Illinois Journal of Mathematics
Volume 48, Number 4, Winter 2004, Pages 1385–1403
S 0019-2082

KILLING FIELDS, MEAN CURVATURE, TRANSLATION MAPS

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ABSTRACT. D. Hoffman, R. Osserman and R. Schoen proved that if the Gauss map of a complete constant mean curvature (cmc) oriented surface M immersed in $\mathbb{R}^3$ is contained in a closed hemisphere of $\mathbb{S}^2$ (equivalently, the function $\langle \eta, v\rangle$ does not change sign on M, where η is a unit normal vector of M and v some non-zero vector of $\mathbb{R}^3$), then M is invariant by a one parameter subgroup of translations of $\mathbb{R}^3$ (the one determined by v). We obtain an extension of this result to the case that the ambient space is a Riemannian manifold N and M is a hypersurface on N by requiring that the function $\langle \eta, V\rangle$ does not change sign on M, where V is a Killing field on N. We also obtain a stability criterium for cmc surfaces in N^3. In the last part of the article we consider a Killing parallelizable Riemannian manifold N and define a translation map $\gamma : M \to \mathbb{R}^n$ of a hypersurface M of N which is a natural extension of the Gauss map of a hypersurface in $\mathbb{R}^n$. Considering the same hypothesis on the image of γ we obtain an extension to this setting of the original Hoffmann-Osserman-Schoen result. Motivated by this extension, we restate in this context a conjecture made by M. P. do Carmo which, in $\mathbb{R}^3$, states that the Gauss image of a complete cmc surface which is not a plane nor a cylinder contains a neighborhood of some equator of the sphere.

1. Introduction

D. Hoffman, R. Osserman and R. Schoen proved that if the Gauss map of a complete constant mean curvature (cmc) oriented surface M immersed in $\mathbb{R}^3$ is contained in a closed hemisphere of $\mathbb{S}^2$, then M is invariant by a one parameter subgroup of translations of $\mathbb{R}^3$; it then follows that M is a circular cylinder or a plane (Theorem 1 of [HOS]). This result may be equivalently stated as follows: Let η be a unit normal vector field to M in $\mathbb{R}^3$. If, for some nonzero vector $V \in \mathbb{R}^3$, the map

$$f(p) := \langle \eta(p), V\rangle\,,\ p \in M, \tag{1.1}$$

Received May 7, 2004; received in final form July 19, 2004.
2000 *Mathematics Subject Classification.* Primary 53. Secondary 58.

does not change sign on M, then M is invariant by the one parameter subgroup of translations determined by V.

This interpretation of the Hoffman-Osserman-Schoen result admits an extension to hypersurfaces of a Lie group, obtained by K. Frensel, N. do Espírito Santo and both authors of the present article, as follows: Let M be a complete cmc n-dimensional hypersurface of a Lie group $\mathbb{G}$ with a bi-invariant metric and assume that the function f in (1.1) does not change sign on M, where V is a left invariant vector field on $\mathbb{G}$. Then, if M is compact or $n = 2$, M is invariant by the one parameter subgroup of isometries of $\mathbb{G}$ determined by V; it follows, when $n = 2$, that M is isometric to a plane, a circular cylinder or a flat torus (see Theorem 4 of [EFFR]).

More recently, the authors realized that this same result holds in a Lie group with only a left invariant metric and for any right invariant vector field V. Considering that such vector fields are Killing fields, one may therefore ask the following general question:

QUESTION. *Let M be a complete constant mean curvature H oriented hypersurface immersed on a $(n+1)$-dimensional Riemannian manifold N. Assume that the function f given by* (1.1) *does not change sign on M, where V is a Killing vector field in N. Is it true that M is invariant by the one parameter subgroup of isometries of N determined by V?*

We will prove (Corollary 1):

If M is compact and the Ricci curvature of N satisfies

$$\mathrm{Ric}(W) \geq -nH^2$$

for any unit tangent vector W of N, then the answer is positive or M is umbilic and N has constant non-positive Ricci curvature $\mathrm{Ric}(\eta) = -nH^2$ *on the η-direction at the points of M. In particular, the answer is positive if N has positive Ricci curvature and M is compact.*

For M complete and simply connected we prove the following (Corollary 2):

When N has dimension 3 *and*

$$\mathrm{Ric}(W) \geq -2H^2,$$

for any unit tangent vector W of N, if the function f given by (1.1) *does not change sign on M, for some Killing field V of N and, either M has the conformal type of the disk or of the sphere or M is conformally the plane and f is bounded on M, then the answer is positive, that is, M is invariant by the one parameter subgroup of isometries of N determined by V , or M is umbilic and N has constant non-positive Ricci curvature* $\mathrm{Ric}(\eta) = -nH^2$ *on the η-direction.*

We note that this result gives the following extension of the Hoffman-Osserman-Schoen theorem also in $\mathbb{R}^3$ (see Section 3 and Corollary 5 for more details):

If the function $\langle \eta, V \rangle$ does not change sign on a complete cmc surface M in $\mathbb{R}^3$, where V is a Killing field of $\mathbb{R}^3$ which is bounded on M, then M is a helicoidal surface. In particular, if V is translational, then M is a cylinder or a plane, and if V is rotational, then M is a Delaunay surface.

