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INFINITE PRODUCT IDENTITIES FOR L-FUNCTIONS

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ABSTRACT. We establish certain infinite product identities for Dirichlet series twisted by Dirichlet characters and give examples where the products have meromorphic continuation to the whole complex plane.

1. Introduction

The purpose of this note is to extend in scope product identities for Dirichlet L-series considered in [12] and [3]. We observe that the product identities are essentially local and hold for arbitrary Dirichlet series with Euler products. We will show that the two sides of the formulæ have natural interpretations in the contexts of Hasse-Weil L-functions and the L-functions of automorphic forms (see 3.1 and 3.2 below). This will allow us to deduce meromorphic continuation for the products in our identities in these cases. We will also prove a variant of our main formula which will allow us to obtain relations between values of L-functions at odd and even integers. Although the proofs are elementary, none of these observations seem to have been recorded in the literature. The authors are indebted to Professor J.-P. Serre for his numerous comments and corrections. This note was written while the first author was visiting the Tata Institute of Fundamental Research which he wishes to thank for its hospitality.

2. The main results

Let $L(s,\pi)$ be a Dirichlet series with an Euler product of the form

$$L(s,\pi) := \prod_p L_p(s,\pi_p) := \prod_p \prod_{j=1}^{m_p} (1-\alpha_j(p)p^{-s})^{-1},$$

where $L_p(s,\pi_p)$ denotes the local Euler factor at the prime p, the $\alpha_j(p)$ are complex numbers satisfying $|\alpha_j(p)| < p^{\delta}$, for some $\delta > 0$, and p runs over the

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set of rational primes. We will assume that there is a fixed $m_0 \in \mathbb{N}$ such that $m_p < m_0$ for all primes p. We define the twisted Dirichlet series

$$L(s, \pi \times \chi) := \prod_p L_p(s, \pi_p \times \chi_p) := \prod_p \prod_{j=1}^{m_p} (1 - \chi(p)\alpha_j(p)p^{-s})^{-1},$$

for any Dirichlet character χ. Here χ_p is the local character at p associated to χ and $L_p(s, \pi_p \times \chi_p)$ denotes the twisted local Euler factor at p. We denote by X_N the set of all Dirichlet characters modulo N, and by E_N the subset of X_N of Dirichlet characters such that $\chi(-1) = 1$. Recall that for all $\chi \in X_N$, $\chi(n) = 0$ if $(n, N) > 1$ (even if $\chi = 1$, the trivial character).

THEOREM 2.1. *With notation as above we have*

$$\prod_{N \geq 1} \prod_{\chi \in X_N} L(s+1, \pi \times \chi) = \frac{L(s, \pi)}{L(s+1, \pi)} \tag{2.1}$$

and

$$\prod_{N \geq 1} \prod_{\chi \in E_N} L(s+1, \pi \times \chi) = \left(\frac{L(s, \pi)}{L_2(s+1, \pi_2) L(s+1, \pi)} \right)^{1/2}, \tag{2.2}$$

for $\mathrm{Re}(s) > 1 + \delta$.

REMARK 2.2. The double product above is not absolutely convergent for any value of s. However, the outer product, that is, the product over $N \geq 1$ is absolutely convergent for $\mathrm{Re}(s) > 1 + \delta$.

REMARK 2.3. Analogues of the identities in Theorem 2.1 can be proved for Dirichlet series over number fields twisted by finite order ideal class characters. For simplicity of exposition, we restrict our attention to Dirichlet characters in this note.

The real content of (2.1) and (2.2) rests on purely local identities for individual Euler factors. We work in the ring of formal power series $\mathbb{C}[[T]]$. For any prime p we define

$$L_p(T, N) = \prod_{\chi \in X_N} (1 - \chi(p)T)^{-1} \quad \text{and} \quad L_p^0(T, N) = \prod_{\chi \in E_N} (1 - \chi(p)T)^{-1}.$$

Note that $L_p(T, N) = 1$ and $L_p^0(T, N) = 1$, if $(N, p) > 1$. We set

$$M_p(T) = \prod_{N \geq 1} L_p(T, N) \quad \text{and} \quad M_p^0(T) = \prod_{N \geq 1} L_p^0(T, N).$$

It is not *a priori* clear that the products above converge even in $\mathbb{C}[[T]]$ but we will show that this is the case below. We extract the following lemma from the proof of the theorem in [3] but give a slightly shorter proof.

