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INFINITE RANK ONE ACTIONS AND NONSINGULAR CHACON TRANSFORMATIONS

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Abstract. Let G be a discrete countable Abelian group. We construct an infinite measure preserving rank one action $T = (T_g)$ of G such that (i) the transformation T_g has infinite ergodic index but $T_g \times T_{2g}$ is not ergodic for any element g of infinite order, (ii) $T_{g_1} \times \cdots \times T_{g_n}$ is conservative for every finite sequence $g_1, \ldots, g_n \in G$. In the case $G = \mathbb{Z}$ this answers a question of C. Silva. Moreover, we show that (i) every weakly stationary nonsingular Chacon transformation with 2-cuts is power weakly mixing and (ii) every weakly stationary nonsingular Chacon* transformation with 2-cuts has infinite ergodic index but is not power weakly mixing.

0. Introduction

Let T be an invertible nonsingular transformation of a Lebesgue space (X, μ). T is *conservative* if for any subset $A \subset X$ of positive measure there exists $n > 0$ such that $\mu(T^n A \cap A) > 0$. If all Cartesian powers of T are ergodic, then T is said to have *infinite ergodic index* (see [KP] for the first example of an infinite measure preserving map with this property). If, moreover, for any finite sequence $n_1, \ldots, n_p$ of nonzero integers, the product $T^{n_1} \times \cdots \times T^{n_p}$ is ergodic, then T is called *power weakly mixing*. Examples of non-power weakly mixing infinite measure preserving transformations with infinite ergodic index are presented in [AFS2] and [G–W]. However, these examples are such that either $T \times T^2$ or $T \times T^2 \times \cdots \times T^7$ is non-conservative. Note that another family of such examples was given in [Da], where for any discrete Abelian group G and AT-flow V, a nonsingular action $T = (T_g)_{g \in G}$ of G was constructed such that

- T_g has infinite ergodic index but $T_g \times T_{2g}$ is non-conservative (and hence T_g is not power weakly mixing) for any element $g \in G$ of infinite order and

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- the associated flow of T is V.

We refer the reader to [HO1] for the definition of the associated flow and to [CW] for the definition of AT-flows.

In this connection C. Silva asks:

(A) *Is there a non-power weakly mixing infinite measure preserving transformation T with infinite ergodic index and such that the Cartesian products $T^{n_1} \times \cdots \times T^{n_p}$ are all conservative?*

We also state a related question of V. Bergelson:

(B) *Is there an infinite measure preserving transformation T with infinite ergodic index but such that $T \times T^{-1}$ is not ergodic?*

In the first section of this paper we demonstrate the following theorem, which answers (A) if we take $G = \mathbb{Z}$. The question (B) remains open.

THEOREM 0.1. *Let G_∞ stand for the set of G-elements of infinite order. There exists an infinite measure preserving (C, F)-action T of G such that the following properties are satisfied:*

(i) *The transformation T_g has infinite ergodic index for every $g \in G_\infty$.*
(ii) *The transformation $T_g \times T_{2g}$ is not ergodic for any $g \in G$.*
(iii) *The transformation $T_{g_1} \times \cdots \times T_{g_n}$ is conservative for every finite sequence $g_1, \dots, g_n$ of elements from G.*
(iv) *T_g is not conjugate to T_g^2 for any $g \in G_\infty$.*

Notice that a topological version of Theorem 0.1 also holds (see Remark 1.3).

Nonsingular Chacon transformations with 2-cuts (i.e., with three copies of the n-th column in the $(n+1)$-th column and a single spacer between the second and the third copies) appeared first in [JuS1] and were studied later in [AFS2], [HaS] and [JuS2]. The main theorem of [HaS] states that every λ-*weakly stationary symmetric* nonsingular Chacon transformation has ergodic Cartesian square, where λ is the equi-distribution on $\{0, 1, 2\}$ (see Section 2 for the definitions). A stronger result was obtained in [AFS2] for *stationary symmetric* Chacon transformations. It was shown that they are power weakly mixing. However, as was noticed by the authors of [AFS2], their methods do not work with the transformations considered in [HaS]. Thus it was an open problem whether or not the higher Cartesian products of weakly stationary Chacon maps are ergodic. We solve this problem in the second (final) part of this paper as follows.