The proofs of the above results are based on the following formula for the Laplacian of f:

$$\Delta f = -n \langle \nabla H, V \rangle - (\mathrm{Ric}(\eta) + \|B\|^2) f. \tag{1.2}$$

This formula has several other implications, one of them being a stability criterium for cmc surfaces. Recall that, by a well known result on the stability of cmc surfaces in $\mathbb{R}^3$ obtained by M. do Carmo and L. Barbosa [BdoC], we have the following criterion of stability: If the Gauss image of a cmc surface M in $\mathbb{R}^3$ lies in a closed hemisphere of the sphere, then M is stable. This result of do Carmo and Barbosa can also be rephrased in terms of a translational Killing field as follows: If M is a cmc surface in $\mathbb{R}^3$ (not necessarily complete) such that, for some vector $V \in \mathbb{R}^3$, the function f does not change sign, then M is stable. We obtain here the following extension (Corollary 3):

Let N be a 3-dimensional Riemannian manifold such that

$$\mathrm{Ric}(W) \geq -2H^2$$

for any unit tangent vector W of N. Let M be a surface of constant mean curvature H (not necessarily complete) and let D be a domain in M such that $\overline{D} \subset \mathrm{int}(M)$. Let V be a Killing vector field on N and assume that f has a sign on $\overline{D}$. Then $\overline{D}$ is stable.

This result can be applied to prove the stability of constant mean curvature graphs in some twisted products, such as warped products (Corollary 4).

We can also make use of formula (1.2) to get an extension of the well known formula

$$\Delta g = -\|B\|^2 g \tag{1.3}$$

satisfied by the Gauss map $g : M \to \mathbb{S}^n$ of an immersed cmc surface in $\mathbb{R}^{n+1}$. In fact, in the last part of the article we use Killing vector fields on a Killing parallelizable Riemannian manifold N (that is, N admits $n+1$ Killing vector fields linearly independent at each point of N $(n+1 = \dim N)$) to define a translation map $\gamma : M \to \mathbb{R}^n$ of a hypersurface M of N (see (4.1) and (4.2)). This map is a natural extension of the Gauss map g of a hypersurface in $\mathbb{R}^n$ and we call it a Killing translation map. We then use (1.2) to prove that M

has cmc if and only if γ satisfies the equation

$$\Delta\gamma = -\left(\mathrm{Ric}(\eta) + \|B\|^2\right)\gamma. \tag{1.4}$$

Formula (1.4) can be used in many different situations. We use it here to obtain a characterization of cmc hypersurfaces from hypotheses on the image of γ. In particular, as we will see, it leads in this context to a natural and interesting extension of a conjecture of M. P. do Carmo which, in $\mathbb{R}^3$, asserts that the Gauss image of a complete cmc surface which is not a plane nor a cylinder contains a neighborhood of an equator of the sphere (see the end of Section 4).

We finally remark that examples of complete Killing parallelizable Riemannian manifolds are Lie groups with a left invariant metric. In particular, the above results apply to any symmetric space of noncompact type since, by the Iwasawa decomposition, any such space is isometric to a Lie group with some left invariant metric.

2. Killing fields and constant mean curvature hypersurfaces

Let N be a Riemannian $(n+1)$-dimensional manifold. Recall that the curvature tensor R of N is defined by

$$R(X,Z)Y, = \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y + \nabla_{[X,Z]}Y$$

and the Ricci tensor of N is defined by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n+1} \langle R(X,W_i)Y, W_i\rangle\,,$$

where X, Y, Z are tangent to N and $\{W_1, \ldots, W_{n+1}\}$ is an orthonormal basis of the tangent space of N. The Ricci curvature of N on the Z-direction is

$$\mathrm{Ric}(Z) = \mathrm{Ric}(Z,Z).$$

We begin by proving formula (1.2). Its proof is somewhat long, but straightforward, so we indicate only the main steps.

Proposition 1. *Let N be a $(n+1)$-dimensional manifold and let V be a Killing vector field of N. Let M be an oriented hypersurface of N and assume that η is a unitary normal vector field to M in N. Then, setting*

$$f(p) := \langle \eta(p), V(p)\rangle\,,\ p \in M,$$

we have

$$\Delta f = -n\,\langle V, \nabla H\rangle - \left(\mathrm{Ric}(\eta) + \|B\|^2\right)f, \tag{2.1}$$

where H denotes the mean curvature function of M with respect to η and ∇H its gradient, $\mathrm{Ric}(\eta)$ *the Ricci curvature of N in the η-direction, $\|B\|$ the norm of the second fundamental form B of M in N, and Δ the Laplacian of M on* *the metric induced by N. In particular, if M has constant mean curvature, then f satisfies*

$$\Delta f = -(\mathrm{Ric}(\eta) + \|B\|^2)f. \tag{2.2}$$

Proof. We fix a point $p \in M$ and take an orthonormal basis $E_1(p), \dots, E_n(p)$ that diagonalizes B in p; let $\lambda_1, \dots, \lambda_n$ be the eigenvalues associated to $E_1(p), \dots, E_n(p)$. Denote by $E_1, \dots, E_n$ a geodesic frame extending $E_1(p), \dots, E_n(p)$ in a neighborhood of p in M. Then, at p,

$$\Delta f = \sum_{i=1}^{n} E_i E_i(f). \tag{2.3}$$

Using that V is a Killing field and that $E_1(p), \dots, E_n(p)$ diagonalizes the second fundamental form of M in p we easily obtain

$$E_i E_i(f) = \langle \nabla_{E_i} \nabla_{E_i} \eta, V \rangle + \langle \eta, \nabla_{E_i} \nabla_{E_i} V \rangle, \ i = 1, \dots, n. \tag{2.4}$$

Extend E_i in a neighborhood of p in N parallel to the geodesics normal to M. Then $[\eta, E_i] = \lambda_i E_i$ at p, so that $\left\langle \nabla_{[\eta, E_i]} V, E_i \right\rangle = 0$ and

$$\sum_{i=1}^{n} \langle \eta, \nabla_{E_i} \nabla_{E_i} V \rangle = -\sum_{i=1}^{n} \langle R(\eta, E_i)V, E_i \rangle = -\mathrm{Ric}(\eta, V). \tag{2.5}$$

