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ON THE STRUCTURE OF THE SET OF SEMIDUALIZING COMPLEXES

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ABSTRACT. We study the structure of the set of semidualizing complexes over a local ring. In particular, we prove that for a pair of semidualizing complexes X_1 and X_2 such that $G_{X_2} \dim X_1 < \infty$ we have $X_2 \simeq X_1 \otimes^{L}_{R} \mathbf{R}\mathrm{Hom}_R(X_1, X_2)$. Specializing to the case of semidualizing modules over artinian rings we obtain a number of quantitative results for rings possessing a configuration of semidualizing modules of special form. For rings with $\mathfrak{m}^3 = 0$ this condition reduces to the existence of a nontrivial semidualizing module and we prove a number of structural results in this case.

1. Introduction

In this paper we study the structure of the set of semidualizing complexes over a commutative local Noetherian ring. The motivation behind this problem is its close relation to various questions about G-dimension, most notably the question about the transitivity of G-dimension, raised by Avramov and Foxby in [AF2].

Trivial examples of semidualizing complexes are the free module of rank one and the dualizing complex when it exists. In all examples of rings known to the author for which there are nontrivial semidualizing complexes the set of these complexes has a very simple partial ordering structure, identical to that of the set of all subsets of a finite set. The ultimate question we would like to answer is whether a similar kind of structure exists in general, but so far even simpler questions, such as whether there is always a finite number of semidualizing complexes, remain unresolved.

In Section 3 we study a binary relation which, conjecturally, can endow the set of the semidualizing modules over a ring with a structure of partial ordering. More precisely, we are interested in pairs of semidualizing complexes X_1 and X_2 such that $G_{X_2} \dim X_1 < \infty$. In particular, we prove (Theorem

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3.1) that in this case X_2 can be represented as the left derived tensor product of X_1 and another semidualizing complex. This splitting result for pairs of semidualizing complexes is easily generalized (Corollary 3.3) to the case of *chains*, i.e., sequences of semidualizing complexes X_i, where for any two consecutive entries we have $G_{X_i} \dim X_{i-1} < \infty$.

In Section 4 we prove a number of quantitative results about Artinian rings with “a large number” of Tor-independent semidualizing modules (the condition conjecturally equivalent to the existence of a corresponding long *chain* of semidualizing modules).

Finally, in Section 5 we study the structure of such rings in more detail; in particular, over rings with $\mathfrak{m}^3 = 0$ we obtain analogues of the structural results of the paper [Y] for rings possessing a nontrivial module of zero G-dimension.

2. Preliminaries

In this section we recall several notions from commutative and homological algebra and fix some notations which will be used throughout the paper.

By a ring R we will always mean a commutative Noetherian local ring with maximal ideal m and with residue class field k. A complex X of R-modules is a collection of modules X_i and homomorphisms $\partial_i^X : X_i \to X_{i-1}$ such that $\partial_i^X \partial_{i+1}^X = 0$. The i-th homology of a complex X is a module $\mathrm{H}_i(X) = \ker \partial_i^X / \operatorname{im} \partial_{i+1}^X$. The following numbers denote the positions of the non-zero homologies of the complex X:

$$\begin{aligned} \sup(X) &= \sup\{i \mid \mathrm{H}_i(X) \neq 0\}, \\ \inf(X) &= \inf\{i \mid \mathrm{H}_i(X) \neq 0\}, \\ \operatorname{amp}(X) &= \sup(X) - \inf(X). \end{aligned}$$

A complex is acyclic ($\mathrm{H}_i(X) = 0$ for every i) if and only if any of the following is true: $\sup(X) = -\infty$, $\inf(X) = \infty$, $\operatorname{amp}(X) = -\infty$. If $\operatorname{amp}(X) < \infty$ (resp. $\inf(X) > -\infty$, $\sup(X) < \infty$) then we say that X is bounded (resp. bounded below, bounded above).

We are working with the derived categories $\mathcal{D}_b^f(R)$ (resp. $\mathcal{D}_+^f(R), \mathcal{D}_-^f(R)$), i.e., the category of bounded (resp. bounded below, bounded above) R-complexes with finite homology, localized at the class of quasi-isomorphisms (see [H] or [GM]). All modules are assumed to be finitely generated, unless otherwise specified.

