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MULTIPLICATIVE MONOTONIC CONVOLUTION

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Abstract. We show that the monotonic independence introduced by Muraki can also be used to define a multiplicative convolution. We also find a method for the calculation of this convolution based on an appropriate form of the Cauchy transform. Finally, we discuss infinite divisibility in the multiplicative monotonic context.

1. Introduction

Consider an algebraic probability space, that is, a pair $(\mathfrak{A}, \varphi)$, where $\mathfrak{A}$ is a unital complex algebra, and $\varphi : \mathfrak{A} \to \mathbb{C}$ is a linear functional satisfying $\varphi(1) = 1$. Muraki [5] introduced the concept of *monotonic independence* for elements of $\mathfrak{A}$, which we will now review. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two subalgebras of $\mathfrak{A}$; it is not assumed that either of these subalgebras contains the unit. These algebras are said to be *monotonically independent* if the following two conditions are satisfied:

(1) for every $x_1, y_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$, we have $x_1x_2y_1 = \varphi(x_2)x_1y_1$;
(2) for every $x_1 \in \mathfrak{A}_1$ and $x_2, y_2 \in \mathfrak{A}_2$, we have $\varphi(x_2x_1y_2) = \varphi(x_2)\varphi(x_1)\varphi(y_2)$, $\varphi(x_2x_1) = \varphi(x_2)\varphi(x_1)$, and $\varphi(x_1y_2) = \varphi(x_1)\varphi(y_2)$.

Proceeding inductively, the algebras $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n$ are said to be monotonically independent if $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_{n-1}$ are monotonically independent, and the algebras $\mathfrak{A}', \mathfrak{A}_n$ are monotonically independent, where $\mathfrak{A}'$ is the (generally nonunital) algebra generated by $\mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_{n-1}$. More generally, if I is a totally ordered set, and $(\mathfrak{A}_i)_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$, this family is said to be monotonically independent if the algebras $\mathfrak{A}_{i_1}, \mathfrak{A}_{i_2}, \dots, \mathfrak{A}_{i_n}$ are monotonically independent for any choice of indices $i_1 < i_2 < \cdots < i_n$. A family $(x_i)_{i\in I}$ of elements of $\mathfrak{A}$ is said to be monotonically independent if the (generally nonunital) subalgebras $\mathfrak{A}_i$ generated by x_i form a monotonically independent family.

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The distribution μ_x of an element x of $\mathfrak{A}$ (a.k.a. a random variable) is the linear functional defined on the polynomial algebra $\mathbb{C}[X]$ by the formula

$$\mu_x(p) = \varphi(p(x)), \quad p \in \mathbb{C}[X].$$

Clearly, μ_x is entirely determined by the sequence $\mu_x(X^n) = \varphi(x^n)$ of moments of x. A functional μ on $\mathbb{C}[X]$ is the distribution of some random variable if and only if $\mu(1) = 1$. The set of these functionals, endowed with the weak* topology, will be denoted $\mathfrak{M}$.

Muraki [6] observed that, given monotonically independent random variables x_1,x_2, the distribution of $x_1 + x_2$ only depends on μ_{x_1}, μ_{x_2}. This gives rise to a binary operation $\triangleright$ on $\mathfrak{M}$, called monotonic convolution. It was also shown in [6] how to calculate monotonic convolutions using moment generating functions.

It is also true that the distribution of x_1x_2 only depends on μ_{x_1}, μ_{x_2} if x_1, x_2 are monotonically independent, but the dependence is rather trivial. Indeed, if $n \geq 1$, property (1) above yields

$$(x_1x_2)^n = \varphi(x_2)^{n-1}x_1^n x_2,$$

and then from property (2)

$$\varphi((x_1x_2)^n) = \varphi(x_2)^n \varphi(x_1^n).$$

In other words, the product x_1x_2 has the same distribution as αx_1, with $\alpha = \varphi(x_2)$.

A more interesting result is obtained by considering x_1,$x_2 \in \mathfrak{A}$ such that the variables $x_1 - c_1, x_2 - c_2$ are monotonically independent, where c_1, c_2 are scalars. It is again easy to see that, under this condition, $\mu_{x_1x_2}$ depends only on the distributions of x_1, x_2 and on the numbers c_1, c_2. This yields a new operation $\circlearrowright$ on $\mathfrak{M} \times \mathbb{C}$, called *multiplicative monotonic convolution*, such that

$$(\mu_{x_1x_2}, c_1c_2) = (\mu_{x_1}, c_1) \circlearrowright (\mu_{x_2}, c_2)$$

if $x_1 - c_1$ and $x_2 - c_2$ are monotonically independent. It is interesting to note that, under this condition, $\mu_{x_1x_2} = \mu_{x_2x_1}$, but the operation $\circlearrowright$ is not commutative, since monotonic independence itself is not a symmetric relation. While $\circlearrowright$ is not an operation on $\mathfrak{M}$ itself, there are two ways in which it induces such an operation. The first one is obtained by identifying $\mathfrak{M}$ with the subset $\{(\mu, 1) : \mu \in \mathfrak{M}\}$; we will use the same notation for the operation induced this way, that is

$$(\mu_1, 1) \circlearrowright (\mu_2, 1) = (\mu_1 \circlearrowright \mu_2, 1), \quad \mu_1, \mu_2 \in \mathfrak{M}.$$

The second one is obtained by identifying $\mathfrak{M}$ with the subset $\{(\mu, \mu(X)) : \mu \in \mathfrak{M}\}$; we will use the notation $\circlearrowright_0$ for this operation, so that

$$(\mu_1, \mu_1(X)) \circlearrowright (\mu_2, \mu_2(X)) = (\mu_1 \circlearrowright_0 \mu_2, \mu_1(X)\mu_2(X)), \quad \mu_1, \mu_2 \in \mathfrak{M}.$$

The operation $\circlearrowright$ on $\mathfrak{M}$ has the advantage that it is easily extended to measures with unbounded supports. On the other hand, $\circlearrowright_0$ has the advantage that convolution with a Dirac point mass has the natural dilation effect.

We will show that multiplicative monotonic convolution can be calculated in terms of an appropriate moment generating series. We will deduce from this that the multiplicative monotonic convolution of two probability measures on the unit circle is again a probability measure on the unit circle. Analogously, the multiplicative monotonic convolution of two compactly supported probability measures on $\mathbb{R}_+ = [0, +\infty)$ is a measure of the same kind. As mentioned above, the operation $\circlearrowright$ extends to arbitrary probability measures on $\mathbb{R}_+$. It is not clear whether the same is true for $\circlearrowright_0$. In the case of probability measures on $\mathbb{R}_+$ and $\mathbb{T}$ we will give a description of one-parameter convolution semigroups and of infinitely divisible measures, at least for compact supports. This was done by Muraki [6] for additive monotonic convolution semigroups of compactly supported measures on $\mathbb{R}$.

Our approach in calculating multiplicative monotonic convolutions is related to the one we used in [2] to approach additive monotonic convolution, rather than the original combinatorial approach of [6].

2. Realization of monotonically independent variables

In order to see that monotonic convolution is defined everywhere on $\mathfrak{M}\times\mathbb{C}$, we need to show that any two random variables have monotonically independent copies in some algebraic probability space. We will extend slightly a construction from [6]. Fix two algebraic probability spaces $(\mathfrak{A}_1, \varphi_1)$, $(\mathfrak{A}_2, \varphi_2)$, and denote by 1_j the unit of $\mathfrak{A}_j$. Denote by $p_j : \mathfrak{A}_j \to \mathfrak{A}_j$ the linear projection defined by $p_j(a_j) = \varphi_j(a_j)1_j$, $a_j \in \mathfrak{A}_j$. Also, for $a_j \in \mathfrak{A}_j$ we denote by m_{a_j} the left multiplication operator defined by $m_{a_j}x_j = a_jx_j$, $x_j \in \mathfrak{A}_j$. The algebra $L(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ of all linear operators on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ becomes a probability space if we define the functional $\varphi : L(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \to \mathbb{C}$ by

$$\varphi(a)(1_1 \otimes 1_2) = (p_1 \otimes p_2)(a(1_1 \otimes 1_2)), \quad a \in L(\mathfrak{A}_1 \otimes \mathfrak{A}_2).$$

We can then define monotonically independent copies $\mathfrak{B}_1, \mathfrak{B}_2 \subset L(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ of $\mathfrak{A}_1, \mathfrak{A}_2$ as follows:

$$\mathfrak{B}_1 = \{m_{a_1} \otimes p_2 : a_1 \in \mathfrak{A}_1\}, \quad \mathfrak{B}_2 = \{I_{\mathfrak{A}_1} \otimes m_{a_2} : a_2 \in \mathfrak{A}_2\}.$$

The reader will have no difficulty verifying their monotonic independence.