To estimate $\langle \nabla_{E_i} \nabla_{E_i} \eta, V \rangle$ it is convenient to write, on M,

$$V = \sum_{j=1}^{n} v_j E_j + f\eta,$$

so that

$$\langle V, \nabla_{E_i} \nabla_{E_i} \eta \rangle = \sum_{j=1}^{n} v_j \langle E_j, \nabla_{E_i} \nabla_{E_i} \eta \rangle + f \langle \eta, \nabla_{E_i} \nabla_{E_i} \eta \rangle .$$

We have

$$\langle E_j, \nabla_{E_i} \nabla_{E_i} \eta \rangle = \langle R(E_i, E_j)E_i, \eta \rangle - E_j \left(\langle \nabla_{E_i} E_i, \eta \rangle \right)$$

and $\langle \eta, \nabla_{E_i} \nabla_{E_i} \eta \rangle = -\lambda_i^2$. It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \langle V, \nabla_{E_i} \nabla_{E_i} \eta \rangle &= \sum_{i,j=1}^{n} v_j \left(\langle R(E_i, E_j)E_i, \eta \rangle - E_j \left(\langle \nabla_{E_i} E_i, \eta \rangle \right) \right) - f\lambda_i^2 \\
&= \sum_{i=1}^{n} \left\{ \langle R\left(E_i, V - f\eta\right) E_i, \eta \rangle - \left(\sum_{j=1}^{n} v_j E_j \right) \left(\langle \nabla_{E_i} E_i, \eta \rangle \right) - f\lambda_i^2 \right\} \\
&= \mathrm{Ric}(\eta, V) - f\,\mathrm{Ric}(\eta) - \left(\sum_{j=1}^{n} v_j E_j \right) (nH) - f \sum_{i=1}^{n} \lambda_i^2,
\end{aligned}$$

so that

$$\sum_{i=1}^{n} \langle V, \nabla_{E_i} \nabla_{E_i} \eta \rangle = \mathrm{Ric}(\eta, V) - f\,\mathrm{Ric}(\eta) - n \langle V, \nabla H \rangle - f \|B\|^2. \tag{2.6}$$

We have used above that

$$nH = \sum_{i=1}^{n} \langle \nabla_{E_i} E_i, \eta \rangle .$$

Substituting (2.4) in (2.3) and using (2.5) and (2.6) we obtain (2.1). □

Remark 1. Formula (1.2) is known in the case when the mean curvature is constant and the ambient space has constant sectional curvature. A proof is given in the paper [Ro] of H. Rosenberg (see the derivation of formula (3.4) and the comments on p. 227 of [Ro]). Formula (1.2) can be easily obtained, by using the same technique, when the mean curvature is constant and without hypothesis on the curvature of the ambient space. However, we could not obtain (1.2) in the general case using this technique. In fact, it seems that the constancy of the mean curvature is essential for the arguments used in [Ro].

Corollary 1. *Let N be a Riemannian $(n+1)$-dimensional manifold. Let M be a compact hypersurface of constant mean curvature H immersed in N and assume that*

$$\mathrm{Ric}(W) \geq -nH^2$$

for any unit tangent vector W of N. Let V be a Killing vector field on N. If the function

$$f = \langle \eta, V \rangle$$

does not change sign on M, then M is invariant by the one parameter subgroup of isometries of N determined by V or M is umbilic and N has constant non-positive Ricci curvature $\mathrm{Ric}(\eta) = -nH^2$ on the η-direction.

Proof. Let us assume that $f \geq 0$. Noting that $\langle V, \nabla H \rangle = 0$ and

$$\mathrm{Ric}(\eta) + \|B\|^2 \geq \mathrm{Ric}(\eta) + nH^2 \geq 0, \tag{2.7}$$

it follows from Proposition 1 that

$$\Delta f \leq 0.$$

Since M is compact, it follows from Hopf's theorem that $f = cte$ and then $\Delta f = 0$. From (1.2) we obtain $f \equiv 0$ or $\mathrm{Ric}(\eta) + \|B\|^2 \equiv 0$. In the first case we conclude that V is a vector field on M so that M is invariant by the one parameter subgroup of isometries determined by V. In the second case, it follows from (2.7) that $\|B\|^2 = nH^2 = -\mathrm{Ric}(\eta)$, and the equality $\|B\|^2 = nH^2$ implies that M is umbilic. This concludes the proof of the corollary. □

Corollary 2. *Let N be a Riemannian 3-dimensional manifold. Let M be a complete, connected, simply connected surface of constant mean curvature H immersed in N such that*

$$\mathrm{Ric}(W) \geq -2H^2$$

for any unit tangent vector W of N. Let V be a Killing vector field on N and assume that the function

$$f = \langle \eta, V \rangle$$

does not change sign on M. If

(a) *M has the conformal type of the disk or the sphere, or*
(b) *M has the conformal type of the plane and $\|V\|$ is bounded on M,*

then M is invariant by the one parameter subgroup of isometries of N determined by V or M is umbilic and N has constant non-positive Ricci curvature $\mathrm{Ric}(\eta) = -2H^2$ on the η-direction at the points of M.