By $\mathbf{R}\mathrm{Hom}_R(-,-)$ $(- \otimes_R^L -)$ we denote the right (left) derived functor of the homomorphism (tensor product) functor of complexes. Note that no boundedness conditions on the arguments are needed by the results of [S], [AF1].

For a complex X bounded below [above] we define the Betti [Bass] numbers as $\beta_i^R(X) = \dim_k(\mathrm{Ext}_R^i(X,k))$ $[\mu_R^i(X) = \dim_k(\mathrm{Ext}_R^i(k,X))]$. The Betti

[Bass] series, aggregating these data, are defined as $P_X^R(t) = \sum_i \beta_i^R(X)t^i$ $[I_R^X(t) = \sum_i \mu_R^i(X)t^i]$.

We use without further comments the standard morphisms of complexes, in particular, the biduality morphism $\omega : M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,N),N)$ and the composition morphism $\varphi : \mathbf{R}\mathrm{Hom}_R(M,N) \otimes_R^L \mathbf{R}\mathrm{Hom}_R(N,K) \to \mathbf{R}\mathrm{Hom}_R(M,K)$.

A complex $K \in \mathcal{D}_b^f(R)$ is called *semidualizing* (see [C], [G]) if the biduality morphism $R \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(R,K),K)$ is an isomorphism. The trivial examples are the ring itself and the dualizing complex when it exists.

For every semidualizing complex K the complex $X \in \mathcal{D}_b^f(R)$ is said to be of finite G-dimension with respect to K, or of finite G_K-dimension, if $\mathbf{R}\mathrm{Hom}_R(X,K) \in \mathcal{D}_b^f(R)$ and the biduality morphism $X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,K),K)$ is an isomorphism. In this case we set $G_K \dim X = -\inf(\mathbf{R}\mathrm{Hom}_R(X,K))) + \inf(K)$; otherwise we define $G_K \dim X = \infty$. Note that the above conditions hold trivially when K is dualizing. When $K = R$ this dimension coincides with the G-dimension of Auslander and Bridger [AB].

3. Semidualizing complexes

Theorem 3.1. *If X_1 and X_2 are semidualizing complexes over a ring R such that $G_{X_2} \dim X_1 < \infty$, and the complex M has finite G-dimension with respect to both X_1 and X_2, then the composition morphism*

$$\varphi : \mathbf{R}\mathrm{Hom}_R(M,X_1) \otimes_R^L \mathbf{R}\mathrm{Hom}_R(X_1,X_2) \to \mathbf{R}\mathrm{Hom}_R(M,X_2)$$

is an isomorphism.

Proof. It suffices to prove that $\operatorname{cone}\varphi$ is acyclic. From the commutative diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,X_1) \otimes_R^L \mathbf{R}\mathrm{Hom}_R(X_1,X_2),X_2) \arrow[d, "\simeq"] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,X_2),X_2) \arrow[l, "\mathbf{R}\mathrm{Hom}_R(\varphi,X_2)"'] \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,X_1),\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X_1,X_2),X_2)) & M \arrow[u, "\simeq"'] \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,X_1),X_1) \arrow[u, "\simeq"'] & M \arrow[l, "\simeq"'] \arrow[u, equal]
\end{tikzcd}$$

it follows that $\mathbf{R}\mathrm{Hom}_R(\varphi,X_2)$ is an isomorphism. Thus the complex $\mathbf{R}\mathrm{Hom}_R(\operatorname{cone}\varphi,X_2)$ is acyclic. Since the complexes $\mathbf{R}\mathrm{Hom}_R(M,X_1) \otimes_R^L \mathbf{R}\mathrm{Hom}_R(X_1,X_2)$ and $\mathbf{R}\mathrm{Hom}_R(M,X_2)$ are bounded below, the complex $\operatorname{cone}\varphi$ is also bounded below. If $\mathrm{H}(\operatorname{cone}\varphi) \neq 0$, then $\inf \operatorname{cone}\varphi$ is finite. We have

$$\begin{aligned} -\infty &= \sup \mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(\operatorname{cone}\varphi, X_2)) \\ &= \sup \mathbf{R}\mathrm{Hom}_R(k \otimes^L_R \operatorname{cone}\varphi, X_2) \\ &= \sup \mathbf{R}\mathrm{Hom}_k(k \otimes^L_R \operatorname{cone}\varphi, \mathbf{R}\mathrm{Hom}_R(k, X_2)) \\ &= \sup \mathbf{R}\mathrm{Hom}_R(k, X_2) - \inf \operatorname{cone}\varphi, \end{aligned}$$

a contradiction, since the right-hand side is finite. □

Definition 3.2. We say that the non-isomorphic semidualizing complexes X_0, $X_1, \dots, X_n$ form a *chain* of length n if $G_{X_i} \dim X_{i-1} < \infty$ for all $i = 1, \dots, n$.