The case which will be of interest for us in finding analytical formulas for multiplicative convolution is more specific. Consider a Hilbert space $\mathfrak{H}$, and a unit vector $e \in \mathfrak{H}$. The algebra $\mathfrak{L}(\mathfrak{H})$ of bounded linear operators on $\mathfrak{H}$ becomes a probability space with the vector functional $\varphi_e(x) = (xe, e)$, $x \in \mathfrak{L}(\mathfrak{H})$. Consider now the Hilbert space tensor product $\mathfrak{H}' = \mathfrak{H} \otimes \mathfrak{H}$ and the unit vector $e' = e \otimes e$. One considers the two monotonically independent copies $\mathfrak{A}_1, \mathfrak{A}_2$ of $\mathfrak{L}(\mathfrak{H})$ in $(\mathfrak{L}(\mathfrak{H}'), \varphi_{e'})$ defined as follows:

$$\mathfrak{A}_1 = \{x \otimes p : x \in \mathfrak{L}(\mathfrak{H})\}, \quad \mathfrak{A}_2 = \{1 \otimes x : x \in \mathfrak{L}(\mathfrak{H})\},$$

where p denotes the rank one orthogonal projection onto the space generated by e, and 1 denotes the identity operator on $\mathfrak{H}$.

For every distribution $\mu \in \mathfrak{M}$ we will consider the formal power series

$$\psi_\mu(z) = \sum_{n=1}^{\infty} \mu(X^n) z^n, \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}.$$

If x is a random variable, we will also use the notation $\psi_x = \psi_{\mu_x}, \eta_x = \eta_{\mu_x}$. The calculation of multiplicative monotonic convolution will involve the series η_μ. For $x \in \mathfrak{L}(\mathfrak{H})$, the formal power series $\psi_x(z), \eta_x(z)$ are actually convergent, at least for $|z| < 1/\|x^{-1}\|$.

Assume now that $\mathfrak{H}$ has an orthonormal basis $(e_j)_{j=1}^{\infty}$, and $e = e_0$. Consider the shift $s \in \mathfrak{L}(\mathfrak{H})$ defined by $se_j = e_{j+1}$ for all j. We will be interested in elements $x \in \mathfrak{L}(\mathfrak{H})$ of the form $x = (1+s)u(s^*)$, where $u \in \mathbb{C}[X]$ is a polynomial. It is easy to see that the distributions μ_x of these operators form a dense subset in $\mathfrak{M}$. Moreover, as shown by Haagerup (see Theorem 2.3.(a) in [4]), the generating function ψ_x is easily related to u.

Lemma 2.1. *If $x = (1+s)u(s^*)$, where u is a polynomial with $u(0) \neq 0$, then*

$$\psi_x\left(\frac{z}{(1+z)u(z)}\right) = z$$

for sufficiently small $|z|$.

We can now state the main result of this section.

Theorem 2.2. *Consider two distributions $\mu_1, \mu_2 \in \mathfrak{M}$, constants $c_1, c_2 \in \mathbb{C}$ and the multiplicative monotonic convolution $(\mu, c_1c_2) = (\mu_1, c_1) \circlearrowright (\mu_2, c_2)$. We have then*

$$\eta_\mu(z) = \eta_{\mu_1}\left(\frac{1}{c_1}\eta_{\mu_2}(c_1 z)\right),$$

if $c_1 \neq 0$, and

$$\eta_\mu(z) = \eta_{\mu_1}(\eta'_{\mu_2}(0)z),$$

if $c_1 = 0$.

Proof. The second formula above is obtained from the first by letting c_1 tend to zero, and using the fact that the operation $\circlearrowright$ is obviously continuous. Continuity also shows that it will suffice to prove the theorem for μ_1, μ_2 in a dense family of distributions, for instance the family of distributions obtained from the random variables $(1+s)u(s^*)$, where u is a polynomial with $u(0) \neq 0$.

Assume then that u_1, u_2 are two polynomials which do not vanish at the origin. We consider the variables x_1, x_2 in $(\mathfrak{L}(\mathfrak{H} \otimes \mathfrak{H}), \varphi_{e_0 \otimes e_0})$ defined by

$$x_1 = c_1 \otimes (1-p) + (1+s)u_1(s^*) \otimes p,$$
$$x_2 = (1 + 1 \otimes s)u_2(1 \otimes s^*) = 1 \otimes [(1+s)u_2(s^*)].$$

These variables have the required property that $x_1 - c_1, x_2 - c_2$ belong to monotonically independent subalgebras. Moreover, it is easy to see that x_1, x_2 have the same distributions as $(1+s)u_1(s^*), (1+s)u_2(s^*)$, so that

$$\psi_{x_1}\left(\frac{z}{(1+z)u_1(z)}\right) = z, \quad \psi_{x_2}\left(\frac{z}{(1+z)u_2(z)}\right) = z$$

for sufficiently small z.

Consider now the vectors $\xi_\lambda = \sum_{n=0}^{\infty} \lambda^n e_n \in \mathfrak{H}$ defined for $|\lambda| < 1$; note that $\xi_0 = e_0$. Since

$$s\xi_\mu = \frac{1}{\mu}(\xi_\mu - \xi_0), \quad u(s^*)\xi_\mu = u(\mu)\xi_\mu$$

for $\mu \neq 0$, we can easily calculate

$$x_2(\xi_\lambda \otimes \xi_\mu) = (1 \otimes (1+s))u_2(\mu)\xi_\lambda \otimes \xi_\mu = u_2(\mu)\xi_\lambda \otimes \left(\xi_\mu + \frac{1}{\mu}(\xi_\mu - \xi_0)\right).$$

Then we obtain for $\lambda \neq 0 \neq \mu$

$$\begin{aligned}
x_1 x_2(\xi_\lambda \otimes \xi_\mu) &= c_1 u_2(\mu)\xi_\lambda \otimes \left(\left(1 + \frac{1}{\mu}\right)(\xi_\mu - \xi_0)\right) \\
&\qquad + u_2(\mu)((1+s) \otimes p)\left[u_1(\lambda)\xi_\lambda \otimes \xi_0\right] \\
&= u_2(\mu)\left[c_1\left(1 + \frac{1}{\mu}\right)\xi_\lambda \otimes (\xi_\mu - \xi_0) + u_1(\lambda)\xi_\lambda \otimes \xi_0 + \frac{u_1(\lambda)}{\lambda}(\xi_\lambda - \xi_0) \otimes \xi_0\right] \\
&= u_2(\mu)\left[c_1\left(1 + \frac{1}{\mu}\right)\xi_\lambda \otimes \xi_\mu\right. \\
&\qquad \left. + \left(u_1(\lambda)\left(1 + \frac{1}{\lambda}\right) - c_1\left(1 + \frac{1}{\mu}\right)\right)\xi_\lambda \otimes \xi_0 - \frac{u_1(\lambda)}{\lambda}\xi_0 \otimes \xi_0\right].
\end{aligned}$$

This equation can be simplified when

$$u_1(\lambda)\left(1 + \frac{1}{\lambda}\right) - c_1\left(1 + \frac{1}{\mu}\right) = 0,$$

in which case it becomes

$$x_1 x_2(\xi_\lambda \otimes \xi_\mu) = \frac{1}{z}(\xi_\lambda \otimes \xi_\mu) - \frac{u_1(\lambda)u_2(\mu)}{\lambda}\xi_0 \otimes \xi_0,$$

with

$$\frac{1}{z} = c_1 u_2(\mu)\left(1 + \frac{1}{\mu}\right).$$

This can then be rewritten as

$$(1 - zx_1x_2)^{-1}\xi_0 \otimes \xi_0 = \frac{\lambda}{zu_1(\lambda)u_2(\mu)}\xi_\lambda \otimes \xi_\mu,$$

so that

$$\varphi((1 - zx_1x_2)^{-1}) = ((1 - zx_1x_2)^{-1}\xi_0 \otimes \xi_0, \xi_0 \otimes \xi_0) = \frac{\lambda}{zu_1(\lambda)u_2(\mu)}.$$

The constant on the right-hand side of this equation is now easily calculated:

$$\frac{\lambda}{zu_1(\lambda)u_2(\mu)} = \frac{\lambda c_1(1 + 1/\mu)}{u_1(\lambda)} = \lambda\left(1 + \frac{1}{\lambda}\right) = \lambda + 1,$$

yielding then

$$\psi_{x_1x_2}(z) = \varphi_{e_0 \otimes e_0}((1 - zx_1x_2)^{-1}) - 1 = \lambda.$$

These calculations hold for $|\lambda| \neq 0$ sufficiently small, because the associated numbers μ and z are also small, and $\mu \neq 0$. Observe now that the identity

$$\frac{1}{c_1z} = u_2(\mu)\left(1 + \frac{1}{\mu}\right)$$

means that $\psi_{x_2}(c_1z) = \mu$, while

$$u_1(\lambda)\left(1 + \frac{1}{\lambda}\right) = c_1\left(1 + \frac{1}{\mu}\right)$$

means that

$$\lambda = \psi_{x_1}\left(\frac{1}{c_1(1 + 1/\mu)}\right).$$

Combining these identities we see that

$$\begin{aligned}\psi_{x_1x_2}(z) &= \psi_{x_1}\left(\frac{1}{c_1(1 + 1/\mu)}\right) = \psi_{x_1}\left(\frac{1}{c_1}\frac{\psi_{x_2}(c_1z)}{1 + \psi_{x_2}(c_1z)}\right)\\ &= \psi_{x_1}\left(\frac{1}{c_1}\eta_{x_2}(c_1z)\right).\end{aligned}$$

The identity above shows that

$$\eta_{x_1x_2}(z) = \eta_{x_1}\left(\frac{1}{c_1}\eta_{x_2}(c_1z)\right)$$

for uncountably many values of z. We deduce that the identity in the statement holds in the generic particular case $\mu_1 = \mu_{x_1}, \mu_2 = \mu_{x_2}$. □

The two convolutions on $\mathfrak{M}$ are now easily described.