THEOREM 0.2. *Every weakly stationary nonsingular Chacon transformation with 2-cuts is power weakly mixing.*

It is worthwhile to note that this theorem not only extends and strengthens the above results from [HaS] and [AFS2] but also provides a new short proof. This is achieved by utilizing the 'algebraic' (C, F)-techniques instead of the classical 'geometric' cutting-and-stacking. The (C, F)-construction was suggested in [Da] and used also in [DaS] to produce rank one infinite measure preserving or nonsingular actions with various (unusual from a 'probability preserving point of view') properties of weak mixing and multiple recurrence. A similar construction was utilized earlier by A. del Junco [Ju] in the probability preserving setting. However, in all our examples of (C, F)-maps with infinite ergodic index from [Da], [DaS] and Section 1 of this paper, we had $\#C_n \to \infty$ (this corresponds to the case of unbounded cuts) and rather complex arrangement of spacers. Thus for the first time in Theorem 0.2, the (C, F)-techniques proved to be effective in analyzing ergodic properties of nonsingular Chacon transformations with *bounded cuts* and 'classical' configuration of spacers.

By slightly modifying the definition of Chacon maps we introduce Chacon* transformations and establish the following result.

THEOREM 0.3. *Every weakly stationary nonsingular Chacon* transformation T_1 with 2-cuts has infinite ergodic index but is not power weakly mixing. (We show that $T_1 \times T_1^3$ is not conservative.)*

We also note that in [AFS1] and [G–W, § 3] (resp. in [AFS2]) some other 2-cuts (and 3-cuts) modifications of the Chacon map with infinite invariant measure were shown to have infinite ergodic index, but be non-power weakly mixing. However, purely nonsingular (i.e., of Krieger type III) examples were not constructed there. At the same time we notice that both classes of Chacon maps from Theorems 0.2 and 0.3 include transformations of any of the Krieger types II_∞, III_λ, $0 < \lambda \le 1$, and a continuum of pairwise non-orbit equivalent maps of type III_0.

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1. Infinite measure preserving (C, F)-actions

Our main purpose in this section is to prove Theorem 0.1. We first recall the construction of (C, F)-actions from [Da]. Two finite subsets C_1 and C_2 of G are called *independent* if

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

This means that if $c_1 + c_2 = c_1' + c_2'$ for some $c_i, c_i' \in C_i$, $i = 1, 2$, then $c_1 = c_1'$ and $c_2 = c_2'$. Let $(C_n)_{n=1}^\infty$ and $(F_n)_{n=0}^\infty$ be two sequences of finite G-subsets such that $F_0 = \{0\}$ and for each $n > 0$ the following properties are satisfied:

$$F_{n-1} + C_n \subset F_n, \ \#C_n > 1, \tag{1.1}$$

$$F_{n-1} \text{ and } C_n \quad \text{are independent.} \tag{1.2}$$

We put $X_n := F_n \times \prod_{k>n} C_k$, endow X_n with the (compact) product topology and define a continuous embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n + c_{n+1}, c_{n+2}, \dots).$$

Then $X_1 \subset X_2 \subset \cdots$. Let $X := \bigcup_n X_n$ stand for the topological inductive limit of the sequence X_n. Clearly, X is a locally compact non-compact totally disconnected metrizable space without isolated points and X_n is clopen in X. Assume in addition that

(1.3) given $g \in G$, there is $m \in \mathbb{N}$ with $g + F_{n-1} + C_n \subset F_n$ for all $n > m$.

Given $g \in G$ and $n \in \mathbb{N}$, we set

$$D_g^{(n)} := (F_n \cap (F_n - g)) \times \prod_{k>n} C_k \qquad \text{and} \qquad R_g^{(n)} := D_{-g}^{(n)}.$$

Clearly, $D_g^{(n)}$ and $R_g^{(n)}$ are clopen subsets of X_n. Moreover, $D_g^{(n)} \subset D_g^{(n+1)}$ and $R_g^{(n)} \subset R_g^{(n+1)}$. Define a map $T_g^{(n)} \colon D_g^{(n)} \to R_g^{(n)}$ by setting

$$T_g^{(n)}(f_n, c_{n+1}, \dots) := (f_n + g, c_{n+1}, \dots).$$

Clearly, this map is a homeomorphism. Put

$$D_g := \bigcup_{n=1}^{\infty} D_g^{(n)} \text{ and } R_g := \bigcup_{n=1}^{\infty} R_g^{(n)}.$$

Then a homeomorphism $T_g \colon D_g \to R_g$ is well defined by $T_g \restriction D_g^{(n)} = T_g^{(n)}$. It follows from (1.3) that $D_g = R_g = X$ for each $g \in G$.