Corollary 3.3. *If the semidualizing complexes* X_0, $X_1, \dots, X_n$ *form a chain, then we have an isomorphism*

$$X_n \simeq X_0 \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_0, X_1) \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_1, X_2) \otimes^L_R \cdots \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_{n-1}, X_n). \tag{3.1}$$

Proof. For each i apply Theorem 3.1 with $M = R$ to the semidualizing complexes X_i and X_{i-1}. □

Remark 3.4. If the semidualizing complexes X_0, $X_1, \dots, X_n \simeq X_0$ form a chain, then Corollary 3.3 implies that

$$X_0 \simeq X_0 \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_0, X_1) \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_1, X_2) \otimes^L_R \cdots \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_{n-1}, X_n).$$

Thus for all i we have $\mathbf{R}\mathrm{Hom}_R(X_i, X_{i+1}) \simeq R$ and $X_i \simeq X_{i+1}$. This is a slight variation of the proofs of [C, Proposition 8.3,(iii)⇒(ii)] and [ATY, Theorem 5.5].

Proposition 3.5. *If* X_1, $X_1 \otimes^L_R X_2$ *are semidualizing complexes over a ring* R, *then* $\varphi : X_1 \to \mathbf{R}\mathrm{Hom}_R(X_2, X_1 \otimes^L_R X_2)$ *is an isomorphism. If, moreover,* X_2 *is semidualizing, then* $\psi : X_2 \to \mathbf{R}\mathrm{Hom}_R(X_1, X_1 \otimes^L_R X_2)$ *is an isomorphism; in particular,* $G_{X_1 \otimes^L_R X_2} \dim X_1 < \infty$.

Proof. Analogously to the proof of Theorem 3.1 note that $\operatorname{cone}\varphi$ is bounded above. Thus, if it is not acyclic, then $\mathbf{R}\mathrm{Hom}_R(X_1, \operatorname{cone}\varphi)$ is also not acyclic, a contradiction. □

Remark 3.6. There are quite a number of questions remaining unresolved about the structure of the set. We note some of them:

Transitivity: If a triple of semidualizing complexes X_1, X_2, X_3 is such that $G_{X_3} \dim X_2 < \infty$ and $G_{X_2} \dim X_1 < \infty$, does this imply that $G_{X_3} \dim X_1 < \infty$?

Existence of a "join": Does there exist, for each pair of semidualizing complexes X_1, X_2, a third semidualizing complex X_3 with the property that $G_{X_3} \dim X_2 < \infty$ and $G_{X_3} \dim X_1 < \infty$? (Note that this holds trivially when the ring possesses a dualizing complex.)

4. Semidualizing modules over Artin rings

In this section we assume that R is Artin and that all modules are finitely generated.

Definition 4.1. The modules $K_1, K_2, \dots, K_n$ are said to be weakly Tor-independent if $\operatorname{amp}(\otimes^L_{1\le i\le n} K_i) = 0$.

Definition 4.2. The modules $K_1, K_2, \dots, K_n$ are said to be strongly Tor-independent if for any subset $I \subset \{1, \dots, n\}$ we have $\operatorname{amp}(\otimes^L_{i\in I} K_i) = 0$.

Remark 4.3. In the case $n = 2$ both notions are equivalent to the classical Tor -independence, i.e., to the condition that $\operatorname{Tor}^R_i(K_1, K_2)$ vanishes for $i > 0$.

Remark 4.4. It is not clear whether weak Tor-independence implies strong Tor-independence if $n > 2$.