Corollary 2.3. *Given measures $\mu_1, \mu_2 \in \mathfrak{M}$, we have*

$$\eta_{\mu_1 \circlearrowright \mu_2}(z) = \eta_{\mu_1}(\eta_{\mu_2}(z)),$$

and

$$\eta_{\mu_1 \circlearrowright_0 \mu_2}(z) = \eta_{\mu_1}\left(\frac{1}{\alpha}\eta_{\mu_2}(\alpha z)\right),$$

with $\alpha = \mu_1(X) = \eta'_{\mu_1}(0)$. *The fraction* $\eta_{\mu_2}(\alpha z)/\alpha$ *must be interpreted as* $\eta'_{\mu_2}(0)z$ *in case* $\alpha = 0$.

3. Measures on the positive half-line

If μ is a probability measure on $\mathbb{R}_+$ one can define

$$\psi_\mu(z) = \int_0^\infty \frac{zt}{1-zt}\, d\mu(t), \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)}$$

for every $z \in \Omega = \mathbb{C} \setminus \mathbb{R}_+$. These functions are analytic, and moreover $\eta_\mu(\Omega) \subset \Omega$. More precisely,

$$\eta_\mu(0-) = 0, \quad \eta_\mu(\overline{z}) = \overline{\eta_\mu(z)}, \quad \text{and} \quad \pi \geq \arg \eta_\mu(z) \geq \arg z, \qquad \text{for } z \in \Omega, \Im z > 0,$$

where $\eta_\mu(0-) = \lim_{t\uparrow 0} \eta_\mu(t)$. Moreover, as seen in [1], these conditions characterize the functions η_μ among all analytic functions defined on Ω. The measure μ is compactly supported if and only if the function η_μ is analytic in a neighborhood of the origin. In this case, μ is entirely determined by the Taylor coefficients of η_μ, and the power series of η_μ at zero is precisely the formal power series denoted by the same symbol in the preceding section, provided that we view μ as an element of $\mathfrak{M}$ by setting $\mu(X^n) = \int_0^\infty t^n\, d\mu(t)$. We can thus identify the collection of compactly supported measures on $\mathbb{R}_+$ with a subset of $\mathfrak{M}$.

Proposition 3.1. *If* μ_1, μ_2 *are compactly supported probability measures on* $\mathbb{R}_+$*, then both* $\mu_1 \circlearrowright \mu_2$ *and* $\mu_1 \circlearrowright_0 \mu_2$ *are compactly supported probability measure on* $\mathbb{R}_+$*.*

Proof. If $\mu_1 = \delta_0$ is Dirac measure at zero, then clearly $\mu_1 \circlearrowright_0 \mu_2 = \delta_0$. Otherwise, the number $\alpha = \eta'_{\mu_1}(0) = \int_0^\infty t\, d\mu_1(t)$ is different from zero, and therefore

$$\eta_{\mu_1 \circlearrowright_0 \mu_2}(z) = \eta_{\mu_1}\left(\frac{1}{\alpha}\eta_{\mu_2}(\alpha z)\right).$$

This shows that $\eta_{\mu_1 \circlearrowright_0 \mu_2}(z)$ makes sense for every $z \in \Omega$, and it is an analytic function of the form η_μ for some compactly supported probability measure μ on $\mathbb{R}_+$. Clearly then $\mu = \mu_1 \circlearrowright_0 \mu_2$. The case of $\mu_1 \circlearrowright \mu_2$ is treated similarly. □

There is a different argument for the preceding result, based on the multiplication of positive random variables. Observe first that the existence of monotonically independent variables is, generally, incompatible with the functional linear φ being a trace. Indeed, if $\mathfrak{A}_1, \mathfrak{A}_2$ are monotonically independent in $(\mathfrak{A}, \varphi)$, and $x_1 \in \mathfrak{A}_1, x_2, y_2 \in \mathfrak{A}_2$, then

$$\varphi(x_2 x_1 y_2) - \varphi(x_1 y_2 x_2) = \varphi(x_1)[\varphi(x_2)\varphi(y_2) - \varphi(x_2 y_2)].$$

Thus, if φ is a trace, either $\varphi|\mathfrak{A}_1$ is identically zero, or $\varphi|\mathfrak{A}_2$ is multiplicative. There is however a remnant of the trace property, for instance when $\mathfrak{A}_1$ is commutative.

LEMMA 3.2. *Assume that $\mathfrak{A}_1, \mathfrak{A}_2$ are monotonically independent in $(\mathfrak{A}, \varphi)$, and $\varphi|\mathfrak{A}_1$ is a trace. Then we have $\varphi(xy) = \varphi(yx)$ for any x in the unital algebra generated by $\mathfrak{A}_1$, and any y in the unital algebra generated by $\mathfrak{A}_1 \cup \mathfrak{A}_2$.*

Proof. Since both sides of the identity to be proved are bilinear in (x, y), it suffices to prove it when $x \in \mathfrak{A}_1$, and y is a product of elements in $\mathfrak{A}_1 \cup \mathfrak{A}_2$, with at least one factor in $\mathfrak{A}_2$. Thus y has the form

$$y = x_1 y_1 \cdots x_n y_n x_{n+1},$$

where $n \geq 1$, $y_1, y_2, \ldots, y_n \in \mathfrak{A}_2$, $x_2, \ldots, x_n \in \mathfrak{A}_1$, and $x_1, x_{n+1} \in \mathfrak{A}_1 \cup \{1\}$. Monotonic independence allows us to calculate

$$\varphi(xy) - \varphi(yx) = [\varphi(x x_1 x_2 \cdots x_{n+1}) - \varphi(x_1 x_2 \cdots x_{n+1} x)] \prod_{j=1}^{n} \varphi(y_j),$$

and the conclusion follows because $\varphi|\mathfrak{A}_1$ is a trace. □

COROLLARY 3.3. *Let x_1, x_2 be two random variables, and $c_1, c_2 \in \mathbb{C}$ be such that $x_1 - c_1$ and $x_2 - c_2$ are monotonically independent. Then the variables $x_1^2 x_2, x_1 x_2 x_1$, and $x_2 x_1^2$ have the same distribution.*

In particular, if the probability space is $(\mathfrak{L}(\mathfrak{H}), \varphi_\xi)$, and x_1, x_2 are selfadjoint, it follows that $x_1^2 x_2$ has the same distribution as the selfadjoint variable $x_1 x_2 x_1$. If μ_1, μ_2 are compactly supported measures on $\mathbb{R}_+$, then we can always find random variables $y_1, y_2 \in \mathfrak{L}(\mathfrak{H})$ such that y_1, y_2 are positive operators, and $\mu_{y_1^2} = \mu_1, \mu_{y_2} = \mu_2$. We can then define new variables

$$x_1 = y_1 \otimes p + c_1^{1/2} \otimes (1 - p), \quad x_2 = 1 \otimes y_2$$

in $(\mathfrak{L}(\mathfrak{H} \otimes \mathfrak{H}), \varphi_{\xi \otimes \xi})$ which have the same distributions as y_1, y_2, and $x_1 - c_1^{1/2}, x_2 - c_2$ are monotonically independent. Considering now $c_1 = c_2 = 1$ or $c_j = \mu_j(X)$, we see that $\mu_1 \circlearrowright \mu_2$, and respectively $\mu_1 \circlearrowright_0 \mu_2$, is the distribution of the positive random variable $x_1 x_2 x_1$. Moreover, the inequality $\|x_1 x_2 x_1\| \leq \|x_1^2\| \|x_2\|$, and the fact that y_1, y_2 can be chosen so that the spectra of y_1^2, y_2 coincide with the supports of μ_1, μ_2, yield the following result.

Corollary 3.4. *Let μ_1, μ_2 be probability measures on $\mathbb{R}_+$ such that the support of μ_j is contained in the interval $[\alpha_j, \beta_j] \subset \mathbb{R}_+$, where $\alpha_j \leq 1 \leq \beta_j$ for $j = 1, 2$. Then the supports of $\mu_1 \circlearrowright \mu_2$ and $\mu_1 \circlearrowright_0 \mu_2$ are contained in $[\alpha_1\alpha_2, \beta_1\beta_2]$.*

We will need an inclusion in the opposite direction. In the following proof we will use the fact that the measure μ can be recovered from the imaginary parts of the limits of the function ψ_μ or η_μ at points on the real line. The relevant fact is as follows: if $(a, b) \subset \mathbb{R}_+$ is an open interval such that $\lim_{\theta\downarrow 0} \arg \eta_\mu(re^{i\theta}) = 0$ for every $r \in (a, b)$, then $\mu((1/b, 1/a)) = 0$.

We will denote by $\operatorname{supp}(\mu)$ the support of a measure μ on $\mathbb{R}_+$.