Proof. The proof is essentially the same as that of Theorem 1 of [HOS]. For completeness we reproduce it here. If M is the sphere, then Corollary 2 reduces to Corollary 1. Let us consider the case when M has the conformal type of the disk. Suppose that $f \leq 0$. Then

$$\Delta f = -(\mathrm{Ric}(\eta) + \|B\|^2) f \geq -(\mathrm{Ric}(\eta) + 2H^2) f \geq 0,$$

so that f is subharmonic. Therefore, if $f = 0$ at some a point of M, then $f \equiv 0$ by the maximum principle, so that M is invariant by V. Let us prove that the case $f < 0$ cannot occur. By contradiction, assume that $f < 0$ everywhere. We have, by the Gauss equation,

$$\|B\|^2 = 4H^2 - 2(K - K_N),$$

where K is the Gauss curvature of M and K_N the sectional curvature of N on the tangent plane of M. From (2.2) we obtain

$$\Delta f - 2Kf + (\mathrm{Ric}(\eta) + 2K_N + 4H^2) f = 0. \tag{2.8}$$

Considering an orthonormal basis E_1, E_2 of the tangent planes of M we obtain

$$\begin{aligned} \mathrm{Ric}(\eta) + 2K_N &= \langle R(\eta, E_1)\eta, E_1 \rangle + \langle R(\eta, E_2)\eta, E_2 \rangle + 2 \langle R(E_1, E_2)E_1, E_2 \rangle \\ &= \langle R(E_1, \eta)E_1, \eta \rangle + \langle R(E_1, E_2)E_1, E_2 \rangle \\ &\quad + \langle R(E_2, \eta)E_2, \eta \rangle + \langle R(E_2, E_1)E_2, E_1 \rangle \\ &= \mathrm{Ric}(E_1) + \mathrm{Ric}(E_2). \end{aligned}$$

Then, from the hypothesis

$$\mathrm{Ric}(\eta) + 2K_N + 4H^2 \geq 0.$$

However, (2.8) contradicts Corollary 3 of [FS], which states that when K is the Gauss curvature of a complete conformal metric on the unit disk there can be no negative solution of equation (2.8) if

$$\mathrm{Ric}(\eta) + 2K_N + 4H^2 \geq 0.$$

Let us assume now that $M = \mathbb{R}^2$ (conformally) and that $\|V\|$ is bounded on M. Then f is subharmonic and bounded on $\mathbb{R}^2$ and it follows that f is constant. Then $\Delta f = 0$ and, from (2.2),

$$(\mathrm{Ric}(\eta) + \|B\|^2)f = 0.$$

The conclusion of the corollary then follows as in the previous result. □

Remark. Corollary 2 applies in an obvious way to a complete, not necessarily simply connected surface, by composing the immersion with the universal covering map of the surface.

The next result is an extension of Theorem 5 of [EFFR] and has the same proof. For the sake of completeness, we reproduce it here.

Corollary 3. *Let N be a Riemannian 3-dimensional manifold and assume that*

$$\mathrm{Ric}(W) \geq -2H^2$$

for any unit tangent vector W of N. Let M be a surface of constant mean curvature H (not necessarily complete) and let D be a domain in M such that $\overline{D} \subset \mathrm{int}(M)$. Let V be a Killing vector field on N and assume that $f = \langle V, \eta\rangle$ has a sign on $\overline{D}$. Then $\overline{D}$ is stable.

Proof. Let us assume that $f > 0$ on $\overline{D}$. From (1.2) we have

$$\Delta f \;=\; -(\mathrm{Ric}(\eta) + \|B\|^2)f \leq -(\mathrm{Ric}(\eta) + nH^2)f \leq 0.$$

Then, from the Corollary 1 of [FS], $\lambda_1(\overline{D}) \geq 0$, where $\lambda_1(\overline{D})$ is the first eigenvalue of the Laplacian operator over $\overline{D}$. We will show that, in fact, $\lambda_1(\overline{D}) > 0$. Since $\overline{D} \subset \mathrm{int}(M)$, there exists a domain $D' \subset M$ such that $\overline{D} \subset D'$, $D' \backslash \overline{D} \neq \emptyset$ and $f|_{D'} > 0$. Again, from Corollary 1 of [FS], $\lambda_1(D') \geq 0$. But, from the Lemma of [FS], $\lambda_1(D') < \lambda_1(\overline{D})$, so $\lambda_1(\overline{D}) > 0$. This proves Corollary 3. □

It follows from this corollary that any radial or horizontal cmc graph in the half space model for the hyperbolic space is stable. (These graphs have been recently studied; see [GE], [N].) In fact, these are particular cases of a more general result stated below.

Given $N = N_0 \times \mathbb{R}$, let $\langle , \rangle$ be a twisted product in N of the form

$$\langle , \rangle = \alpha^2 \langle , \rangle_{N_0} + \beta^2 \langle , \rangle_{\mathbb{R}} ,$$

where $\langle , \rangle_{N_0}$ is a Riemannian metric on N_0, $\langle , \rangle_{\mathbb{R}}$ the multiplication on $\mathbb{R}$ and α, β are positive functions on M. When $\alpha \equiv 1$, N is a warped product (see [MRS]). We use below the notation $N = N_0 \times_{(\alpha,\beta)} \mathbb{R}$ for this Riemannian metric on N. A graph on N is a hypersurface of the form $(x, u(x))$ for x in some open subset Ω of N_0, where u is a smooth function on Ω.

Corollary 4. *Let $N = N_0 \times_{(\alpha,\beta)} \mathbb{R}$ be as above and assume that*

$$\mathrm{Ric}(W) \geq -2H^2$$

for any unit tangent vector W of N. Let M be a constant mean curvature H graph on N and let D be a domain in M such that $\overline{D} \subset \mathrm{int}(M)$. Then $\overline{D}$ is stable.