Theorem 4.5. *If the modules $K_1, K_2, \dots, K_n$ are non-free and strongly* Tor*-independent, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Betti series of k has the form $1/\prod_{i=1}^n (1 - d_i t)$ for some positive integers d_i.*

Proof. Let $Y_i = \operatorname{Syz}_1(K_i)$. Note that if we take, for each i, a module $X_i \in \{K_i, Y_i\}$, then the modules X_i are still strongly Tor-independent. Suppose $\mathfrak{m}^n = 0$. We prove by induction that $\mathfrak{m}^{n-j} \otimes_{1\le i\le j} Y_i = 0$. If $j = 1$ this is clear, since $Y_1 \subset \mathfrak{m}R^{\beta^R_0(K_1)}$. If this holds for $j = l$, then taking the exact sequence

$$0 \to Y_{l+1} \to R^{\beta^R_0(K_{l+1})} \to K_{l+1} \to 0,$$

tensoring it by $\otimes_{1\le i\le l} Y_i$ and using strong Tor-independence, we get $\otimes_{1\le i\le l+1} Y_i \subset \mathfrak{m}\ (\otimes_{1\le i\le l+1} Y_i)^{\beta^R_0(K_{l+1})}$, which, using the induction hypothesis, gives the desired statement. Applying this result with $j = n-1$ we get $\mathfrak{m}(\otimes_{1\le i\le n-1} Y_i) = 0$, i.e., $\otimes_{1\le i\le n-1} Y_i$ is a vector space over the residue field of R. Since $\operatorname{Tor}^R_1(\otimes_{1\le i\le n-1} Y_i, K_n) = 0$, the module K_n is free. Thus $\mathfrak{m}^n \neq 0$.

Now if $\mathfrak{m}^{n+1} = 0$, the same reasoning shows that $\mathfrak{m}^2(\otimes_{1\le i\le n-1} Y_i) = 0$, $\mathfrak{m}(\otimes_{1\le i\le n} Y_i) = 0$. The first isomorphism implies that there exists an exact sequence of the form

$$0 \to k^{a_n} \to \otimes_{1\le i\le n-1} Y_i \to k^{b_n} \to 0.$$

Tensoring this sequence by K_n and using the fact that, by the long exact sequence of Tor's, $\operatorname{Tor}^R_j(\otimes_{1\le i\le n-1} Y_i, K_n) = 0$ for all $j > 0$, we get $\mathrm{Tor}_i^R(K_n,k)^{a_n} \simeq \mathrm{Tor}_{i+1}^R(K_n,k)^{b_n}$ for $i > 0$. It follows that $\mathrm{Tor}_i^R(Y_n,k)^{a_n} \simeq \mathrm{Tor}_{i+1}^R(Y_n,k)^{b_n}$ for $i \geq 0$. Thus the Betti series of Y_n is $\mathrm{P}_{Y_n}^R(t) = c_n/(1-(a_n/b_n)t)$. Analogously, we see that the Betti series of the other modules Y_i have the same form and we get $\mathrm{P}^R_{\otimes_{1\leq i\leq n}Y_i}(t) = \prod_{i=1}^n c_i/\prod_{i=1}^n(1-(a_i/b_i)t)$. Finally, since $\otimes_{1\leq i\leq n}Y_i$ is a vector space over k and $\beta_0^R(k)=1$, the claim follows. ☐

Conjecture 4.6. If the semidualizing modules $K_0, K_1, \ldots, K_n$ form a chain, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1}=0$, then the Betti series of k has the form $1/\prod_{i=1}^n(1-d_it)$ for some d_i.

Remark 4.7. Note that the conditions of the conjecture imply, by Corollary 3.3, that the modules K_0, $\mathrm{Hom}_R(K_0,K_1)$, $\mathrm{Hom}_R(K_1,K_2), \ldots, \mathrm{Hom}_R(K_{i-1},K_i)$ are weakly Tor-independent for every $i \leq n$. It is not known whether these conditions imply strong Tor-independence, which would be enough to prove the conjecture.

Theorem 4.8. *Conjecture 4.6 holds for $n \leq 3$.*

Proof. First note that if the semidualizing modules $K_0, K_1, \ldots, K_n$ form a chain, then the modules $R, K_1, \ldots, K_{n-1}, D$ (where D is dualizing) also form a chain. Thus we can assume that we have a chain of this form. For $n=1$ the statement hold trivially. The existence of two non-isomorphic semidualizing modules already implies that $\mathfrak{m} \neq 0$, and the statement about the Betti series of the residue field holds for *all* rings with $\mathfrak{m}^2 = 0$. For $n=2$ the modules K_1 and $\mathrm{Hom}(K_1,D)$ are Tor-independent and we are in the situation of Theorem 4.5. For $n=3$, by Theorem 4.5 everything would follow from the strong Tor-independence of the modules $K_1, \mathrm{Hom}_R(K_1,K_2), \mathrm{Hom}_R(K_2,D)$. The weak Tor-independence follows from Corollary 3.3. To prove that any two of these modules are Tor-independent it remains to apply Theorem 3.1 to the triples $(M,X_1,X_2) = (R,K_1,K_2)$, (K_1,K_2,D), $(\mathrm{Hom}_R(K_1,K_2),K_2,D)$. ☐