Proposition 3.5. *For any compactly supported probability measures μ_1, μ_2 on $\mathbb{R}_+$, we have*

$$\operatorname{supp}(\mu_2) \subset \operatorname{supp}(\mu_1 \circlearrowright \mu_2),$$

and

$$\left(\int_0^\infty t\, d\mu_1(t)\right) \operatorname{supp}(\mu_2) \subset \operatorname{supp}(\mu_1 \circlearrowright_0 \mu_2).$$

Proof. We provide the argument for $\mu = \mu_1 \circlearrowright_0 \mu_2$. Assume that an interval (a, b) is disjoint from the support of μ, so that the function η_μ is analytic and real-valued on the interval $(1/b, 1/a)$. Now

$$\eta_\mu(z) = \eta_{\mu_1}\left(\frac{1}{\alpha}\eta_{\mu_2}(\alpha z)\right)$$

for $z \notin \mathbb{R}_+$, with

$$\alpha = \eta'_{\mu_1}(0) = \int_0^\infty t\, d\mu_1(t).$$

We deduce that

$$\lim_{\theta\downarrow 0} \arg \eta_{\mu_2}(\alpha re^{i\theta}) = \lim_{\theta\downarrow 0} \arg \frac{1}{\alpha}\eta_{\mu_2}(\alpha re^{i\theta}) \leq \lim_{\theta\downarrow 0} \arg \eta_\mu(re^{i\theta}) = 0$$

for every $r \in (1/b, 1/a)$. As noted before the statement of the proposition, this implies that the support of the measure μ_2 contains no points in $(a/\alpha, b/\alpha)$. In other words,

$$\operatorname{supp}(\mu_2) \subset \frac{\operatorname{supp}(\mu_1 \circlearrowright_0 \mu_2)}{\int_0^\infty t\, d\mu_1(t)},$$

as claimed. □

It is now fairly easy to find the multiplicative monotonic convolution semigroups. These are simply families $\{\mu_\tau : \tau \geq 0\}$ of compactly supported probability measures on $\mathbb{R}_+$ such that $\mu_0 = \delta_1$, $\mu_{\tau+\tau'} = \mu_\tau \circlearrowright \mu_{\tau'}$ (or $\mu_{\tau+\tau'} = \mu_\tau \circlearrowright_0 \mu_{\tau'}$) for $\tau, \tau' \geq 0$, and the map $\tau \mapsto \mu_\tau$ is continuous. The topology on probability measures will be the one inherited from $\mathfrak{M}$, but in this case it is precisely the topology of weak convergence of probability measures. Indeed, in the case of $\circlearrowright$-semigroups the support of μ_τ is contained in the support of μ_1 for $\tau \le 1$, and it is immediate that the map $\tau \mapsto \mu_\tau$ is continuous when we consider the weak topology on the collection of probability measures. Similarly, in the case of $\circlearrowright_0$-semigroups, observe first that the function $\alpha(\tau) = \int_0^\infty t\,d\mu_\tau(t)$ is continuous, and $\alpha(\tau+\tau') = \alpha(\tau)\alpha(\tau')$. We conclude that $\alpha(\tau) = e^{a\tau}$, with $a = \log\alpha(1) \in \mathbb{R}$. The preceding result now shows that the support of μ_τ is uniformly bounded when τ runs in a bounded set. Indeed, we see that

$$\operatorname{supp}(\mu_\tau) \subset \frac{\operatorname{supp}(\mu_T)}{e^{a(T-\tau)}}$$

for $\tau \in [0,T]$. It is again easy to conclude that the map $\tau \mapsto \mu_\tau$ is continuous when we consider the weak topology on the collection of probability measures.

THEOREM 3.6. *Consider a $\circlearrowright_0$-semigroup $\{\mu_\tau : \tau \ge 0\}$ of compactly supported probability measures on $\mathbb{R}_+$, and let $a \in \mathbb{R}$ be such that*

$$\int_0^\infty t\,d\mu_\tau(t) = e^{a\tau}, \quad \tau \ge 0.$$

There is a neighborhood V of $0 \in \mathbb{C}$ such that the map $\tau \mapsto \eta_{\mu_\tau}(z)$ is differentiable at $\tau = 0$ for every $z \in \Omega \cup V$, and the derivative

$$A(z) = \left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0}$$

is an analytic function of z. Moreover, we can write $A(z) = z(B(z)+a)$, where B is analytic in $\Omega \cup V$, $B(0) = 0$, $B(\overline{z}) = \overline{B(z)}$ and $\Im B(z) \ge 0$ whenever $\Im z > 0$.

Conversely, for any $a \in \mathbb{R}$, and any analytic function B defined in a set of the form $\Omega \cup V$, with V a neighborhood of 0, satisfying the conditions above, there exists a unique $\circlearrowright_0$-semigroup $\{\mu_\tau : \tau \ge 0\}$ of compactly supported probability measures on $\mathbb{R}_+$ such that

$$\left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0} = z(B(z)+a), \quad z \in \Omega,$$

and $\int_0^\infty t\,d\mu_\tau(t) = e^{a\tau}$ for $\tau \ge 0$. Moreover, $\eta_{\mu_t}(z) = u_\tau(e^{a\tau}z)$, where $u_\tau(z)$ is the solution of the initial value problem

$$\frac{du_\tau(z)}{d\tau} = u_\tau(z)B(u_\tau(z)), \quad u_0(z) = z \in \Omega.$$

This solution exists for all $\tau \ge 0$.

Proof. Start first with a semigroup $\{\mu_\tau : \tau \ge 0\}$, and define functions $u_\tau : \Omega \to \Omega$ by setting $u_\tau(z) = \eta_{\mu_\tau}(e^{-a\tau}z)$ for $z \in \Omega$. Clearly then $u_\tau(z)$ depends continuously on z, and the semigroup property can be translated into $u_{\tau+\tau'}(z) = u_\tau(u_{\tau'}(z))$. As shown by Berkson and Porta [3, Theorem 1.1], these conditions imply that $u_\tau(z)$ is a differentiable function of τ, and it satisfies the equation

$$\frac{du_\tau(z)}{d\tau} = C(u_\tau(z)),$$

where $C(z) = (du_\tau(z)/d\tau)|_{\tau=0}$. The initial condition $u_0(z) = z$ comes from the identity $u_0 = \eta_{\mu_0} = \eta_{\delta_1}$, and this last function is easily seen to be the identity function on Ω. Clearly

$$\left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0} = C(z) + az, \quad z \in \Omega.$$

Let us observe next that $\tau \mapsto \arg u_\tau(z)$ is an increasing function for $\Im z > 0$, and therefore

$$\Im \frac{C(z)}{z} = \left.\frac{d \log u_\tau(z)}{d\tau}\right|_{\tau=0} \geq 0,$$

so that indeed $C(z) = zB(z)$, where B is an analytic function with positive imaginary part in the upper half-plane $\mathbb{C}^+$. Moreover, the fact that $u_\tau(0) = 0$ yields $C(0) = 0$, so that B is analytic in a neighborhood of zero. We also have $u_\tau'(0) = 1$, which shows that C also has zero derivative at $z = 0$, and therefore $B(0) = 0$ as well.

Conversely, assume that we are given an analytic function B in $\Omega \cup V$, with $B(0) = 0$, and with positive imaginary part in $\mathbb{C}^+$. We show first that the initial value problem

$$\frac{du_\tau(z)}{d\tau} = u_\tau(z)B(u_\tau(z)), \quad u_\tau(0) = z \in \Omega \tag{3.1}$$

has a solution defined for all positive τ. In order to do this we apply another result of [3] (see Theorem 2.6, and the description of the class $\mathcal{G}_2(\mathcal{H})$ for $b = 0$), which we reformulate for the upper half-plane $\mathbb{C}^+$ and the left half-plane $i\mathbb{C}^+$: Let $C : \mathbb{C}^+ \to \mathbb{C}$ (resp., $C : i\mathbb{C}^+ \to \mathbb{C}$) be an analytic function such that $C(z)/z^2 \in \mathbb{C}^+$ (resp., $-C(z)/z^2 \in i\mathbb{C}^+$) for every z. Then for every z (in the relevant domain), the initial value problem $du_\tau(z)/d\tau = C(u_\tau(z))$, $u_0(z) = z$, has a solution defined for all positive τ. The function $C(z) = zB(z)$ satisfies the hypotheses of both of these results. Indeed, the fact that B has positive imaginary part in $\mathbb{C}^+$ allows us to write B in Nevanlinna form

$$B(z) = \beta + \gamma z + \int_{-\infty}^{\infty} \frac{1 + zt}{t - z}\, d\rho(t), \quad z \in \mathbb{C}^+,$$

where β is a real number, $\gamma \geq 0$, and ρ is a finite, positive Borel measure on $\mathbb{R}$. The fact that B is real and analytic on $(-\infty, \varepsilon]$ for some $\varepsilon > 0$ shows that the support of ρ is contained in $[\varepsilon, +\infty)$, and the condition $B(0) = 0$ yields the value

$$\beta = -\int_0^\infty \frac{1}{t}\, d\rho(t).$$

We conclude that

$$C(z) = z^2 \left(\gamma + \int_0^\infty \frac{t^2+1}{t(t-z)}\, d\rho(t) \right), \quad z \in \Omega.$$