Proof. We have only to note that the vector field $V = d/dr$ is a Killing vector field on N and, since M is a graph, V is everywhere transversal to M. □

3. The spaces of constant sectional curvature

In this section we analyze in greater details Corollaries 2 and 3 when N is a simply connected 3-dimensional space of constant sectional curvature, that is, $\mathbb{R}^3$, $\mathbb{S}^3$, or $\mathbb{H}^3$. Our first observations however apply to general Riemannian manifolds. Thus, let us assume that N is a $(n+1)$-dimensional Riemannian manifold and let $G = \mathrm{ISO}(N)$ be the Lie group of isometries of N. Denote by $\mathcal{G}$ the Lie algebra of G. Let

$$\exp : \mathcal{G} \to G$$

be the exponential map. Recall that, given $g \in G$, the adjoint map $\mathrm{Ad}_g : \mathcal{G} \to \mathcal{G}$ is given by

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\left[g(\exp tX)g^{-1}\right]_{t=0}, \; X \in \mathcal{G}.$$

We can verify that it satisfies

$$g(\exp X)g^{-1} = \exp \mathrm{Ad}_g(X).$$

A Killing field X of N is given as follows: Taking $X^* \in \mathcal{G}$ we have, at $p \in N$,

$$X(p) = \frac{d}{dt}(\exp tX^*)(p)\bigg|_{t=0}.$$

In what follows, we denote X^* also by X.

We say that a submanifold M of N is X-invariant if $X(p) \in T_pM$ for all $p \in M$.

PROPOSITION 2. *Let X, Y be Killing vector fields on N such that $X = \operatorname{Ad}_g Y$ for some $g \in G = \operatorname{ISO}(N)$. Then any submanifold M of N which is X-invariant is congruent to a Y-invariant submanifold of N.*

Proof. Let S be a Y-invariant submanifold of N. Then

$$\begin{aligned}(\exp tX)(g(S)) &= g(g^{-1}(\exp tX)g)(S)\\ &= g(\exp t \operatorname{Ad}_{g^{-1}} X)(S)\\ &= g(\exp tY)(S) = g(S),\end{aligned}$$

proving the proposition. □

We now specialize to the 3-dimensional spaces of constant sectional curvature. We will use some standard facts about semi-simple Lie algebras, which can be found in [W].

The Euclidean case. The isometry group $G = \operatorname{ISO}(\mathbb{R}^3)$ of $\mathbb{R}^3$ can be interpreted as the matrix group

$$G = \left\{ \begin{bmatrix} & & & u \\ & A & & v \\ & & & w \\ 0 & 0 & 0 & 1 \end{bmatrix};\ A \in O(3),\ u, v, w \in \mathbb{R} \right\}$$

acting on $\mathbb{R}^3$ as

$$\begin{bmatrix} & & & u \\ & A & & v \\ & & & w \\ 0 & 0 & 0 & 0 \end{bmatrix}(x, y, z) = \begin{bmatrix} & & & u \\ & A & & v \\ & & & w \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

The Lie algebra $\mathcal{G}$ of G is the semi-simple rank 1 Lie algebra given by

$$\mathcal{G} = \left\{ \begin{bmatrix} 0 & x & y & u \\ -x & 0 & z & v \\ -y & -z & 0 & w \\ 0 & 0 & 0 & 0 \end{bmatrix}; x, y, z,\ u, v, w \in \mathbb{R} \right\}$$

with Lie bracket $[X, Y] = XY - YX$ having a Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \alpha \in \mathbb{R} \right\}.$$

Given $X \in \mathcal{G}$, by the Jordan-Chevalley decomposition theorem we may write

$$X = X_S + X_N,$$

where X_S is semi-simple (X_S is diagonalizable over $\mathbb{C}$) and X_N is nilpotent, with $[X_S, X_N] = 0$. If $X_S = 0$, then $X = X_N$ is nilpotent and it is easy to see that X induces a translation on $\mathbb{R}^3$ and that there is $g \in G$ such that

$$\mathrm{Ad}_g(X) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $X_S \neq 0$, then, by Cartan's subalgebras conjugation theorem, there is $g \in G$ such that

$$\mathrm{Ad}_g(X_S) = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $\alpha \neq 0$. Since

$$[\mathrm{Ad}_g(X_S), \mathrm{Ad}_g(X_N)] = \mathrm{Ad}_g([X_S, X_N]) = 0,$$

a direct computation shows that

$$\mathrm{Ad}_g(X_N) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $\beta \in \mathbb{R}$, so that

$$\mathrm{Ad}_g(X) = \mathrm{Ad}_g(X_S + X_N) = \mathrm{Ad}_g(X_s) + \mathrm{Ad}_g(X_N) = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have therefore proved that any Killing field in $\mathbb{R}^3$ is Ad-conjugated to a Killing field of the form

$$X_{\alpha,\beta} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\phi_{a,\beta}$ be the subgroup of isometries of $\mathbb{R}^3$ determined by $X_{\alpha,\beta}$, that is,

$$\phi_{\alpha,\beta}(t)(x, y, z) = \begin{bmatrix} \cos\alpha t & \sin\alpha t & 0 & 0 \\ -\sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & 1 & t\beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \ (x, y, z) \in \mathbb{R}^3.$$

We may then use Proposition 2 and Corollary 2 to obtain:

Corollary 5. *Let M be a complete surface of constant mean curvature immersed in $\mathbb{R}^3$. Assume that the function*

$$f = \langle \eta, V \rangle$$

does not change sign on M, where V is a Killing field of $\mathbb{R}^3$. If

(a) *M has the conformal type of the disk or the sphere, or*

(b) *M has the conformal type of the plane and $\|V\|$ is bounded on M,*

then M, up to congruence, is $\phi_{\alpha,\beta}$-invariant for some $\alpha, \beta \in \mathbb{R}$.

If $\alpha = 0$ and $\beta \neq 0$, then $X_{0,\beta}$ is a translation on $\mathbb{R}^3$, which is of course bounded in the whole space. It is easy to prove that the only cmc $X_{0,\beta}$-invariant surfaces are the right circular cylinder and the plane. This is the case considered in Theorem 1 of [HOS].