Definition 4.9. An Artin ring R is called *SD*(n)-full if the following conditions are satisfied:

(1) $\mathfrak{m}^{n+1}=0$.
(2) There are strongly Tor-independent non-free semidualizing modules $K_1, K_2, \ldots, K_n$ such that for any subset $I \subset \{1,\ldots,n\}$ the module $\otimes_{i\in I}K_i$ is semidualizing.

Remark 4.10. If the set of semidualizing modules satisfies condition (2) in this definition, then the semidualizing modules $X_0 = R$, $X_k = \otimes_{1\leq i\leq k}K_i$ form a chain by Proposition 3.5.

EXAMPLE 4.11. All non-Gorenstein rings with $\mathfrak{m}^2 = 0$ are $SD(1)$-full. A ring with $\mathfrak{m}^3 = 0$ is $SD(2)$-full iff there exists a nontrivial semidualizing R-module. The ring $\otimes_k^{1\leq i\leq n} k \ltimes k^{a_i}$, where $a_i > 1$, is $SD(n)$-full according to [G2].

PROPOSITION 4.12. *For an $SD(n)$-full ring R the module $\otimes K_i$ is dualizing.*

Proof. Suppose $\otimes K_i$ is not dualizing. We prove that the semidualizing modules $K_1, K_2, \ldots, K_n, \mathrm{Hom}(\otimes_R K_i, D)$ are strongly Tor-independent, which, by Theorem 4.5, contradicts the first condition in the definition of an $SD(n)$-full ring . Taking $X_1 = \otimes_{i\in I} K_i$, $X_2 = \otimes_{i\notin I} K_i$ we obtain the following isomorphisms:

$$\begin{aligned} X_1 \otimes^L_R &\mathbf{R}\mathrm{Hom}_R(X_1 \otimes^L_R X_2, D) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X_2, X_1 \otimes^L_R X_2) \otimes^L_R \mathbf{R}\mathrm{Hom}_R(X_1 \otimes^L_R X_2, D) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X_2, D). \end{aligned}$$

The first isomorphism is due to Proposition 3.5 and the second one is due to Theorem 3.1. Hence

$$\mathrm{amp}(\otimes^L_{i\in I} K_i \otimes^L \mathrm{Hom}(\otimes^L_R K_i, D)) = \mathrm{amp}(\mathrm{Hom}(\otimes^L_{i\notin I} K_i, D)) = 0. \qquad \square$$

REMARK 4.13. If the conditions of Conjecture 4.6 hold for a ring R and if $n \leq 3$ and $m^{n+1} = 0$, then R is $SD(n)$-full.

5. Semidualizing modules over $SD(n)$-full rings

The proofs of this section closely mimic that of the paper [Y]. We start with basic facts about modules M with $\mathfrak{m}^2 M = 0$ having finite G_K-dimension with respect to a non-dualizing semidualizing module K. Throughout this section we denote the module $\mathrm{Hom}(M, K)$ by M^*.

PROPOSITION 5.1. *If R is Artin, $\mathfrak{m}^2 M = 0$ and $G_K \dim M = 0$, then there exists an integer c such that for the Bass numbers $\mu^i(K)$ we have $\mu^{i+1}(K) = c\mu^i(K)$ for all $i > 0$.*

Proof. Starting with the short exact sequence $0 \to k^a \to M \to k^b \to 0$, writing down the long exact sequence for $\mathrm{Ext}^i_R(-, K)$, and using the fact that $\mathrm{Ext}^i_R(M, K) = 0$ for all $i > 0$, we obtain isomorphisms $\mathrm{Ext}^i_R(k, K)^a \simeq \mathrm{Ext}^{i+1}_R(k, K)^b$ for all $i > 0$. Since K is not dualizing, $\mathrm{Ext}^1_R(k, K) \neq 0$. Thus, $(a/b)^n \dim_k(\mathrm{Ext}^1_R(k, K)) = \dim_k(\mathrm{Ext}^{n+1}_R(k, K))$ is a positive integer for each $n \geq 0$, which implies $b|a$. $\square$