It is now easy to see that the integral above has positive imaginary part if $z \in \mathbb{C}^+$, and positive real part for $z \in i\mathbb{C}^+$. We conclude that the equation (3.1) has a solution defined for $\tau \geq 0$ for initial values z in $\mathbb{C}^+ \cup i\mathbb{C}^+$, and by symmetry for all $z \in \Omega$. This equation will also have a solution defined for small τ given an initial value $z > 0$ sufficiently close to zero. We deduce that, for small values of τ, the function u_τ is also analytic in a neighborhood of zero. The equation $u_{\tau+\tau'} = u_\tau \circ u_{\tau'}$ shows that the same is true for all values of τ, and $u_\tau(0) = 0$. The fact that B has positive imaginary part in $\mathbb{C}^+$ implies that the function $\tau \mapsto \arg u_\tau(z)$ is an increasing function of τ, and therefore $\arg u_\tau(z) \geq \arg z$ for $z \in \mathbb{C}^+$. (Note that $u_\tau(\mathbb{C}^+) \subset \mathbb{C}^+$ by the theorem of Berkson and Porta.) We conclude that there exist compactly supported probability measures μ_τ on $\mathbb{R}_+$ such that $\eta_{\mu_\tau}(z) = u_\tau(e^{at}z)$ for $z \in \Omega$ and $\tau \geq 0$. It is easy to verify now that these measures form a multiplicative monotone convolution semigroup satisfying the required conditions. The uniqueness of the semigroup is a consequence of the uniqueness of solutions of ordinary differential equations with a locally Lipschitz right-hand side. $\square$

The results of Berkson and Porta [3] can also be formulated, via conformal maps, for the entire region Ω. The corresponding formulation however does not reflect the additional symmetries present in our particular case.

The representation of the function C found in the preceding proof provides a bijection between $\circlearrowright_0$-convolution semigroups and triples (γ, ρ, a), where a is a real number, $\gamma \geq 0$, and ρ is a finite, positive Borel measure on some interval $[\varepsilon, +\infty)$. The representation of the function A can be written more compactly if we use the measure ν defined on the interval $[0, 1/\varepsilon]$ by the requirements that $\nu(\{0\}) = \gamma$ and $d\nu(t) = (t^2+1)d\rho(1/t)$ on $(0, 1/\varepsilon]$. We have then

$$A(z) = az + z^2 \int_0^\infty \frac{1}{1-zt} d\nu(t),$$

with $a \in \mathbb{R}$ and ν a positive, Borel, compactly supported measure on $\mathbb{R}_+$. The constant a is equal to zero if the measures μ_τ have first moment equal to one, in which case the functions $\eta_{\mu_\tau} = u_\tau$ simply form a semigroup relative to composition of functions on Ω.

It is difficult to find explicit formulas for these semigroups. One case when this is possible is $A(z) = \gamma z^2$ for some $\gamma > 0$. In this case the differential equation is easily solved, and it yields

$$\eta_{\mu_\tau}(z) = \frac{z}{1-\gamma\tau z}, \quad \psi_{\mu_\tau}(z) = \frac{z}{1-(1+\gamma\tau)z}, \quad z \in \Omega,$$

so that

$$\mu_\tau = \frac{\gamma\tau}{1+\gamma\tau}\delta_0 + \frac{1}{1+\gamma\tau}\delta_{1+\gamma\tau}, \quad \tau \geq 0.$$

As in the case of additive monotone convolution [6], the preceding parametrization of semigroups also yields a parametrization of $\circlearrowright_0$-infinitely divisible measures. Naturally, a compactly supported probability measure μ on $\mathbb{R}_+$ is said to be $\circlearrowright_0$-*infinitely divisible* if, for every positive integer n, there exists a compactly supported probability measure $\mu_{1/n}$ on $\mathbb{R}_+$ such that

$$\mu = \underbrace{\mu_{1/n} \circlearrowright_0 \mu_{1/n} \circlearrowright_0 \cdots \circlearrowright_0 \mu_{1/n}}_{n \text{ times}}.$$

As seen in [6] in the additive case, the measure $\mu_{1/n}$ is unique. Indeed, let ν be any compactly supported measure on $\mathbb{R}_+$ such that

$$\underbrace{\nu \circlearrowright_0 \nu \circlearrowright_0 \cdots \circlearrowright_0 \nu = \mu}_{n \text{ times}}.$$

This relation can be written as a system of equations in the Taylor coefficients of η_ν, and it suffices to show that this system has a unique solution. Let us write

$$\eta_\mu(z) = \sum_{n=1}^{\infty} \alpha_n z^n, \quad \eta_\nu(z) = \sum_{n=1}^{\infty} \beta_n z^n$$

in a neighborhood of zero, with $\alpha_1 = \mu(X)$ and $\beta_1 = \nu(X)$. Identifying the coefficients of z in the equation

$$\underbrace{\eta_\nu \circ \eta_\nu \circ \cdots \circ \eta_\nu}_{n \text{ times}} = \eta_\mu,$$

we obtain $\beta_1^n = \alpha_1$, which yields $\beta_1 = \alpha_1^{1/n}$ since $\beta_1 > 0$. For the coefficients of z^2 we obtain the equation

$$\sum_{j=0}^{n} \beta_1^{n-1+j}\beta_2 = \alpha_2$$

which uniquely determines β_2. The general pattern is that the kth equation contains only $\beta_1, \ldots, \beta_k$, and β_k appears only at first power with a positive coefficient at least equal to β_1^{n-1}.

Theorem 3.7. *Let $\mu \neq \delta_0$ be a $\circlearrowright_0$-infinitely divisible, compactly supported, probability measure on $\mathbb{R}_+$. There exists a unique $\circlearrowright_0$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of compactly supported probability measures on $\mathbb{R}_+$ such that $\mu_1 = \mu$.*

Proof. Replacing the measure μ by the measure $d\mu(t/b)$, with $b = \int_0^\infty t d\mu(t)$ allows us to restrict ourselves to measures with first moment equal to one. In this case it is clear that the measures $\mu_{1/n}$ satisfy the same property, and

$$\eta_\mu = \underbrace{\eta_{\mu_{1/n}} \circ \eta_{\mu_{1/n}} \circ \cdots \circ \eta_{\mu_{1/n}}}_{n \text{ times}}.$$

As seen above, the measures $\mu_{1/n}$ are uniquely determined, so that we can further define

$$\mu_{m/n} = \underbrace{\mu_{1/n} \circlearrowright_0 \mu_{1/n} \circlearrowright_0 \cdots \circlearrowright_0 \mu_{1/n}}_{m \text{ times}}$$

for arbitrary positive integers m, n. Clearly we have $\mu_{\tau+\tau'} = \mu_\tau \circlearrowright_0 \mu_{\tau'}$ for rational $\tau, \tau' > 0$. It is then seen from Proposition 3.5 that the measures $\mu_{m/n}$ have uniformly bounded supports if m/n varies in a bounded set of rational numbers. We can now verify that $\mu_{\tau_k} \to \mu_\tau$ weakly when the rational numbers τ_k converge to a rational $\tau = m/n$. Assume indeed that ν is the weak limit of a subsequence of μ_{τ_k}. The continuity of multiplicative monotone convolution implies that

$$\underbrace{\nu \circlearrowright_0 \nu \circlearrowright_0 \cdots \circlearrowright_0 \nu}_{n \text{ times}} = \mu_m,$$

and the uniqueness of roots gives then $\nu = \mu_{m/n}$. On the other hand, if $\tau_k \to 0$ and μ_{τ_k} tends to ν, the measure $\nu \circlearrowright_0 \mu$ is the weak limit of $\mu_{1+\tau_k}$, so that $\nu \circlearrowright_0 \mu = \mu$. In other words, $\eta_\nu \circ \eta_\mu = \eta_\mu$, which shows that η_ν must be the identity function, and hence $\nu = \delta_1$. It is now easy to see that η_τ can be defined for arbitrary $\tau > 0$ by continuity. Indeed, consider two sequences of positive rational numbers $\tau_k \to \tau, \tau'_k \to \tau$ such that the sequences $\mu_{\tau_k}, \mu_{\tau'_k}$ tend weakly to measures ν, ν'. By dropping to subsequences (and possibly switching the two sequences), we may assume that $\tau_k > \tau'_k$ for all k. Since $\mu_{\tau_k} = \mu_{\tau'_k} \circlearrowright_0 \mu_{\tau_k - \tau'_k}$, and $\tau_k - \tau'_k \to 0$, we deduce that $\nu = \nu'$. The uniqueness of the semigroup obtained this way follows immediately from the uniqueness of $\mu_{1/n}$. $\square$

There are analogous results for $\circlearrowright$-semigroups.

THEOREM 3.8. *Consider a $\circlearrowright$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of compactly supported measures on $\mathbb{R}_+$. The map $\tau \mapsto \eta_{\mu_\tau}(z)$ is differentiable for every $z \in \Omega$, and*

$$\frac{d\eta_{\mu_\tau}(z)}{d\tau} = A(\eta_{\mu_\tau}(z)), \quad \tau \geq 0, z \in \Omega,$$

where

$$A(z) = \left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0}, \quad z \in \Omega.$$

The function A can be written as $A(z) = zB(z)$, where B is analytic in Ω and in a neighborhood of zero, and $\Im B(z) \geq 0$ for $z \in \mathbb{C}^+$.