LEMMA 2.4. *In the ring* $\mathbb{C}[[T]]$ *we have for any prime* p

$$\log M_p(T) = \log\left(\frac{1-T}{1-pT}\right), \tag{2.3}$$

and

$$\log M_p^0(T) = \begin{cases} \frac{1}{2}\log\left(\frac{1-T}{1-pT}\right) & \text{for } p > 2, \\ -\frac{1}{2}\log(1-2T) & \text{for } p = 2, \end{cases} \tag{2.4}$$

Moreover, $M_p(T)$ *and* $M_p^0(T)$ *converge in a circle of radius* p^{-1} *about the origin in* $\mathbb{C}$.

Proof. We first prove (2.3). The product giving $L_p(T,N)$ is finite, so we may take logarithms, expand in power series, differentiate and change the order of summation to obtain

$$\frac{d}{dT}\log L_p(T,N) = \sum_{\chi \pmod N}\sum_{m\geq 1}(\chi(p))^m T^{m-1} = \sum_{\substack{m\geq 1 \\ p^m\equiv 1 \pmod N}} \phi(N)T^{m-1},$$

We note that all the coefficients occurring in the last power series above are non-negative so we may change the order of summation freely below. Summing over N we get

$$\frac{d}{dT}\log M_p(T) = \sum_{N\geq 1}\sum_{\substack{m\geq 1 \\ p^m\equiv 1 \pmod N}} \phi(N)T^{m-1} = \sum_{l\geq 1} b_l T^{l-1}, \tag{2.5}$$

where $b_l = \sum_{N\,|p^l-1}\phi(N) = p^l - 1$. This already gives the convergence in $\mathbb{C}[[T]]$ since the l-th term is $b_l T^l$ and l tends to (plus) infinity. The radius of convergence can also easily be seen to be p^{-1} about the origin. From (2.5) we obtain

$$\frac{d}{dT}\log M_p(T) = \frac{p}{1-pT} - \frac{1}{1-T},$$

and (2.3) follows immediately upon integration. The proof of (2.4) is almost identical. We use only the additional fact that the number of even Dirichlet characters modulo N is $\phi(N)/2$ if $N > 2$. This proves the lemma. □

Recall that $\alpha_j(p)$ is a complex number such that $|\alpha_j(p)| < p^\delta$. If $T = \alpha_j(p)p^{-(s+1)}$ for s in $\mathbb{C}$, the condition $|T| < p^{-1}$ is satisfied if $\mathrm{Re}(s) > \delta$. Substituting this value of T for $j = 1, 2, \ldots, m_p$ and taking the product over j we obtain:

PROPOSITION 2.5. *With notation as above we have*

$$\prod_{N\geq 1}\prod_{\chi\in X_N} L_p(s+1, \pi_p\times\chi_p) = \frac{L_p(s,\pi_p)}{L_p(s+1,\pi_p)} \tag{2.6}$$

for $\mathrm{Re}(s) > \delta$ *and for all* p, *and*

$$\prod_{N\geq 1}\prod_{\chi\in E_N} L_p(s+1,\pi_p\times\chi_p) = \begin{cases} \left(\frac{L_p(s,\pi_p)}{L_p(s+1,\pi_p)}\right)^{1/2} & \textit{for } p>2,\\ L_2(s,\pi_2)^{1/2} & \textit{for } p=2,\end{cases} \tag{2.7}$$

for $\mathrm{Re}(s) > \delta$.

We may view the right hand sides of (2.6) and (2.7) as giving meromorphic continuations to the whole plane for the infinite products on the left hand sides. From the proposition above we can deduce (2.1) and (2.2) immediately by taking the product over all primes p. It can be easily checked (see [3]) that the order in which the products are taken over χ and p can be switched for $\mathrm{Re}(s) > 1+\delta$. This completes the proof of Theorem 2.1.

3. Examples

The formulations in (2.6) and (2.1) lend themselves easily to L-functions in the different contexts in which they arise such as Hasse-Weil zeta functions, automorphic L-functions, Galois representations, and l-adic representations. The purely local nature of the formulæ allows us to treat both local and global L-functions. We give one example in each case where the quotients of L-functions appearing in (2.6) and (2.1) have natural interpretations.