PROPOSITION 5.2. *If R is Artin, $\mathfrak{m}^2 M = 0$ and $G_K \dim M = 0$, then $\mathrm{l}(M^*) = \mathrm{l}(M)$ and $\mu^1(K) = \mu^0(K)^2 - 1$.*

Proof. Starting with the short exact sequence

$$0 \to k^a \to M \to k^b \to 0,$$

applying $\operatorname{Hom}(-,K)$, and using the fact that $\operatorname{Ext}^i_R(M,K)=0$ for all $i>0$, we obtain the short exact sequence

$$0 \to k^{b\mu^0(K)} \to M^* \to k^{a\mu^0(K)-b\mu^1(K)} \to 0.$$

Counting the lengths gives

$$\mathrm{l}(M^*) = (a+b)\mu^0(K) - b\mu^1(K) = \mathrm{l}(M)\mu^0(K) - b\mu^1(K). \tag{5.1}$$

Analogously, starting with the sequence

$$0 \to k^{b\mu^0(K)} \to M^* \to k^{a\mu^0(K)-b\mu^1(K)} \to 0,$$

we get

$$\mathrm{l}(M^{**}) = \mathrm{l}(M^*)\mu^0(K) - (a\mu^0(K) - b\mu^1(K))\mu^1(K). \tag{5.2}$$

Finally, we have

$$a+b = \mathrm{l}(M) = \mathrm{l}(M^{**}). \tag{5.3}$$

Eliminating a and b from these equalities gives

$$\mathrm{l}(M^*)\mu^1(K) = \mathrm{l}(M)(\mu^0(K)^2 - 1)$$

and

$$\mathrm{l}(M)\mu^1(K) = \mathrm{l}(M^*)(\mu^0(K)^2 - 1),$$

and the proposition follows. □

Remark 5.3. If R is $SD(n)$-full, then taking $K_1, K_2, \ldots, K_n$ to be the corresponding set of nontrivial semidualizing modules and setting $Y_i = \operatorname{Syz}_1(K_i)$, it is easy to see from the proof of Theorem 4.5 that for every $i \in \{1,\ldots,n\}$ the module $\otimes_{j\neq i} Y_j$ is annihilated by $\mathfrak{m}^2$ and has finite $G_{K_{-i}}$-dimension, where $K_{-i} = \otimes_{j\neq i} K_j$.

Proposition 5.4. *If R is $SD(n)$-full, $K_1, K_2, \ldots, K_n$ are the corresponding semidualizing modules, then for each i the Bass series of K_{-i} is $I^{K_{-i}}(t) = (\mu^0(K_{-i}) - t)/(1-\mu^0(K_{-i})t)$ and the Betti series of K_i is $P_{K_i}(t) = (\beta_0(K_i) - t)/(1-\beta_0(K_i)t)$.*

Proof. The previous two propositions imply that

$$I^{K_{-i}}(t) = \frac{\mu^0(K_{-i}) - \mu^0(K_{-i})ct + \mu^0(K_{-i})^2 t - t}{1-ct},$$

where $\mu^{j+1}(K_{-i}) = c\mu^j(K_{-i})$ for $j>0$. It remains to prove that $c = \mu^0(K_{-i})$. By Remark 5.3 there exists an R-module M, annihilated by $\mathfrak{m}^2$, which has finite $G_{K_{-i}}$-dimension. Dualizing the exact sequence

$$0 \to k^a \to M \to k^b \to 0$$

we get the exact sequence

$$0 \to k^{b\mu^0(K_{-i})} \to M^* \to k^{a\mu^0(K_{-i})-b\mu^1(K_{-i})} \to 0.$$

As in the proof of Proposition 5.1, from these two exact sequences we get that

$$a/b = c = b\mu^0(K_{-i})/(a\mu^0(K_{-i}) - b\mu^1(K_{-i})).$$

Substituting, by Proposition 5.2, $\mu^1(K_{-i}) = \mu^0(K_{-i})^2 - 1$, and rearranging terms, we obtain the equality

$$(\mu^0(K_{-i})b - a)(b - \mu^0(K_{-i})a) = 0.$$

Since a/b is an integer, $a = \mu^0(K_{-i})b$. The statement about the Betti series follows from the isomorphism $\mathbf{R}\mathrm{Hom}_R(K_i, K_i \otimes K_{-i}) \simeq K_{-i}$ and Proposition 4.12, which implies that $P_{K_i}(t) = I^{K_{-i}}(t)$. ☐

Next we specialize to the case of $SD(2)$-full algebras over a field. Denote a nontrivial semidualizing module by K.

