{\int_P v^2(x) \, dx} \\
&\leq \sup_{u \in H^1(P;\mathcal{R})} \frac{\alpha \int_{\partial P} u^2(x) \, d\sigma_x - \int_P |\nabla u(x)|^2 \, dx}{\int_P u^2(x) \, dx} \; = \; \lambda_1^P(\alpha).
\end{aligned}$$

Case 3: If $\partial\widetilde{P} \cap \partial P = \emptyset$, then $\widetilde{P} = B(\mathbf{0}, r_+) \subset P$, with $r_+ > 1$, and $\phi_1^{B(\mathbf{0},1)}$ verifies the Dirichlet problem on $B(\mathbf{0}, r_+)$. Therefore $\lambda_1^{B(\mathbf{0},1)}(\alpha) = \lambda_D^{B(\mathbf{0},r_+)} \leq \lambda_1^{B(\mathbf{0},r_+)}(\alpha)$. We then look at $\widetilde{P} = P \cap \{\phi_1^{B(\mathbf{0},r_+)}(\alpha;x) > 0\}$, and repeat our division into three possible cases. After a finite number of steps we are either in Case 1 or in Case 2.

REMARK 4. If P is a triangle or a regular polygon, the previous theorem provides a sharp bound for $\lambda_1^P(\alpha)$ for α small, in view of the limit (5), since one can take as B the inscribed ball.

COROLLARY 3. *Let $\alpha < 0$ and let $B \subset \mathcal{R}^n, n \geq 2$, denote a ball. If Ω is a $C^{0,1}$ convex domain such that $B \subset \Omega$, then*

$$\lambda_1^B(\alpha) \leq \lambda_1^\Omega(\alpha) < 0.$$

Proof. This corollary is a consequence of Theorem 2 and Corollary 3.7 in [5], where Dancer and Daners present domain perturbation results for the Robin boundary condition.

REMARK 5. If a polyhedron P circumscribes a ball B, then a similar result holds for $\alpha > 0$ (see [6], and recall Figure 1). However, this cannot be generalized to convex domains following the idea of Corollary 3, since, for fixed $\alpha > 0$ and Ω, the small perturbations of the boundary results, in the spirit of the study for $\alpha < 0$ done by Dancer and Daners in [5], would not hold. In fact, if we consider the rectangle $R_\epsilon = (-\epsilon, \epsilon) \times (-1, 1)$, separation of variables yields $\lambda_1^{R_\epsilon}(\alpha) = \lambda_1^{(-\epsilon,\epsilon)}(\alpha) + \lambda_1^{(-1,1)}(\alpha)$. Fix $\alpha > 0$. The scaling property of the principal eigenvalue tells us that

$$\lambda_1^{(-\epsilon,\epsilon)}(\alpha) = \frac{\lambda_1^{(-1,1)}(\alpha\epsilon)}{\epsilon^2} = \frac{\alpha}{\epsilon} \, \frac{\lambda_1^{(-1,1)}(\alpha\epsilon)}{\alpha\epsilon},$$

and using equation (5) we see that

$$\lim_{\epsilon \to 0^+} \lambda_1^{R_\epsilon}(\alpha) = \infty.$$

An immediate corollary of Theorem 1 and Corollary 3 is the monotonicity of the eigenvalues for convex domains that are separated by a ball.

COROLLARY 4. *Let $\alpha < 0$, and let Ω_1, Ω_2 be convex domains in $\mathcal{R}^n$, $n \geq 2$, if there exists a ball B such that $\Omega_1 \subset B \subset \Omega_2$, then $\lambda_1^{\Omega_1}(\alpha) \leq \lambda_1^{\Omega_2}(\alpha)$.*

References

[1] D. E. Amos, *Computation of modified Bessel functions and their ratios*, Math. Comp. **28** (1974), 239–251. MR 0333287 (48 #11612)

[2] M. S. Ashbaugh and R. D. Benguria, *A second proof of the Payne-Pólya-Weinberger conjecture*, Comm. Math. Phys. **147** (1992), 181–190. MR 1171765 (93k:33002)

[3] R. F. Bass and K. Burdzy, *On domain monotonicity of the Neumann heat kernel*, J. Funct. Anal. **116** (1993), 215–224. MR 1237993 (94k:60120)

[4] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. MR 990239 (90e:35123)

[5] E. N. Dancer and D. Daners, *Domain perturbation for elliptic equations subject to Robin boundary conditions*, J. Differential Equations **138** (1997), 86–132. MR 1458457 (98e:35017)

[6] T. Giorgi & R. Smits, *Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity*, Z. Angew. Math. Phys., to appear.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1983. MR 737190 (86c:35035)

[8] J. Hersch, *Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum*, Z. Angew. Math. Phys. **11** (1960), 387–413. MR 0125319 (23 #A2622)

[9] A. A. Lacey, J. R. Ockendon, and J. Sabina, *Multidimensional reaction diffusion equations with nonlinear boundary conditions*, SIAM J. Appl. Math. **58** (1998), 1622–1647. MR 1637882 (99g:35066)

[10] N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972. MR 0350075 (50 #2568)

[11] L. E. Payne and P. W. Schaefer, *Eigenvalue and eigenfunction inequalities for the elastically supported membrane*, Z. Angew. Math. Phys. **52** (2001), 888–895. MR 1872006 (2002k:35232)

[12] R. G. Smits, *Spectral gaps and rates to equilibrium for diffusions in convex domains*, Michigan Math. J. **43** (1996), 141–157. MR 1381604 (97d:35037)

[13] R. P. Sperb, *Untere und obere Schranken für den tiefsten Eigenwert der elastisch gestützten Membran*, Z. Angew. Math. Phys. **23** (1972), 231–244. MR 0312800 (47 #1355)

DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NM 88003-8001, USA

E-mail address, T. Giorgi: tgiorgi@nmsu.edu

E-mail address, R. Smits: rsmits@nmsu.edu