Conversely, if A is an analytic function in Ω with the above properties, there exists a unique $\circlearrowleft$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of compactly supported measures on $\mathbb{R}_+$ such that $A(z) = d\eta_{\mu_\tau}(z)/d\tau|_{\tau=0}$ for $z \in \Omega$.

Proof. The differentiability of the map $\tau \mapsto \eta_{\mu_\tau}(z)$ follows from Theorem 1.1 of [3], and the fact that B has positive imaginary part follows as before from the fact that the map $\tau \mapsto \arg \eta_{\mu_\tau}(z)$ is increasing when $z \in \mathbb{C}^+$. The uniqueness of the semigroup μ_τ is an immediate consequence of the uniqueness of solutions to differential equations (with locally Lipschitz right-hand side). The only thing that requires attention is the fact that, given a function A with the properties in the statement, the initial value problem

$$\frac{du}{d\tau} = A(u), \quad u(0) = z \in \Omega$$

has a solution defined for all $\tau \geq 0$. We will show that this is in fact true whenever $B(z) = A(z)/z$ has positive imaginary part in $\mathbb{C}^+$(without assuming that B is analytic at zero). To do this we write B in Nevanlinna form

$$B(z) = \beta + \gamma z + \int_0^\infty \frac{1 + zt}{t - z}\, d\rho(t), \quad z \in \Omega,$$

with $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_+$, and ρ a positive Borel measure on $\mathbb{R}_+$. We will distinguish three cases, according to the behavior of the function B on the interval $(-\infty, 0)$. Note that B is increasing on this interval, so that it could be negative on $(-\infty, 0)$, positive on $(-\infty, 0)$, or vanish at some point in $(-\infty, 0)$. The first situation, $B(z) \leq 0$ for all $z \in (-\infty, 0)$, amounts to $B(0-) \leq 0$, which implies that $\int_0^\infty \frac{1}{t}\, d\rho(t)$ is finite. After rewriting the above formula as

$$B(z) = \beta + \int_0^\infty \frac{1}{t}\, d\rho(t) + \gamma z + z \int_0^\infty \frac{t^2 + 1}{t(t - z)}\, d\rho(t), \quad z \in \Omega,$$

we deduce that

$$\beta + \int_0^\infty \frac{1}{t}\, d\rho(t) \leq 0.$$

It is then easy to verify that $A(z)/z^2 \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$ and $-A(z)/z^2 \in i\mathbb{C}^+$ for $z \in i\mathbb{C}^+$. We deduce as in the proof of Theorem 3.6 that the solution to our initial value problem extends to all $\tau \geq 0$. Assume next that $B(z) \geq 0$ for all $z \in (-\infty, 0)$. Since

$$\int_0^\infty \frac{1 + zt}{t - z}\, d\rho(t) = o(z)$$

as $z \downarrow -\infty$, this is only possible when $\gamma = 0$. In this case

$$\lim_{z \downarrow -\infty} B(z) = \beta - \int_0^\infty t\, d\rho(t),$$

and we conclude that $\int_0^\infty t\,d\rho(t) < \infty$, and $\beta \geq \int_0^\infty t\,d\rho(t)$. Setting $\alpha = \beta - \int_0^\infty t\,d\rho(t)$, we have

$$A(z) = z\left(\alpha + \int_0^\infty \frac{t^2+1}{t-z}\,d\rho(t)\right).$$

Using this formula, the inequality $\alpha \geq 0$, and the fact that

$$\frac{z}{t-z} = -1 + \frac{t}{t-z},$$

it is easy to see that $A(z) \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$, and $A(z) \in i\mathbb{C}^+$ for $z \in i\mathbb{C}^+$. The results of Berkson and Porta show again that the solution u of the initial value problem extends to $\tau \geq 0$ for every $z \in \mathbb{C}^+ \cup i\mathbb{C}^+$ and, by symmetry, for every $z \in \Omega$. (Note that in this case the relevant Denjoy-Wolff point is infinity, which corresponds with the family $\mathcal{G}_1(\mathcal{H})$ in the notation of [3].) Finally, assume that $B(-a) = 0$ for some $a > 0$. This yields the value

$$\beta = -\gamma a - \int_0^\infty \frac{1-at}{t+a}\,d\rho(t),$$

yielding the formula

$$A(z) = z(z+a)\left(\gamma + \int_0^\infty \frac{t^2+1}{(t+a)(t-z)}\,d\rho(t)\right), \quad z \in \Omega.$$

As in the preceding case, it will suffice to show that the initial value problem for u has a solution defined for all $\tau \geq 0$ if $z \in \mathbb{C}^+ \cup i\mathbb{C}^+$. Using the results of [3] (specifically, the classes $\mathcal{G}_2(\mathbb{C}^+)$ and $\mathcal{G}_3(i\mathbb{C}^+)$), we see that A must satisfy the following conditions:

$$\frac{A(z)}{(z+a)^2} \in \mathbb{C}^+ \quad \text{for} \quad z \in \mathbb{C}^+,$$

and

$$\frac{A(z)}{(z+a)(z-a)} \in -i\mathbb{C}^+ \quad \text{for} \quad z \in i\mathbb{C}^+.$$

For the first of these conditions we write

$$\frac{A(z)}{(z+a)^2} = \frac{\gamma z}{z+a} + \int_0^\infty \frac{t^2+1}{t+a} \cdot \frac{z}{(t-z)(z+a)}\,d\rho(t),$$

which allows the calculation of the imaginary part

$$\Im\frac{A(z)}{(z+a)^2} = \Im z\left(\frac{\gamma a}{|z+a|^2} + \int_0^\infty \frac{t^2+1}{t+a} \cdot \frac{ta+|z|^2}{|t-z|^2|z+a|^2}\,d\rho(t)\right).$$

This is clearly positive for $z \in \mathbb{C}^+$. For the second condition we have

$$\frac{A(z)}{(z+a)(z-a)} = \frac{\gamma z}{z-a} + \int_0^\infty \frac{t^2+1}{t+a} \cdot \frac{z}{(t-z)(z-a)}\,d\rho(t),$$

and

$$\Re\frac{A(z)}{(z+a)(z-a)} = \frac{\gamma(|z|^2 - a\Re z)}{|z-a|^2} + \int_0^\infty \frac{t^2+1}{t+a}\cdot\frac{t|z|^2 + a|z|^2 - (ta+|z|^2)\Re z}{(t-z)(z-a)}\,d\rho(t).$$

This is clearly positive when $\Re z < 0$.

We have thus shown that the initial value problem has a solution defined for all $\tau \geq 0$. Denote by $\eta_\tau(z)$ this solution. This is an analytic function of z, and it extends analytically to a neighborhood of zero if, in addition, B is analytic at zero; moreover, $\eta_\tau(0) = 0$ in this case. It is shown now as in the proof of Theorem 3.6 that $\eta_t = \eta_{\mu_\tau}$ for some compactly supported measure μ_τ on $\mathbb{R}^+$, and these measures form a $\circlearrowright$-semigroup. $\square$

As in the case of the operation $\circlearrowright$, δ_0 is $\circlearrowright_0$-infinitely divisible. All other $\circlearrowright$-infinitely divisible measures belong to a $\circlearrowright$-semigroup.

Theorem 3.9. *Let $\mu \neq \delta_0$ be a $\circlearrowright$-infinitely divisible measure, compactly supported, probability measure on $\mathbb{R}^+$. There exists a unique $\circlearrowright$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of compactly supported probability measures on $\mathbb{R}_+$ such that $\mu_1 = \mu$.*

Proof. The argument is virtually identical with that of Theorem 3.7, except that we need not start by normalizing the measures μ. The details are left to the interested reader. $\square$

4. Measures on the unit circle

If μ is a probability measure on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, the formal power series ψ_μ, η_μ converge in the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and their sums are given by

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{z\zeta}{1-z\zeta}\,d\mu(\zeta), \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)}, \quad z \in \mathbb{D}.$$

An analytic function $\eta : \mathbb{D} \to \mathbb{C}$ is of the form η_μ, for some probability measure on $\mathbb{T}$, if and only if $|\eta(z)| \leq |z|$ for all $z \in \mathbb{D}$ (cf., for instance, [1]). As in the case of compactly supported measures on $\mathbb{R}_+$, the collection of probability measures on $\mathbb{T}$ is identified with a subset of $\mathfrak{M}$. The topology of $\mathfrak{M}$, restricted to this subset, is exactly the topology of weak convergence of probability measures. One should note that an element of $\mathfrak{M}$ may correspond to a measure on $\mathbb{T}$, or to a measure on $\mathbb{R}_+$, and these two measures may be quite different. The simplest occurrence is the equality $\eta_{\delta_0} = \eta_m = 0$, where δ_0 is a unit mass at the origin, while m is normalized arclength (or Haar) measure on $\mathbb{T}$.