Remark 5.5. Any finite algebra R with $\mathfrak{m}^3 = 0$ is naturally graded ([Y, Proof of Theorem 3.1, Step 7]), as are R-modules that are annihilated by $\mathfrak{m}^2$.

Proposition 5.6. *If R is an $SD(2)$-full ring and K a nontrivial semidualizing module, then* $\mathrm{l}(K) = \mathrm{l}(R)$.

Proof. Dualizing with respect to K the exact sequence

$$0 \to \mathrm{Syz}_1(K) \to R^{\beta_0(K)} \to K \to 0$$

we get the exact sequence

$$0 \to R \to K^{\beta_0(K)} \to \mathrm{Syz}_1(K)^* \to 0.$$

From Lemma 5.2 it follows that $\mathrm{l}(\mathrm{Syz}_1(K)) = \mathrm{l}(\mathrm{Syz}_1(K)^*)$. Thus, counting the lengths gives

$$\beta_0(K)\,\mathrm{l}(R) - \mathrm{l}(K) = \beta_0(K)\,\mathrm{l}(K) - \mathrm{l}(R),$$

which implies that $\mathrm{l}(K) = \mathrm{l}(R)$. ☐

Lemma 5.7. *If R is an $SD(2)$-full ring and K a nontrivial semidualizing module, then* $\operatorname{socle} R = \mathfrak{m}^2$ *and for $i \geq 2$ we have* $\mathfrak{m}\,\mathrm{Syz}_i(K) = \mathfrak{m}^2 R^{\beta_{i-1}(K)}$. *In particular, there exists a natural grading on the minimal free resolution of a module* $\mathrm{Syz}_1(K)$.

Proof. [HSV, Remark 2.4] applied to $M = K, N = \mathrm{Hom}(K, D)$, where D is a dualizing module and K is a nontrivial semidualizing module, implies $\operatorname{socle} R = \mathfrak{m}^2$. For the second statement we proceed as in the proof of [HSV, Remark 2.4]. The inclusion $\mathfrak{m}\,\mathrm{Syz}_i(K) \subset \mathfrak{m}^2 R^{\beta_{i-1}(K)}$ is obvious. Suppose $x \in \mathfrak{m}^2 R^{\beta_{i-1}(K)} \setminus \mathfrak{m}\,\mathrm{Syz}_i(K)$. Since $\mathrm{Syz}_{i-1}(K)$ is annihilated by $\mathfrak{m}^2$, $x \in \mathrm{Syz}_i(K) \setminus \mathfrak{m}\,\mathrm{Syz}_i(K)$. Since the Tor's of $\mathrm{Syz}_i(K)$ and $\mathrm{Hom}(K, D)$ vanish,

$\mathrm{Syz}_i(K)$ has no k's as direct summands. Thus x is not annihilated by m, a contradiction. $\square$

PROPOSITION 5.8. *Let R be an $SD(2)$-full ring, and let K be a nontrivial semidualizing module. Then we have the equalities*

1. $\dim_k \mathfrak{m}^2 = \mu^0(K)\beta_0(K)$,
2. $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \mu^0(K) + \beta_0(K)$,
3. $\dim_k \mathfrak{m}^2 K = \mu^0(K)$.