Given $f \in F_n$, we set $[f]_n := \{x = (x_i)_{i \ge n} \in X_n \mid x_n = f\}$ and call $[f]_n$ an *n-cylinder*. Clearly, $[f]_n = \bigsqcup_{c \in C_{n+1}} [f + c]_{n+1}$, where $\bigsqcup$ denotes the union of disjoint subsets.

To make this paper independent of [Da] we list here all the properties of $(T_g)_{g \in G}$ that will be used in the sequel. They are rather simple and the reader can easily prove them (see also [Da, Proposition 1.3]).

(P1) $T = (T_g)_{g \in G}$ is a minimal free action of G on X. We call this action the *(C, F)-action associated with $(C_n, F_n)_{n \ge 1}$*.

(P2) $T_g[f]_n = [g + f]_n$ for all $g \in G$ and $f \in F_n \cap (F_n - g)$, $n \ge 0$.

(P3) Two points $x, y \in X$ are T-orbit equivalent if and only if they have the same 'tails', i.e., there are $n \le m$ with $x = (x_i)_{i \ge n}, y = (y_i)_{i \ge n} \in X_n$ and $x_i = y_i$ for all $i \ge m$. Furthermore, $y = T_g x$ for $g = \sum_{i \ge n} (y_i - x_i)$.

(P4) There is a unique (ergodic) σ-finite non-atomic T-invariant measure on X such that $\mu(X_0) = 1$. We call this measure the *Haar* measure for T. The Haar measure of every cylinder is finite.

(P5) μ is finite if and only if

$$\lim_{n \to \infty} \frac{\# F_n}{\# C_1 \cdots \# C_n} < \infty.$$

(P6) The family of T-'towers' $\{T_g[0]_n \mid g \in F_n\}$, $n \in \mathbb{N}$, generates the topology and hence the Borel σ-algebra on X. Hence (X, μ, T) is funny rank one (by definition).

REMARK 1.1. In the case $G = \mathbb{Z}$, it is easy to notice a similarity between the (C, F)-construction and the classical cutting-and-stacking construction of rank one transformations. Indeed, F_{n-1} (or, more precisely, the set of $(n-1)$-cylinders) corresponds to the levels of the $(n-1)$-tower and C_n corresponds to the locations of the copies of F_{n-1} inside the n-th tower F_n. (The copies $F_{n-1}+c$, $c \in C_n$, are disjoint by (1.2) and they sit inside F_n by (1.1).) The remaining part of F_n, i.e., $F_n \setminus (F_{n-1}+C_n)$, is the set of spacers in the n-th tower. Next, (1.3) says that the number of spacers over the highest copy of F_{n-1} in F_n and the number of spacers under the lowest copy of F_{n-1} in F_n tend both to infinity as $n \to \infty$. Notice that it is not common for the classical construction to put spacers under the lowest copy of towers. However, because of this (i.e., due to (1.3)) the (C, F)-actions have some advantages. They are defined everywhere (not only a.e.) on a 'good' topological space and are continuous. In the next section, which is devoted to nonsingular Chacon transformations, we will change slightly the (C, F)-construction to imitate the classical Chacon map. For those transformations, (1.3) will not be satisfied. (See also Remark 2.3 below.) However, in this section we work with general countable Abelian groups G. Then the (C, F)-construction may be regarded as a 'modified' arithmetical equivalent of the cutting-and-stacking construction. It works even when the classical geometrical concept of tower seems to have no sense, for instance, if G has no invariant ordering like any group with torsions or if G has infinitely many generators like $\mathbb{Q}$, $\mathbb{Z}_0^\infty$, etc. As for other arguments in favor of (C, F)-actions we mention that their orbit structure is explicit (see (P3)). This fact facilitates the study of conservativeness and ergodicity of the actions since these properties are orbit equivalent, i.e., they are properties of the orbit equivalence relation. Finally, since this paper is about dynamical properties of Cartesian products of transformations, it is worthwhile to note that the (C, F)-construction 'respects' Cartesian products. Namely, the product of two (C, F)-actions $(T_g^{(i)})_{g \in G_i}$ associated with $(C_n^{(i)}, F_n^{(i)})_n$, $i = 1, 2$, is the (C, F)-action of $G_1 \times G_2$ associated with $(C_n^{(1)} \times C_n^{(2)}, F_n^{(1)} \times F_n^2)_n$.

The following lemma is a particular case of [Da, Lemma 2.4]. However, for the reader's convenience we sketch its proof. Recall that $\mathbb{R}_+^*$ stands for the set of (strictly) positive reals.