PROPOSITION 4.1. *If μ_1, μ_2 are probability measures on $\mathbb{T}$, then $\mu_1 \circlearrowright \mu_2$ and $\mu_1 \circlearrowright_0 \mu_2$ are also probability measures on $\mathbb{T}$.*

Proof. If $|\alpha| \leq 1$ is a complex number, we have

$$\left|\frac{1}{\alpha}\eta_{\mu_2}(\alpha z)\right| \leq |z|, \quad z \in \mathbb{D},$$

where the left-hand side must be interpreted as $|\eta'_{\mu_2}(0)z|$ when $\alpha = 0$. We deduce that

$$\left|\eta_{\mu_1}\left(\frac{1}{\alpha}\eta_{\mu_2}(\alpha z)\right)\right| \leq \left|\frac{1}{\alpha}\eta_{\mu_2}(\alpha z)\right| \leq |z|, \quad z \in \mathbb{D},$$

showing that the formal power series $\eta_{\mu_1 \circlearrowright_0 \mu_2}(z)$ corresponds indeed with a probability measure on $\mathbb{T}$. The measure $\mu_1 \circlearrowright \mu_2$ is treated similarly. □

The part of the preceding result concerning $\circlearrowright$ can be viewed as a consequence of the fact that the product of two unitary operators is again a unitary operator. Indeed, given probability measures μ_1, μ_2 on $\mathbb{T}$, we can find unitary operators x_1, x_2 such that $x_1 - 1, x_2 - 1$ are monotonically independent, and the distribution of x_j is μ_j for $j = 1, 2$. It would be nice to also understand the part concerning $\circlearrowright_0$ in the same manner, but it is not clear how to construct unitary operators x_1, x_2, with given distributions, such that $x_1 - \varphi(x_1), x_2 - \varphi(x_2)$ are monotonically independent. Such operators are easily seen not to exist in the standard realization used in Section 2.

Monotonic convolution semigroups of probability measures on $\mathbb{T}$ are defined as in the case of the half-line, and the following result is the analogue of Theorem 3.6 in this context.

THEOREM 4.2. *Consider a $\circlearrowright_0$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of probability measures on $\mathbb{T}$. The map $\tau \mapsto \eta_{\mu_\tau}(z)$ is differentiable for every $z \in \mathbb{D}$, and the derivative*

$$A(z) = \left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0}$$

is an analytic function of z. Moreover, we can write $A(z) = zB(z)$, where B is analytic in $\mathbb{D}$ and $\Re B(z) \leq 0$ for $z \in \mathbb{D}$.

Conversely, for any analytic function B defined in $\mathbb{D}$, with $\Re B(z) \leq 0$ for $z \in \mathbb{D}$, there exists a unique $\circlearrowright_0$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of probability measures on $\mathbb{T}$ such that

$$\left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0} = zB(z), \quad z \in \mathbb{D}.$$

This semigroups satisfies $\int_{\mathbb{T}} \zeta \, d\mu_\tau(\zeta) = e^{B(0)\tau}$ for $\tau \geq 0$. Moreover, $\eta_{\mu_t}(z) = u_\tau(e^{B(0)\tau}z)$, where $u_\tau : e^{B(0)\tau}\mathbb{D} \to \mathbb{D}$ is an analytic functions satisfying the initial value problem

$$\frac{du_t(z)}{dt} = u_t(z)(B(u_t(z)) - B(0)), \quad u_0(z) = z \in e^{B(0)\tau}\mathbb{D}.$$

This solution exists and belongs to $\mathbb{D}$ *for all* $t \in [0, \tau]$.

Proof. The numbers $\alpha(\tau) = \int_{\mathbb{T}} \zeta \, d\mu_\tau(\zeta)$ depend continuously on τ, $\alpha(\tau + \tau') = \alpha(\tau)\alpha(\tau')$, and $|\alpha(\tau)| \le 1$ for all τ. It follows that $\alpha(\tau) = e^{a\tau}$ for some complex number a with $\Re a \le 0$. Define now functions $u_\tau : e^{a\tau}\mathbb{D} \to \mathbb{D}$ by $u_\tau(z) = \eta_{\mu_\tau}(e^{-a\tau}z)$ for $z \in e^{a\tau}\mathbb{D}$. These functions are analytic, and they satisfy the equation

$$u_\tau(u_{\tau'}(z)) = u_{\tau+\tau'}(z), \quad z \in e^{a(\tau+\tau')}\mathbb{D}.$$

Moreover, the map $t \mapsto u_t(z)$ is easily seen to be continuous on the interval $[0, \tau]$, provided that $z \in e^{a\tau}\mathbb{D}$. The argument in Theorem 1.1 of [3] applies in this situation as well, and it implies that the map $t \mapsto u_t(z)$ is in fact differentiable, and the function

$$F(z) = \left.\frac{du_\tau(z)}{d\tau}\right|_0, \quad z \in \mathbb{D}$$

is analytic. It follows that the map $\tau \mapsto \eta_{\mu_\tau}(z)$ is differentiable as well, and the function A in the statement is analytic. In fact, we have $A(z) = F(z) - az$ since $\eta_{\mu_0}(z) = z$. In order to show that A has the required form, let us also consider the function $v_\tau(z) = e^{a\tau}u_\tau(z) = e^{a\tau}\eta_{\mu_\tau}(e^{-a\tau}z)$ defined in $e^{a\tau}\mathbb{D}$, for which

$$\left.\frac{dv_\tau(z)}{d\tau}\right|_0 = az + \left.\frac{du_\tau(z)}{d\tau}\right|_0 = A(z), \quad z \in \mathbb{D}.$$

For this function we have $|v_\tau(z)| \le |z| = |v_0(z)|$, so that indeed

$$\Re\frac{A(z)}{z} = \left.\frac{d\Re\log v_\tau(z)}{d\tau}\right|_{\tau=0} = \left.\frac{d\log|v_\tau(z)|}{\tau}\right|_{\tau=0} \le 0, \quad z \in \mathbb{D} \setminus \{0\}.$$

Let us then write $A(z) = zB(z)$, and verify that $a = -B(0)$. Indeed, all the functions $(u_\tau(z) - z)/\tau$ have a double zero at the origin, and therefore so does their limit $F(z)$; therefore $B(z) + a$ must be zero for $z = 0$.

Conversely, assume that B is an analytic function with negative real part in $\mathbb{D}$. It will suffice to show that the initial value problem

$$\frac{du_t(z)}{dt} = u_t(z)(B(u_t(z)) - B(0)), \quad u_t(0) = z \in e^{B(0)\tau}\mathbb{D}$$

has a solution defined on the entire interval $[0, \tau]$, and that

$$|u_\tau(z)| \le e^{-\Re B(0)\tau}|z|, \quad z \in e^{B(0)\tau}\mathbb{D}.$$

Indeed, once this is done, we can define the functions $\eta_\tau : \mathbb{D} \to \mathbb{D}$ by $\eta_\tau(z) = u_\tau(e^{B(0)\tau}z)$, and these functions will be of the form $\eta_\tau = \eta_{\mu_\tau}$ for some probability measures μ_τ which are easily seen to form a $\circlearrowright_0$-semigroup. The existence of the solutions u_t on the stated interval is easy to deduce from the general theory of ordinary differential equations. We sketch a somewhat more direct argument based on an appropriate approximation scheme. Namely, define functions $w_\varepsilon : \mathbb{D} \to \mathbb{C}$ by

$$w_\varepsilon(z) = ze^{\varepsilon(B(z)-B(0))} \quad z \in \mathbb{D}, \varepsilon > 0.$$

These functions satisfy $|w_\varepsilon(z)| \le e^{-\varepsilon B(0)}|z|$. We then define $u_\tau^{(n)} : e^{B(0)\tau}\mathbb{D} \to \mathbb{D}$ by

$$u_\tau^{(n)} = \underbrace{w_{\tau/n} \circ w_{\tau/n} \circ \cdots \circ w_{\tau/n}}_{n \text{ times}};$$

it is easy to see that $u_\tau^{(n)}$ is indeed defined in $e^{B(0)\tau}\mathbb{D}$. There exists a positive number δ such that $u_\tau^{(n)}|\delta\mathbb{D}$ converge uniformly as $n \to \infty$ to the solution u_τ of our initial value problem, provided that $\tau \le \delta$. Now, the functions $u_\tau^{(n)}$ are analytic and uniformly bounded on $e^{B(0)\delta}\mathbb{D}$ for $\tau \le \delta$, and therefore $\lim_{n\to\infty} u_\tau^{(n)}$ will exist (by the Vitali-Montel theorem) on the entire disk $e^{B(0)\delta}\mathbb{D}$ for all such τ. In an analogous fashion, we deduce that $u_\tau(z) = \lim_{n\to\infty} u_\tau^{(n)}(z)$ exists for all $z \in e^{B(0)\tau}\mathbb{D}$ if $\tau \le \delta$. Observe now the equality

$$u_\tau^{(n)} \circ u_{\tau'}^{(n')} = u_{\tau+\tau'}^{(n+n')} \quad \text{when} \quad \frac{\tau}{n} = \frac{\tau'}{n'},$$

which shows now that the convergence of $u_\tau^{(n)}$ can be extended from the interval $[0, \delta]$ to arbitrary $\tau > 0$, yielding a function u_τ defined in the common domain of $u_\tau^{(n)}$. Clearly these functions will solve the initial value problem in the required range. $\square$

The preceding result yields a parametrization of all $\circlearrowleft_0$-semigroups on the unit circle. In fact, every analytic function B with negative real part on $\mathbb{D}$ can be written using the Herglotz formula

$$B(z) = i\beta - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\rho(\zeta), \quad z \in \mathbb{D},$$

where β is a real number, and ρ is a finite positive Borel measure on $\mathbb{T}$. The constant $a = B(0)$ is then given by

$$a = i\beta - \rho(\mathbb{T}),$$

and the differential equation for u_τ is

$$\frac{du_t(z)}{dt} = 2u_t(z)^2 \int_{\mathbb{T}} \frac{d\rho(\zeta)}{u_t(z) - \zeta}, \quad u_0(z) = z \in e^{a\tau}\mathbb{D}.$$

As in the case of the half-line, the solutions of this equation can seldom be calculated explicitly. The case $\rho = 0$ corresponds with semigroups where each μ_τ is a point mass. In all cases when $\rho \ne 0$, it is easy to see that the measures μ_τ converge weakly to Haar measure m as $\tau \to \infty$. One semigroup which can be calculated explicitly corresponds with $B(z) = z^n - 1$, where $n \geq 1$ is an integer. We just mention the following formula:

$$u_\tau(z) = \frac{z}{(1-(n+1)z^n\tau)^{1/n}}, \quad z \in e^{-\tau}\mathbb{D},$$

where the root is chosen to be equal to one at the origin.