If $\alpha \neq 0$ and $\beta = 0$, then the $X_{a,0}$-invariant surfaces are the surfaces of revolution. The cmc revolution surfaces are called Delaunay surfaces and are well known. They were first studied in 1841 by Charles Delaunay [D], who discovered the rolling method of construction of the generating curve of these surfaces.

In general, that is, for arbitrary values of α and β, the $X_{\alpha,\beta}$-invariant surfaces are known as helicoidal surfaces. Helicoidal surfaces with cmc were studied by M. P. do Carmo and M. Dajczer [doCD].

The spherical case. It follows from the maximal torus theorem that any Killing field of $\mathbb{S}^3$ is Ad-conjugated to a Killing field of the form

$$X_{\alpha,\beta} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix} \in \mathrm{so}(4),$$

where so(4) is the Lie algebra of isometry group of $\mathbb{S}^3$. Let $\phi_{a,\beta}$ be the subgroup of isometries of $\mathbb{S}^3$ determined by $X_{\alpha,\beta}$, that is,

$$\phi_{\alpha,\beta}(t) = \begin{bmatrix} \cos\alpha t & \sin\alpha t & 0 & 0 \\ -\sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & \cos\beta t & \sin\beta t \\ 0 & 0 & -\sin\beta t & \cos\beta t \end{bmatrix}.$$

Using again Proposition 2 and Corollary 2, and taking into account that any Killing vector field in $\mathbb{S}^3$ is obviously bounded, we obtain:

Corollary 6. *Let M be a complete surface of constant mean curvature immersed in $\mathbb{S}^3$. Assume that the function*

$$f = \langle \eta, V \rangle$$

does not change sign on M, where V is a Killing field of $\mathbb{S}^3$. Then M, up to congruence, is $\phi_{\alpha,\beta}$-invariant for some $\alpha, \beta \in \mathbb{R}$.

If $\alpha = 0$ or $\beta = 0$ but $X_{\alpha,\beta} \neq 0$, then a $X_{a,\beta}$-invariant surface is a surface of revolution in $\mathbb{S}^3$. We give below a brief description of these surfaces. If

$$\phi_t = \exp tX_{0,1},$$

then a ϕ_t-invariant surface is given by

$$S_\gamma := \{\phi_t(\gamma) \mid t \in \mathbb{R}\},$$

where γ is a regular curve on the orbit space for ϕ_t:

$$\mathbb{S}^2_+ = \{(x, y, z, 0) \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}.$$

Using the coordinates

$$x = \sqrt{1 - z^2} \cos\theta,$$
$$y = \sqrt{1 - z^2} \sin\theta,$$

for $\mathbb{S}^2_+$ we see that if γ is parametrized by arc length in the coordinates $z = z(t)$, $\theta = \theta(t)$ with $\theta(0) = 0$, then S_γ has cmc H if and only if

$$\theta(t) = \int_0^t \frac{\sqrt{1 - z^2(s) - (z'(s))^2}}{1 - z^2(s)} ds$$

and $z(t)$ satisfies the ODE

$$zz'' + (z')^2 + 2z^2 + 2Hz\sqrt{1 - z^2 - (z')^2} - 1 = 0.$$

This equation can be completely integrated; in fact, assuming that a certain initial condition is satisfied, the solutions are given by

$$z(t) = \sqrt{A \sin\left(2\sqrt{1 + H^2}t + \frac{\pi}{2}\right) - \frac{H^2 + H\sqrt{1/4 - c^2}}{1 + H^2} + \frac{1}{2}},$$

where

$$A = \frac{\sqrt{H^2 - 4(H^2 - 1)c^2 \pm 8Hc\sqrt{1/4 - c^2}}}{2(1 + H^2)}$$

and c is a parameter that can vary on the interval $[-1/2, 1/2)$.

We note that the rolling constructions are also valid for the cmc surfaces of revolution in $\mathbb{S}^3$. This fact was proved by I. Sterling [S].

With respect to the $X_{\alpha,\beta}$-invariant surfaces with cmc in $\mathbb{S}^3$ we observe that if $\alpha = \beta$, then $X_{\alpha,\alpha}$ is a Hopf vector field and the only cmc $X_{\alpha,\alpha}$-invariant surfaces are the Clifford tori (see the proof of Theorem 4 of [EFFR]). We did not find in the literature a description of the cmc $X_{\alpha,\beta}$-invariant surfaces in $\mathbb{S}^3$ when $\alpha \neq \beta$ and $\alpha \neq 0 \neq \beta$, with the exception of a paper of W. Y. Hsiang and B. Lawson [HL] describing the minimal $X_{\alpha,\beta}$-invariant surfaces for α and β integers.

The hyperbolic case. The isometry group $O(3,1) = \mathrm{ISO}(\mathbb{H}^3)$ is isomorphic to the group of linear transformations of $\mathbb{R}^4$ that preserves the quadratic form

$$q(x) = -x_1^2 + x^2 + x_3^2 + x_4^2$$

and whose Lie algebra $o(3,1)$ is given by

$$o(3,1) = \left\{ \begin{bmatrix} 0 & a & b & c \\ a & 0 & x & y \\ b & -x & 0 & z \\ c & -y & -z & 0 \end{bmatrix} ;\ a,b,c,x,y,z \in \mathbb{R} \right\}.$$

It is a basic fact that $o(3,1)$ is a semisimple Lie algebra of rank 2. We may then see that