LEMMA 1.2. *Let T be a (C, F)-action of G on (X, μ) and μ the Haar measure for T. Let $\delta : G \to \mathbb{R}_+^*$ be a map with $\sum_{h \in G} \delta(h) < 0.5$ and $g \in G$.*

If for infinitely many—say 'good'—numbers n and every pair $f, f' \in F_n$ there exist a subset $A \subset [f]_n$ and $l \in \mathbb{Z}$ such that $T_{lg}A \subset [f']_n$ and $\mu(A) > \delta(f - f')\mu([f]_n)$, then the transformation T_g is ergodic.

Proof. Let $A_1, A_2 \subset X$ be subsets of positive finite measure. Since any subset of finite measure is approximated in measure up to any positive number by a finite union of cylinders (see (P6)), there exist $n > 0$ and $f_1, f_2 \in F_n$ such that $[f_i]_n$ is 0.99-full of A_i, i.e., $\mu(A_i \cap [f_i]_n) > 0.99\mu([f_i])$, $i = 1, 2$. For $m > n$, we consider the partition of $[f_i]_n$ into m-cylinders: $[f_i]_n = \bigsqcup_{c \in C}[f_i + c]_m$, where $C := C_{n+1} + \cdots + C_m$. When m increases, these partitions refine and generate the entire Borel σ-algebra on $[f_i]_n$. Fix $\epsilon > 0$. By a standard measure theoretical fact, the total measure of those m-cylinders in $[f_i]_n$ that are $(1-\epsilon)$-full of A_i goes to $\mu(A_i \cap [f_i]_n)$ as $m \to \infty$. Since this limit is strictly greater than $0.5\mu([f_i]_n)$, there are subsets $D_1, D_2 \subset C$ ('names' of these m-cylinders) such that $\#D_i > 0.5\#C$ and $\mu(A_i \cap [f_i + c]_m) > (1-\epsilon)\mu([f_i + c]_m)$ for all $c \in D_i$. Now let $\epsilon := 0.5\delta(f_1 - f_2)$ and m be large and 'good'. Take any $d \in D_1 \cap D_2$ (the intersection is not empty) and apply the condition of the lemma to $f_1 + d$ and $f_2 + d$, which belong both to F_m by (1.1). Then there are a subset $A \subset [f_1 + d]_m$ and $l \in \mathbb{Z}$ such that $T_{lg}A \subset [f_2 + d]_m$ and $\mu(A) > \delta(f_1 - f_2)\mu([f_1 + d]_m)$. By the choice of ϵ, it is easy to deduce that $\mu(A \cap A_1) > 0$ and $\mu(T_{lg}(A \cap A_1) \cap A_2) > 0$. □

Proof of Theorem 0.1. To define T we are going to construct in a special way the corresponding sequences $(F_n)_{n>0}$ and $(C_n)_{n>0}$. This will be done inductively. Suppose that we already have $C_1, F_1, \ldots, C_{n-1}, F_{n-1}$. Our purpose is to construct C_n and F_n. Let $F_{n-1} - F_{n-1} = \{f_i^{(n)} | i = 1, \ldots, k_n\}$ with $f_1^{(n)} = 0$. Fix a map $\delta : G \to \mathbb{R}_+^*$ with $\sum_{g \in G} \delta(g) < 0.5$. Select integers $d_0^{(n)}, \ldots, d_{k_n}^{(n)}$ in such a way that $d_0^{(n)} = 0$ and $d_i^{(n)} > \delta(f_i^{(n)})d^{(n)}$, $i = 1, \ldots, k_n$, where $d^{(n)} := d_1^{(n)} + \cdots + d_{k_n}^{(n)}$. Let $\mathfrak{S} = \{\sigma_p \mid p \in \mathbb{N}\}$ stand for the set of finite sequences of elements (possibly equal) from G_∞. Consider the n-th sequence $\sigma_n = (g_1^{(n)}, \ldots, g_{l_n}^{(n)})$. We will distinguish two cases.

(I) Suppose first that there are unequal elements among $g_j^{(n)}$, $j = 1, \ldots, l_n$. (In fact, this condition will not be used while constructing C_n and F_n. We will need it later for the proof of (iii).) Then we set

$$
\begin{aligned}
&A_j := \{0, q_n g_j^{(n)}\}, \ j = 1, \ldots, l_n, \quad \text{and} \\
&C_n := \bigcup_{j=1}^{l_n} (h_{j,n} g_j^{(n)} + A_j),
\end{aligned}
$$

where q_n and $h_{j,n}$ are some integers chosen so that (1.2) and the following two conditions hold:

$$\text{the sets } h_{j,n}g_j^{(n)} + A_j,\ j = 1, \dots, l_n,\ \text{are mutually disjoint (hence } \#C_n = 2l_n.), \tag{1.4}$$

$$2C_n - C_n \text{ is independent of } \sum_{i<n}(2C_i - C_i). \tag{1.5}$$

Recall that for any subset $E \subset G$, we let $2E := \{2g \mid g \in E\}$. To find such integers we first observe that

$$C_n - C_n = \bigcup_j B_j \cup \bigcup_{j \neq j'} (h_{j,n}g_j^{(n)} - h_{j',n}g_{j'}^{(n)} + \cdots) \ \text{ and} \tag{1.6}$$

$$\begin{aligned} &2C_n - C_n - 2C_n + C_n \\ &\quad = \bigcup_{j_1,j_2} B_{j_1,j_2} \cup \bigcup_{(j_1,j_2)\neq(j_1',j_2')} (\underbrace{g_{j_1,j_2,j_1',j_2'} + \cdots}), \end{aligned} \tag{1.7}$$

where $B_j := A_j - A_j$, $B_{j_1,j_2} := 2B_{j_1} - B_{j_2}$, $g_{j_1,j_2,j_1',j_2'} := 2h_{j_1,n}g_{j_1}^{(n)} - h_{j_2,n}g_{j_2}^{(n)} - 2h_{j_1',n}g_{j_1'}^{(n)} + h_{j_2',n}g_{j_2'}^{(n)}$ and the sets that we replaced by '...' are all contained in

$$B := \bigcup_{j_1,j_2,j_1',j_2'} (2A_{j_1} - A_{j_2} - 2A_{j_1'} + A_{j_2'}).$$

Denote the set $(F_{n-1} - F_{n-1}) \cup \sum_{i<n}(2C_i - C_i - 2C_i + C_i)$ by L_n.

The sets B_j and B_{j_1,j_2} do not depend on $h_{s,n}$, $s = 1, \dots, l_n$. Moreover, it is easy to see that their elements can be written as $q_n(ag_j^{(n)} + bg_{j'}^{(n)})$ for some $a, b \in \mathbb{Z}$, $|a| \le 2$, $|b| \le 1$ and $j, j' \in \{1, \dots, l_n\}$. Hence we can select q_n in such a way that

$$\begin{cases} \text{either } \ q_n(ag_j^{(n)} + bg_{j'}^{(n)}) = 0, \\ \text{or } \ q_n(ag_j^{(n)} + bg_{j'}^{(n)}) \notin L_n \end{cases} \tag{1.8}$$

for all a, b, j, j' as above. Now we consider the other sets that arise in the decompositions (1.6) and (1.7), i.e., the sets which depend on $h_{j,n}$. Every such set is the translation of a subset of B by $g_{j_1,j_2,j_1',j_2'}$ for some j_1, j_2, j_1', j_2' with $j_1 \neq j_1'$ or $j_2 \neq j_2'$, i.e., $(j_1, j_2) \neq (j_1', j_2')$. It is an easy exercise for the reader to show that, given any finite subset $K \subset G$, there exists a family of integers $0 = h_{1,n} < h_{2,n} < \cdots < h_{l_n,n}$ such that (1.4) holds and $g_{j_1,j_2,j_1',j_2'} \notin K$ for all j_1, j_2, j_1', j_2' with $(j_1, j_2) \neq (j_1', j_2')$. In our case we take $K := B + L_n$. It follows that the sets from (1.6) and (1.7) that are under consideration do not intersect $F_{n-1} - F_{n-1}$ and $\sum_{i<n}(2C_i - C_i - 2C_i + C_i)$, respectively. Due to this fact and (1.8) we deduce (1.2) and (1.5) from (1.6) and (1.7), respectively.

(II) Now we consider the second case, where $g_1^{(n)} = \cdots = g_{l_n}^{(n)} =: g^{(n)}$. Let

$$A'_i := \{0, q_n g^{(n)} + f_i^{(n)}\},\ i = 1, \dots, k_n, \quad \text{and}$$
$$C_n := \bigcup_{i=1}^{k_n} \bigcup_{s=d_0^{(n)}+\cdots+d_{i-1}^{(n)}}^{d_1^{(n)}+\cdots+d_i^{(n)}-1} (h_{s,n} g^{(n)} + A'_i),$$

where q_n and $h_{s,n}$ are some integers chosen so as to satisfy (1.2), (1.5) and the following condition:

$$\text{(1.9)} \quad \text{the sets } h_{s,n} g^{(n)} + A'_i,\ d_0^{(n)} + \cdots + d_{i-1}^{(n)} \le s < d_1^{(n)} + \cdots + d_i^{(n)},$$
$$i = 1, \dots, k_n, \text{ are mutually disjoint (and hence } \#C_n = 2d^{(n)}).$$