Infinite divisibility can also be characterized in terms of semigroups in the case of the circle. As for the half-line (where δ_0 is $\circlearrowright_0$-infinitely divisible, but not part of a semigroup), there is an exception, namely the Haar measure m which satisfies $m \circlearrowright_0 m = m \circlearrowright m = m$. More generally, we have the following result.

Lemma 4.3. *If μ_1, μ_2 are probability measures on $\mathbb{T}$, and $\int_\mathbb{T} \zeta\, d\mu_1(\zeta) = \int_\mathbb{T} \zeta\, d\mu_2(\zeta) = 0$, then $\mu_1 \circlearrowright_0 \mu_2 = m$.*

Proof. We have $\eta_{\mu_1 \circlearrowright \mu_2}(z) = \eta_{\mu_1}(\eta'_{\mu_2}(0)z) = \eta_{\mu_1}(0) = 0$ since $\eta'_{\mu_1}(0) = \eta'_{\mu_2}(0) = 0$. Alternatively, one observes that two monotonically independent variables x_1, x_2 such that $\varphi(x_1) = \varphi(x_2) = 0$ must satisfy $\varphi((x_1x_2)^n) = 0$ for all $n \geq 1$. □

We conclude that a $\circlearrowright_0$-infinitely divisible probability measure μ on $\mathbb{T}$ with first moment zero must in fact coincide with m. Indeed, $\mu = \mu_{1/2} \circlearrowright_0 \mu_{1/2}$, and the measure $\mu_{1/2}$ must also have first moment equal to zero.

Theorem 4.4. *Let $\mu \neq m$ be a $\circlearrowright_0$-infinitely divisible probability measure on $\mathbb{T}$. There exists a $\circlearrowright_0$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of probability measures on $\mathbb{T}$ such that $\mu_1 = \mu$.*

Proof. As noted before the statement, we can write $\int_\mathbb{T} \zeta\, d\mu(\zeta) = \rho e^{i\theta}$ with $\theta \in \mathbb{R}$ and $\rho > 0$. Choose for each integer $n \geq 1$ a measure ν_n such that $\mu = \nu_n^{\circlearrowright_0 2^n}$; these measures are no longer uniquely determined, but (possibly after an appropriate rotation) can be assumed to satisfy $\int_\mathbb{T} \zeta\, d\nu_n(\zeta) = \rho^{1/2^n} e^{i\theta/2^n}$. There exists a sequence $n_1 < n_2 < \cdots$ with the property that the each sequence $\{\nu_{n_j}^{\circlearrowright_0 2^{n_j-n}} : j \geq n\}$ has a weak limit; call this limit $\mu_{1/2^n}$. These measures will then satisfy

$$\int_\mathbb{T} \zeta\, d\mu_{1/2^n}(\zeta) = \rho^{1/2^n} e^{i\theta/2^n}, \quad \mu_{1/2^n}^{\circlearrowright_0 2^n} = \mu, \quad \text{and}$$
$$\mu_{1/2^n}^{\circlearrowright_0 2^m} = \mu_{1/2^{n-m}} \quad \text{for } m < n.$$

Note that the measures $\mu_{1/2^n}$ converge weakly to δ_1 as $n \to \infty$; indeed, their first moments converge to 1, and δ_1 is the only probability measure on $\mathbb{T}$ with first moment equal to one. We can now define

$$\mu_{m/2^n} = \mu_{1/2^n}^{\circlearrowright_0 m}$$

for m, n positive integers, and this is a good definition, i.e., it depends only on the fraction $m/2^n$ and not on the value of n. With this definition, it is still true that μ_τ tends weakly to δ_1 if $\tau \to 0$ is dyadic. Let now τ be an arbitrary positive number, and choose numbers τ_k, τ_k' of the form $m/2^n$ such that $\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \tau_k' = \tau$, and the sequences $\{\mu_{\tau_k}, k \geq 1\}, \{\mu_{\tau_k'}, k \geq 1\}$ have weak limits ν, ν'. Dropping to subsequences we can assume that $\tau_k < \tau_k'$ for all k. The equality $\mu_{\tau_k'} = \mu_{\tau_k' - \tau_k} \circlearrowright_0 \mu_{\tau_k}$ yields then $\nu' = \delta_1 \circlearrowright_0 \nu = \nu$. This unique limit can then be denoted μ_τ. It is easy to verify that the measures μ_τ form a multiplicative monotonic convolution semigroup, and $\mu_1 = \mu$. □

The semigroup provided by the preceding theorem is never unique. Thus, if the semigroup is generated (in the sense of Theorem 4.2) by the function $zB(z)$, then the function $z(B(z) + 2\pi i)$ will generate a new semigroup with $\mu_1 = \mu$. Of course, the only difference between these semigroups is a rotation of angle $2\pi\tau$ of the measure μ_τ. It is fairly easy to see that this is the only possible kind of nonuniqueness. More precisely, we have the following result.

PROPOSITION 4.5. *If $\mu, \mu_1, \mu_2 \in \mathfrak{M}$ are such that $\mu_1 \circlearrowright_0 \mu_1 = \mu_2 \circlearrowright_0 \mu_2 = \mu$ and $\mu_1(X) = \mu_2(X) \neq 0$, then $\mu_1 = \mu_2$. The same result is true for the operation $\circlearrowright$.*

Proof. If $\mu_1(X) = \mu_2(X) = 1$, then we have $\eta_{\mu_1} \circ \eta_{\mu_1} = \eta_{\mu_2} \circ \eta_{\mu_2} = \eta_\mu$. In this case the result follows from the argument of Proposition 5.4 in [6]. The general case reduces to this particular one by considering the new distributions $\nu_j(p(X)) = \mu_j(p(X/\alpha))$, $p \in \mathbb{C}[X]$, where $\alpha = \mu_1(X) = \mu_2(X)$. □

This result shows that in fact the measures ν_n in the proof of Theorem 4.4 are uniquely determined, and therefore there is precisely one semigroup for every choice of the argument of $\int_{\mathbb{T}} \zeta \, d\mu(\zeta)$.

The analogue of Theorem 4.2 for $\circlearrowright$-semigroups is obtained directly from the results of Berkson and Porta [3]. Indeed, the corresponding functions η_{μ_τ} simply form a composition semigroup of analytic maps of the disk, fixing the origin. We record the result below.

THEOREM 4.6. *Consider a $\circlearrowright$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of probability measures on $\mathbb{T}$. The map $\tau \mapsto \eta_{\mu_\tau}(z)$ is differentiable for every $z \in \mathbb{D}$, and the derivative*

$$A(z) = \left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0}$$

is an analytic function of z. Moreover, we can write $A(z) = zB(z)$, where B is analytic in $\mathbb{D}$ and $\Re B(z) \leq 0$ for $z \in \mathbb{D}$.

Conversely, for any analytic function B defined in $\mathbb{D}$, with $\Re B(z) \leq 0$ for $z \in \mathbb{D}$, there exists a unique $\circlearrowright$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of probability measures *on $\mathbb{T}$ such that*

$$\left.\frac{d\eta_{\mu_\tau}(z)}{d\tau}\right|_{\tau=0} = zB(z), \quad z \in \mathbb{D}.$$

The functions η_{μ_τ} satisfy the initial value problem

$$\frac{d\eta_{\mu_\tau}(z)}{d\tau} = \eta_{\mu_\tau}(z)B(\eta_{\mu_\tau}(z)), \quad \eta_{\mu_\tau}(0) = z \in \mathbb{D}.$$

Infinite divisibility is also characterized in terms of semigroups, and the remarks about uniqueness made about $\circlearrowright_0$-divisible measures apply here as well. The proofs given above are easily converted to this setting.

THEOREM 4.7. *Let $\mu \neq m$ be a $\circlearrowright$-infinitely divisible probability measure on $\mathbb{T}$. There exists a $\circlearrowright$-semigroup $\{\mu_\tau : \tau \geq 0\}$ of probability measures on $\mathbb{T}$ such that $\mu_1 = \mu$.*

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