Example 3.1 (Hasse-Weil L-functions). Let X be a scheme of finite type over $\mathbb{Z}$. We recall that at each prime p we have the associated Hasse-Weil zeta function $L_p(s,X)$ (if we wish to consider the zeta function only at a single prime p then it is enough to consider schemes of finite type over $\mathbb{F}_p$). Here we only wish to note that $L_p(s,X)$ consists of a finite product of quotients of functions of the form $(1-\alpha(p)p^{-s})$, and hence, (2.6) is applicable in this context. We can verify easily that

$$L_p(s+1, X\times \mathbf{G}_m) = \frac{L_p(s,X)}{L_p(s+1,X)},$$

where $\mathbf{G}_m$ is the multiplicative group. Thus, (2.6) gives an infinite product identity for $L_p(s+1,X\times\mathbf{G}_m)$ in terms of the twists of the original zeta function by Dirichlet characters. We will denote by $L_p(s,\chi,X)$ the twist of the Hasse-Weil L-function by a character χ.

Following a suggestion of J.-P. Serre it is natural to view the above situation as follows. We view $S=\mathbf{G}_m$ as a scheme over $\mathbb{Z}$. For $N\geq 1$, we define the family of subschemes $S(N)$ by

$$S(N) = \mathrm{Spec}(\mathbb{Z}[T,1/N]/(\Phi_N(T))),$$

where $\Phi_N(T)$ is the N-th cyclotomic polynomial. For any scheme Y of finite type over $\mathbb{Z}$ we denote by $\tilde{Y}$ its atomization, that is, the set of closed points of Y viewed as a discrete topological space and equipped with the sheaf of fields

$k(y)$, where $k(y)$ denotes the residue field at the point y. With this notation it is easy to see that

$$\tilde{S} = \coprod_{N\geq 1}^{\infty} \widetilde{S(N)}.$$

As a consequence (see p. 85 of [9]) we can check that

$$L_p(s, X \times \mathbb{G}_m) = \prod_{N=1}^{\infty} L_p(s, X \times S(N)), \tag{3.1}$$

with absolute convergence for $\mathrm{Re}(s) > \dim X$, where $\dim X$ is the dimension of X (recall that $\mathrm{Re}(s) > \dim X$ is the domain of convergence of $L_p(s, X)$ viewed as a Dirichlet series).

We further note that $(\mathbb{Z}/N\mathbb{Z})^*$ acts freely on $S(N)$, and hence on $X \times S(N)$. This may be viewed as a special case of a finite group G acting on a scheme Y such that the quotient Y/G exists. In this case the L-function of Y naturally factors into the L-functions of Y/G twisted by the irreducible characters of G. In our case the group is $G = (\mathbb{Z}/N\mathbb{Z})^*$, so its character group is X_N. We thus have the formula (see equation (13), p. 88 of [9])

$$L_p(s, X \times S(N)) = \prod_{\chi \in X_N} L_p(s, \chi, X). \tag{3.2}$$

Combining equations (3.1) and (3.2) we see that we get a natural interpretation for the product in equation (2.6) in this case. The outer product (i.e., the product over N) simply corresponds to the decomposition of $\tilde{X} \times \tilde{S}$ as the disjoint union given in (3.1), while the inner products (i.e., the products over the $\chi \in X_N$) arise from the actions of the groups $(\mathbb{Z}/N\mathbb{Z})^*$ on $S(N)$.

More generally, let $P \to X$ be a principal bundle for a torus T over $\mathbb{Z}$ (or again over $\mathbb{F}_p$ if we are interested only in a single prime p) and let $L_p(s, P)$ be its Hasse-Weil zeta function. Then $L_p(s, P)$ can be expressed as a product of factors of the form $L_p(s, X)/L_p(s + 1, X)$. For the case of abelian varieties with complex multiplication see [12].

We note that since $L_p(s, X)$ is known to be a rational function of p^{-s} it can be meromorphically continued to the whole complex plane. We may thus view (2.6) as giving the meromorphic continuation of the infinite product to the whole complex plane.