We will show how to make such a choice. As in (I), we write

$$\text{(1.10)} \qquad C_n - C_n = \bigcup_i B'_i \cup \bigcup_{s \ne s'} ((h_{s,n} - h_{s',n}) g^{(n)} + \cdots),$$

$$\text{(1.11)} \qquad 2C_n - C_n - 2C_n + C_n = \bigcup_{i_1, i_2} B'_{i_1,i_2} \cup$$
$$\bigcup_{(s_1,s_2) \ne (s'_1,s'_2)} ((2h_{s_1,n} - h_{s_2,n} - 2h_{s'_1,n} + h_{s'_2,n}) g^{(n)} + \cdots),$$

where $B'_i := A'_i - A'_i$, $B'_{i_1,i_2} := 2B'_{i_1} - B'_{i_2}$ and all the subsets that we replaced by '$\dots$' are contained in $B' := \bigcup_{i_1,i_2,i'_1,i'_2} (2A'_{i_1} - A'_{i_2} - 2A'_{i'_1} + A'_{i'_2})$. The sets B'_i and B'_{i_1,i_2} do not depend on $h_{s,n}$, $s = 0, \dots, d^{(n)} - 1$. Moreover, any nontrivial element of either of these sets can be written as $a q_n g^{(n)} + f'$, where $a \in \mathbb{Z}$, $0 < |a| \le 3$ and f' belongs to the sum of three copies of $F_{n-1} - F_{n-1}$. Since $a \ne 0$, we can choose q_n large so that

$$B'_i \cap (C_n - C_n) = B'_{i_1,i_2} \cap \sum_{j<n} (2C_j - C_j) = \{0\}$$

for all i, i_1, i_2. Now consider the other sets that arise in the decompositions (1.10) and (1.11). Every such set is a translation of a subset of B' by $(2h_{s_1,n} - h_{s_2,n} - 2h_{s'_1,n} + h_{s'_2,n}) g^{(n)}$ for some s_1, s_2, s'_1, s'_2 with $(s_1, s_2) \ne (s'_1, s'_2)$. Now we let $h_{s,n} := \widetilde{q}_n (10^s - 1)$, $0 \le s < d^{(n)}$, where $\widetilde{q}_n$ is an integer to be specified below. It is straightforward to check that

$$|\, 2h_{s_1,n} + h_{s_2,n} - 2h_{s'_1,n} + h_{s'_2,n}| < \widetilde{q}_n \text{ implies } s_1 = s'_1 \text{ and } s_2 = s'_2.$$

It remains to select $\widetilde{q}_n$ large enough so that (1.9) holds and the sets from (1.10) and (1.11) that are under consideration do not intersect $F_{n-1} - F_{n-1}$ and $\sum_{i<n} (2C_i - C_i - 2C_i + C_i)$, respectively. Then (1.2) and (1.5) follow in the same way as in (I). We need (1.5) to be satisfied in both cases for the proof of (ii).

Thus we constructed C_n. Now any sufficiently 'large' finite set satisfying (1.1) and such that $\frac{\#F_n}{\#C_1\cdots\#C_n} > n$ can be taken as F_n (see (P5)). Denote by T the corresponding (C,F)-action and by μ the Haar measure for T. We claim that (i)–(iv) from the statement of Theorem 0.1 hold for T.

(i) Fix $p \in \mathbb{N}$ and $g \in G_\infty$. As we already mentioned in Remark 1.1, it is easy to see that $T \times \cdots \times T$ (p times) is the (C,F)-action of G^p associated with the sequence $(C_n^p, F_n^p)_n$. The upper indices here denote the Cartesian powers. Hence we can apply Lemma 1.2 to establish that the transformation $T_g \times \cdots \times T_g$ (p times) is ergodic. Take any n so that $\sigma_n = (\underbrace{g', \dots, g'}_{p \text{ times}})$, where g' is a multiple of g. Notice that there exist infinitely many such numbers n. Given any $f = (f^1, \dots, f^p)$ and $\hat f = (\hat f^1, \dots, \hat f^p) \in F_{n-1}^p$, there exist $i_1, \dots, i_p \in \{1, \dots, k_n\}$ such that $f - \hat f = (f_{i_1}^{(n)}, \dots, f_{i_p}^{(n)})$. Then we define a subset $A \subset [f]_{n-1} \subset X^p$ as a union of n-cylinders

$$A := \bigsqcup_{s_1,\dots,s_p} [f^1 + h_{s_1,n} g']_n \times \cdots \times [f^p + h_{s_p,n} g']_n,$$

where the indices $s_1, \dots, s_p$ run as follows:

$$d_0^{(n)} + \cdots + d_{i_m-1}^{(n)} \le s_m < d_1 + \cdots + d_{i_m}^{(n)}, \ m = 1, \dots, p.$$