Proof. The first equality follows from the fact that $\mathrm{Hom}(K,K) \simeq R$ (and thus $\dim_k \operatorname{socle} K \dim_k K/\mathfrak{m}K = \dim_k \operatorname{socle} R$) and Lemma 5.7. For the second, consider the sequence

$$0 \to \mathrm{Syz}_2(K)/\mathfrak{m}\,\mathrm{Syz}_2(K) \to (R/\mathfrak{m}^2R)^{\beta_1(K)} \to \mathrm{Syz}_1(K) \to 0,$$

which is exact by Lemma 5.7. Counting the lengths gives

$$\begin{aligned}\dim_k \mathrm{Syz}_2(K)/\mathfrak{m}\,\mathrm{Syz}_2(K) &= \beta_1(K)(1+\dim_k \mathfrak{m}/\mathfrak{m}^2)\\ &\qquad - \dim_k \mathrm{Syz}_1(K)/\mathfrak{m}\,\mathrm{Syz}_1(K)\dim_k \mathfrak{m}\,\mathrm{Syz}_1(K)\\ &= \beta_1(K)(1+\dim_k \mathfrak{m}/\mathfrak{m}^2) - \beta_1(K) - \beta_1(K)\mu^0(K)\\ &= \beta_1(K)(\dim_k \mathfrak{m}/\mathfrak{m}^2 - \mu^0(K)),\end{aligned} \tag{5.4}$$

where in the second equality we used the fact that $\dim_k \mathfrak{m}\,\mathrm{Syz}_1(K) = \dim_k(\mathrm{Syz}_1(K)/\mathfrak{m}\,\mathrm{Syz}_1(K))\mu^0(K)$ from the proof of Proposition 5.4. On the other hand, from Lemma 5.7 we have

$$\dim_k \mathfrak{m}\,\mathrm{Syz}_2(K) = \beta_1(K)\dim_k \mathfrak{m}^2 = \beta_1(K)\mu^0(K)\beta_0(K) \tag{5.5}$$

Finally, note that the module $\mathrm{Syz}_2(K)$ also has finite G_K-dimension. Thus, as in the proof of Proposition 5.4,

$$\dim_k \mathfrak{m}\,\mathrm{Syz}_2(K) = \mu^0(K)\dim_k \mathrm{Syz}_2(K)/\mathfrak{m}\,\mathrm{Syz}_2(K). \tag{5.6}$$

Combining (5.4), (5.5) and (5.6) gives $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \mu^0(K)+\beta_0(K)$. To obtain the third equality of the proposition, take the short exact sequence

$$0 \to \mathrm{Syz}_2(L)/\mathfrak{m}\,\mathrm{Syz}_2(L) \to (R/\mathfrak{m}^2R)^{\beta_1(L)} \to \mathrm{Syz}_1(L) \to 0,$$

where L is the semidualizing module $\mathrm{Hom}(K,D)$, and tensor it by K. The sequence

$$0 \to (\mathrm{Syz}_2(L)/\mathfrak{m}\,\mathrm{Syz}_2(L)) \otimes K \to (K/\mathfrak{m}^2K)^{\beta_1(L)} \to \mathrm{Syz}_1(L)\otimes K \to 0$$

is also exact, by Remark 4.7. Counting the lengths gives

$$\beta_1(L)(\mathrm{l}(K) - \mathrm{l}(\mathfrak{m}^2K)) = \beta_2(L)\beta_0(K) + (\beta_0(L)-1)l(R), \tag{5.7}$$

where the equality $l(\mathrm{Syz}_1(L)\otimes K) = (\beta_0(L)-1)l(R)$ follows from counting the lengths in the short exact sequence

$$0 \to \mathrm{Syz}_1(L)\otimes K \to K^{\beta_0(L)} \to D \to 0$$

and using Proposition 5.6.

Rearranging (5.7) and using $\beta_0(L) = \mu^0(K)$, $\beta_1(L) = \mu^0(K)^2 - 1$ and $\beta_2(L) = (\mu^0(K)^2 - 1)\mu^0(K)$, which follows from Proposition 5.4, and $l(R) = (1 + \mu^0(K))(1 + \beta_0(K))$, we obtain the desired statement. □

Theorem 5.9. *$SD(2)$-full rings are Koszul, i.e., satisfy $\operatorname{Ext}^i_R(k,k)_j = 0$ for $i \neq j$.*

Proof. For $M = \operatorname{Syz}_1(K), \operatorname{Syz}_1(\operatorname{Hom}(K,D))$ we have $\operatorname{Ext}^i_R(M,k)_j = 0$ for $i \neq j$. Noting that the modules $\operatorname{Syz}_1(K)$ and $\operatorname{Syz}_1(\operatorname{Hom}(K,D))$ are Tor-independent and their tensor product is annihilated by $\mathfrak{m}$, we obtain the desired statement. □

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