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix} ;\ \alpha,\beta \in \mathbb{R} \right\}$$

is a Cartan subalgebra of $o(3,1)$. Given $X \in o(3,1)$, we use the Jordan-Chevalley decomposition theorem to write

$$X = X_S + X_N$$

with X_S semisimple and X_N nilpotent and $[X_S, X_N] = 0$. If $X_S = 0$, then $X = X_N$ and we may prove that X_N is Ad-conjugated to a Killing field of the form

$$X_\gamma = \begin{bmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & \gamma & 0 \\ \gamma & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $\gamma \in \mathbb{R}$. If $X_S \neq 0$, then there is $g \in O(3,1)$ such that $\mathrm{Ad}_g(X_S) \in \mathcal{H}$ and we may see that $\mathrm{Ad}_g(X_N)$ commutes with some nonzero element of $\mathcal{H}$. A direct computation then shows that $\mathrm{Ad}_g(X_N) \in \mathcal{H}$, and this implies that $\mathrm{Ad}_g(X_N) = 0$ that is, $X_N = 0$.

We have therefore proved that any Killing field on $\mathbb{H}^3$ is Ad-conjugated to a Killing field X_γ as above or of the form

$$X_{\alpha,\beta} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix}.$$

Let ϕ_1 and $\phi_{a,\beta}$ be the one parameter subgroups of isometries of $\mathbb{H}^3$ determined by X_1 and $X_{\alpha,\beta}$, that is,

$$\phi(t) = I + tX_1 + (t^2/2)X_1^2,$$

and

$$\phi_{\alpha,\beta}(t) = \begin{bmatrix} \cosh \alpha t & \sinh \alpha t & 0 & 0 \\ -\sinh \alpha t & \cosh \alpha t & 0 & 0 \\ 0 & 0 & \cos \beta t & \sin \beta t \\ 0 & 0 & -\sin \beta t & \cos \beta t \end{bmatrix}.$$

As before, we have:

Corollary 7. *Let M be a complete surface of constant mean curvature immersed in $\mathbb{H}^3$. Assume that the function*

$$f = \langle \eta, V \rangle$$

does not change sign on M, where V is a Killing field of $\mathbb{H}^3$. If

(a) *M has the conformal type of the disk or the sphere, or*

(b) *M has the conformal type of the plane and $\|V\|$ is bounded on M,*

then M, up to congruence, is ϕ_1 or $\phi_{\alpha,\beta}$-invariant for some $\alpha, \beta \in \mathbb{R}$.

The $X_{\alpha,0}$-invariant surfaces ($\alpha \neq 0$) are the surfaces of revolution in $\mathbb{H}^3$, and $X_{0,\beta}$-invariant surfaces are known as hyperbolic surfaces. The $X_{\alpha,\beta}$-invariant surfaces with $\alpha \neq \beta$ and $\alpha \neq 0 \neq \beta$ are analogous to the helicoidal surfaces of $\mathbb{R}^3$. The X_1-invariant surfaces are known as parabolic surfaces. We have not found in the literature explicit equations describing these surfaces.

4. The Killing translation map

Let N be a $(n+1)$-dimensional Riemannian manifold. We say that N is a Killing parallelizable Riemannian manifold if there are $n+1$ Killing vector fields $V_1, \ldots, V_{n+1}$ which are linearly independent at each point of N. The set of vector fields $\mathcal{B} = \{V_1, \ldots, V_{n+1}\}$ is called a Killing basis of TN.

Associated to a Killing basis $\mathcal{B}$ there is a "Killing translation" on TN,

$$\Gamma : TN \to \mathbb{R}^{n+1},$$

defined by setting, for $p \in N$ and $v \in T_pN$,

$$\Gamma_p(v) = \sum_{i=1}^{n+1} \langle v, V_i(p) \rangle \, e_i. \tag{4.1}$$

Note that $\Gamma_p : T_pN \to \mathbb{R}^{n+1}$ is a linear isomorphism, for any given $p \in N$.

Let M be an orientable hypersurface of N and let η be a unitary normal vector field to M on N. We define the normal Killing translation map $\gamma : M \to \mathbb{R}^{n+1}$ associated to the Killing basis $\mathcal{B}$ by setting

$$\gamma(p) = \Gamma_p(\eta(p)). \tag{4.2}$$

When $N = \mathbb{R}^3$ and $V_1 = (1,0,0)$, $V_2 = (0,1,0)$, $V_3 = (0,0,1)$, then γ is the usual Gauss map, say g, of M. In this particular case, we have the well known formula

$$\Delta g = -2\nabla H - \|B\|^2 g.$$

We shall show below that γ satisfies a similar formula.

Remark. Using the same technique as in [Ri], one may use the map γ to prove that if M is a hypersurface of a Killing parallelizable $(n+1)$-dimensional Riemannian manifold N having positive principal curvatures, then M is diffeomorphic to the n-dimensional sphere.

Theorem 1. *Let N be a $(n+1)$-dimensional Killing parallelizable Riemannian manifold, and let $\mathcal{B}$ be a Killing basis of TN. Let M be an orientable hypersurface immersed in N. Then the normal Killing translation map $\gamma : M \to \mathbb{R}^{n+1}$ of M associated to $\mathcal{B}$ satisfies the formula*

$$\Delta\gamma(p) = -n\Gamma_p(\nabla H) - (\mathrm{Ric}(\eta) + \|B\|^2)\gamma(p)$$

for all $p \in M$. In particular, M has constant mean curvature if and only if γ satisfies the equation

$$\Delta\gamma = -\left(\mathrm{Ric}(\eta) + \|B\|^2\right)\gamma.$$

Proof. Assume that $\mathcal{B} = \{V_1, \dots, V_{n+1}\}$. Since

$$\gamma(p) = \sum_{i=1}^{n+1} \langle \eta, V_i\rangle\, e_i,$$

it follows by (1.2) that, given $p \in M$,

$$\begin{aligned}\Delta\gamma(p) &= \sum_{i=1}^{n+1} \Delta(\langle \eta, V_i\rangle)(p) e_i \\ &= -n\sum_{i=1}^{n+1} \langle V_i, \nabla H\rangle(p) e_i - (\mathrm{Ric}(\eta) + \|B\|^2)\sum_{i=1}^{n+1}\langle \eta, V_i\rangle(p)e_i \\ &= -n\Gamma_p(\nabla H) - (\mathrm{Ric}(\eta) + \|B\|^2)\gamma(p),\end{aligned}$$

proving the theorem. □

Remark. We note that Theorem 1 of [EFFR] is an immediate consequence of Theorem 1.