(For consistency of notation we let $i_0 := 0$.) Counting the number of these n-cylinders, we get

$$\frac{\mu^p(A)}{\mu^p([f]_{n-1})} = \frac{d_{i_1}^{(n)} \cdots d_{i_p}^{(n)}}{2^p (d^{(n)})^p} \ge \delta(f_{i_1}^{(n)}) \cdots \delta(f_{i_p}^{(n)}).$$

Moreover, by (P2) and the definition of C_n (see case (II)),

$$\begin{aligned}(T_{g'}^{q_n} \times \cdots \times T_{g'}^{q_n})A = \bigsqcup_{s_1,\dots,s_p} &[\hat f^1 + (h_{s_1,n} g' + q_n g' + f_{i_1}^{(n)})]_n \times \cdots \\ &\times [\hat f^p + (h_{s_p,n} g' + q_n g' + f_{i_p}^{(n)})]_n \subset [\hat f]_{n-1},\end{aligned}$$

where $s_1, \dots, s_p$ run as above. To apply Lemma 1.2, it remains to define a map $\widehat\delta : G^p \to \mathbb{R}_+^*$ by setting $\widehat\delta(g_1, \dots, g_p) := \delta(g_1) \cdots \delta(g_p)$ for all $g_1, \dots, g_p \in G$. Clearly, $\sum_{g \in G^p} \widehat\delta(g) < 0.5^p \le 0.5$. Hence by Lemma 1.2, $T_g \times \cdots \times T_g$ (p times) is ergodic. Thus, T_g has infinite ergodic index.

(ii) We will show now that given $g \in G$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $d \in C_n \setminus \{0\}$, the subsets $(T_{kg} \times T_{2kg})([0]_n \times [d]_n)$ and $[0]_n \times [0]_n$ are disjoint. Indeed, if this is not true, then there exist $x, x', y' \in [0]_n$ and $y \in [d]_n$ such that $T_{kg} x = x'$ and $T_{2kg} y = y'$. By (P3), for some $r > n$, we have

$$\begin{cases} kg + x_{n+1} + \cdots + x_r = x'_{n+1} + \cdots + x'_r, \\ 2kg + d + y_{n+1} + \cdots + y_r = y'_{n+1} + \cdots + y'_r, \end{cases}$$

where $x_i, y_i, x'_i, y'_i \in C_i$ are the i-th coordinates of x, y, x', y', respectively, $n < i \le r$. It follows that

$$\sum_{i=n+1}^{r} (2x_i - y_i) = d + \sum_{i=n+1}^{r} (2x'_i - y'_i).$$

We deduce from (1.5)—recall once more that this is satisfied in both cases (I) and (II)—that $d = 0$, a contradiction.

(iii) First of all we notice that a transformation is conservative whenever a power of it is conservative. Secondly, a transformation is conservative if and only if its direct product with the identity transformation is conservative. In view of this, it is enough to demonstrate that $T_{g_1} \times \cdots \times T_{g_p}$ is conservative for every $(g_1, \dots, g_p) \in \mathfrak{S}$. If $g_1 = \cdots = g_p$, then $T_{g_1} \times \cdots \times T_{g_p}$ is ergodic by (i) and hence conservative since μ is non-atomic (see (P4)). Hence it remains to consider the case when there are unequal elements among $g_1, \dots, g_p$. We find n such that $\sigma_n = (g'_1, \dots, g'_p)$ is a multiple of $(g_1, \dots, g_p)$. (Notice that there exist infinitely many such n. The corresponding families of n-cylinders generate the entire Borel σ-algebra on X^p.) Recall that the corresponding C_n was defined in (I). Take any $f = (f^1, \dots, f^p) \in F^p_{n-1}$ and set

$$A := \bigsqcup_{j_1,\dots,j_p=1}^{p} [f^1 + h_{j_1,n} g'_1]_n \times \cdots \times [f^p + h_{j_p,n} g'_p]_n.$$

Then $A \subset [f]_{n-1}$ and $\mu^p(A) = 2^{-p} \mu^p([f]_{n-1})$. Moreover, by (P2) and the definition of C_n, we get $(T_{g'_1} \times \cdots \times T_{g'_p}) A \subset [f]_{n-1}$. It is now standard to conclude that $T_{g_1} \times \cdots \times T_{g_p}$ is conservative.