Example 3.2 (Automorphic L-functions). We describe the case of automorphic representations in some detail. For simplicity and to illustrate our point clearly we specialise to the following situation. We consider the group $G_p = SL_2(\mathbb{Q}_p)$. We let T be the maximal torus, B the subgroup of upper triangular matrices and N its unipotent radical and N^- the unipotent radical of the lower triangular subgroup. Let χ_p be an unramified character of T and view it as a representation of B by extending it trivially on N. For any complex number s we define $\pi_p = I(\chi_p|\ |^s) = \mathrm{Ind}_{TN}^G(\chi_p|\ |^s\,{\delta_B}^{1/2})$, where δ_B is the modular character on B. The local L-factor of π_p is given by

$$L_p(s,\pi_p) = (1-\chi(p)p^{-(s+1/2)})^{-1}(1-\chi(p)^{-1}p^{-(s-1/2)})^{-1}.$$

Note that χ_p is given by a complex number s_0. If w is the Weyl group element of order 2, then $w(\chi_p|\ |^s)$ corresponds to the parameter $-s_0-s$. Let f be a spherical function in $I(\chi_p|\ |^s)$ and $\tilde{f}$ a spherical function in $I(w(\chi_p|\ |^s))$. It can be checked that

$$A(w,\chi_p)f = \int_{N^-} f(nw^{-1}g)dn \,:\, I(\chi_p|\ |^s) \to I(w(\chi_p|\ |^s))$$

acts on f by the formula $A(w,\chi_p)f = \lambda\tilde{f}$, where $\lambda = L_p(s,\pi_p)/L_p(s+1,\pi_p)$. We see that λ is exactly the quotient that appears in the right hand side of (2.6).

Globally, let $\pi = \otimes_p'\pi_p$ be a cuspidal automorphic representation of $SL_2/\mathbb{Q}$ and let S be a finite set of places containing the archimedean places and all the finite ramified places (it is known that we may take $\delta = 1/9$ in this case; see [5]). Applying (2.1) in this setting we see that the right hand side of this equation is nothing but $L_S(s,\pi)/L_S(s+1,\pi)$, where $L_S(\pi) = \prod_{p\notin S} L_p(s,\pi_p)$. This last quotient arises in the constant term of an Eisenstein series associated to π and can thus be shown to be meromorphic on the whole complex plane (see pp. 37–45 of [7]).

The above ideas work in great generality. If $\pi = \otimes_p'\pi_p$ is a cuspidal automorphic representation of a quasi-split reductive group $\mathbf{G}$ over $\mathbb{Q}$, the local L-factors are products of factors of the form $(1-\alpha(p)p^{-s})^{-1}$. The local quotients in the right hand side of (2.1) can be interpreted in terms of intertwining operators, while the global quotients occur in the constant term of an Eisenstein series. Using the Langlands-Shahidi method one can show the meromorphy (on the whole complex plane) of the global quotients for the following cuspidal automorphic representations: those of GL_n, $GL_m \times G'$, where G' is a split classical group, and the symmetric square, cube and fourth power lifts of cuspidal automorphic representations on GL_2 (see [2], [10], [11], [8], [1], [6], [4]). In fact, in most of the above cases it is known that the L-function $L(s,\pi)$ is itself actually entire.

4. Relations between odd and even L-values

Let O_N denote the set of odd Dirichlet characters in X_N. Rewriting (2.2) and combining it with (2.1) we obtain the following equations:

$$L(s,\pi)^{1/2} = L_2(s+1,\pi_2)^{1/2}L(s+1,\pi)^{1/2}\prod_{N\geq 1}\prod_{\chi\in E_N} L(s+1,\pi\times\chi) \tag{4.1}$$

and

$$L(s,\pi)^{1/2} = L_2(s+1,\pi_2)^{-1/2}L(s+1,\pi)^{1/2}\prod_{N\geq 1}\prod_{\chi\in O_N} L(s+1,\pi\times\chi), \tag{4.2}$$

for $\mathrm{Re}(s) > 1 + \delta$. Note that the equations above allow us to express odd (resp. even) L-values in terms of products of even (resp. odd) L-values. For instance, when π is the trivial character and for $s = 2m+1$, $m \geq 1$, (4.1) yields

$$\zeta(2m+1)^{1/2} = (1 - 2^{-2m-2})^{1/2}\zeta(2m+2)^{1/2} \prod_{N\geq 1} \prod_{\chi \in E_N} L(2m+2, \chi).$$

We thus obtain a formula for the odd values of the Riemann ζ-function in terms of the values at even integers of L-functions of even characters. Note that these last values can be explicitly computed in terms of Bernoulli numbers, so they are well understood. The formulæ above generate such relations for all L-functions of automorphic forms on any of the groups described in the previous section, in particular, for those on $GL_n/\mathbb{Q}$.

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