Corollary 8. *Let N be a $(n+1)$-dimensional Killing parallelizable Riemannian manifold, and let $\mathcal{B}$ be a Killing basis of TN. Assume that*

$$\mathrm{Ric}(W) \geq -nH^2,$$

for any unit tangent vector W of N. Let M be a complete hypersurface immersed in N with constant mean curvature H and assume that $\gamma(M)$ is contained in a half space of $\mathbb{R}^{n+1}$, where γ is the normal Killing translation map associated to $\mathcal{B}$. We then have:

(i) *If M is compact, then M is invariant by a one parameter subgroup of isometries of N or M is umbilic and N has constant non-positive Ricci curvature $\mathrm{Ric}(\eta) = -nH^2$ on the η-direction.*

(ii) *If $n = 2$, M is simply connected and either M has the conformal type of the disk or M is conformally the plane and γ is bounded on M, then M is invariant by a one parameter subgroup of isometries of N or M is umbilic and N has constant non-positive Ricci curvature $\mathrm{Ric}(\eta) = -nH^2$ on the η-direction.*

Proof. By hypothesis, there is $v \in \mathbb{R}^{n+1}$, $v \neq 0$, such that $\langle v, \gamma(p)\rangle \geq 0$ for all $p \in M$. Let V be the Killing field of N given by

$$V = \sum_{i=1}^{n+1} \langle v, e_i\rangle V_i.$$

Then

$$\begin{aligned}\langle \eta, V\rangle &= \left\langle \eta, \sum_{i=1}^{n+1} \langle v, e_i\rangle V_i \right\rangle = \sum_{i=1}^{n+1} \langle \eta, V_i\rangle \langle v, e_i\rangle \\ &= \langle \gamma(p), v\rangle \geq 0.\end{aligned}$$

Corollary 8 is then a consequence of Corollaries 1 and 2. □

In the case when $N = \mathbb{R}^3$, Corollary 8 gives the following extension of the original Hoffman-Osserman-Schoen result:

Corollary 9. *Let $\mathcal{B}$ be a basis of Killing vector fields of $T\mathbb{R}^3$. Let M be a complete cmc surface immersed in $\mathbb{R}^3$ and let $\gamma : M \to \mathbb{R}^3$ be the normal Killing translation map associated to $\mathcal{B}$. If $\gamma(M)$ is contained in a half space of $\mathbb{R}^3$ and γ is bounded on M, then M is a helicoidal surface.*

The map γ can also be used to provide another criterium of stability (in addition to that of Corollary 3):

Corollary 10. *Let N be a 3-dimensional Killing parallelizable Riemannian manifold, and let $\mathcal{B}$ be a Killing basis of TN. Assume that*

$$\mathrm{Ric}(W) \geq -nH^2$$

for any unit tangent vector W of N. Let M be a surface of constant mean curvature H (not necessarily complete) and let D be a domain in M such that $\overline{D} \subset \mathrm{int}(M)$. Assume that $\gamma(M)$ is contained in a half space of $\mathbb{R}^3$, where γ is the normal Killing translation map associated to $\mathcal{B}$. Then $\overline{D}$ is stable.

Proof. The result follows by using the function $f = \langle \eta, V \rangle$ of Corollary 8 and applying Corollary 3. $\square$

Remark. Particular applications of the above results arise when N is a Lie group with a left invariant metric, where the right invariant vector fields in N are Killing fields. It then follows that the translation Γ can be taken as the right translation to the identity, that is,

$$\Gamma_p(v) = d(R_p)^{-1}(v),$$

where $R_p(x) = xp$. In particular, the above results apply to any symmetric space of noncompact type.

Corollary 8, in the special case when $N = \mathbb{R}^3$ and the normal Killing translation map is the usual Gauss map, is at the basis of a conjecture of M. P. do Carmo which asserts that the Gauss map of a complete cmc surface in $\mathbb{R}^3$ which is not a cylinder nor a plane must contain a neighborhood of an equator of the sphere. This property in fact holds for the Delaunay surfaces and for the cmc complete helicoidal surfaces.

One may see that this property also holds for the normal Killing translation map γ associated to a Killing basis $\mathcal{B}$ of $\mathbb{R}^3$ and for many examples of Delaunay and helicoidal cmc surfaces (considering the radial projection of γ on the unit sphere).Therefore, in view of Corollary 8, it is natural to give the following extension of do Carmo's conjecture:

Conjecture. *Let N be a $(n+1)$-dimensional Killing parallelizable Riemannian manifold and let $\mathcal{B}$ be a Killing basis of TN. Let M be a complete constant mean curvature hypersurface immersed in N and let $\gamma : M \to \mathbb{R}^{n+1}$ be the normal Killing translation map associated to $\mathcal{B}$. If M is not invariant by a Killing field generated (over the real numbers) by $\mathcal{B}$, then the radial projection of $\gamma(M)$ on the unit sphere covers a neighborhood of some equator of the sphere.*

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