(iv) follows immediately from (i) and (ii). □

Remark 1.3. Notice that we also established the following topological properties of T:

(i) The homeomorphism $T_g \times \cdots \times T_g$ (p times) is topologically transitive (i.e., the orbit of every open set is dense) for every $g \in G_\infty$ and $p \in \mathbb{N}$.
(ii) The homeomorphism $T_g \times T_{2g}$ is not topologically transitive for any $g \in G$.
(iii) The homeomorphism $T_{g_1} \times \cdots \times T_{g_n}$ is topologically recurrent for every finite sequence $g_1, \dots, g_n$ of elements from G.
(iv) T_g is not topologically conjugate to T_{2g} for any $g \in G_\infty$.

2. Nonsingular Chacon transformations

In this section we will prove Theorems 0.2 and 0.3. Recall first the definition of nonsingular (C, F)-actions from [Da]. For sequences $(C_n)_{n=1}^\infty$ and $(F_n)_{n=0}^\infty$ satisfying (1.1)–(1.3), let κ_n and τ_n be measures on C_n and F_n, respectively, such that $\kappa_n(C_n) = 1$ and $\kappa_n(c) > 0$ for all $c \in C_n$. Moreover, let

$$\tau_{n-1} * \kappa_n = \tau_n \restriction F_{n-1}, \ n \in \mathbb{N}. \tag{2.1}$$

The convolution here is well defined if we consider τ_{n-1} and κ_n as measures on G supported on F_{n-1} and C_n, respectively. It follows from (1.2) that $(\tau_{n-1} * \kappa_n)(f + c) = \tau_n(f)\kappa_n(c)$ for all $f \in F_{n-1}$ and $c \in C_n$. We will assume always that the infinite product measure $\bigotimes_{n=1}^{\infty} \kappa_n$ is non-atomic, i.e., $\prod_{n=1}^{\infty} \max_{c\in C_n} \kappa_n(c) = 0$. Now equip $X_n = F_n \times \prod_{k>n} C_k$ with the product measure $\mu_n := \tau_n \otimes \bigotimes_{k>n} \kappa_k$. It follows from (2.1) that the canonical embeddings $(X_n, \mu_n) \to (X_{n+1}, \mu_{n+1})$ are all measure preserving. Hence an inductive limit μ of the system $(\mu_n)_{n=1}^{\infty}$ is well defined. Clearly, μ is a σ-finite measure on X. It is worthwhile to notice that the equivalence class of μ is determined completely by $(\kappa_n)_{n=1}^{\infty}$. It does not depend on a particular choice of τ_n, $n \in \mathbb{N}$, satisfying (2.1). Moreover, for any sequence of $(\kappa_n)_{n=1}^{\infty}$ there exists a sequence $(\tau_n)_{n=0}^{\infty}$ satisfying (2.1).

It is easy to verify that the (C, F)-action T is μ-nonsingular and ergodic. Moreover, in view of (P3),

$$\frac{d\mu \circ T_g}{d\mu}(x) = \frac{\tau_n(y_n)}{\tau_n(x_n)} \cdot \prod_{k>n} \frac{\kappa_k(y_n)}{\kappa_k(x_n)}, \tag{2.2}$$

if $x = (x_k)_{k\geq n}$ and $T_g x = (y_k)_{k\geq n}$ belong to X_n. Notice that only finitely many multiplies in this product are different of 1. We call (X, μ, T) the *nonsingular* (C, F)*-action associated with* (C_n, F_n, κ_n).

We need a nonsingular counterpart of Lemma 1.2 (cf. [Da, Lemma 2.4]).

LEMMA 2.1. *Let $\delta, \beta : G \to \mathbb{R}_+^*$ be two maps and $g \in G$. If for all n and any pair $f, f' \in F_n$ there exist a subset $A \subset [f]_n$ and $l \in \mathbb{Z}$ such that $T_{lg}A \subset [f']_n$, $\mu(A) > \delta(f - f')\mu([f]_n)$ and*

$$\frac{d\mu \circ T_{lg}}{d\mu}(x) \geq \beta(f - f')\frac{\mu([f']_n)}{\mu([f]_n)} \quad \textit{for a.a. } x \in A,$$

then the transformation T_g is ergodic.

Proof. Let $A_1, A_2 \subset X$ be subsets of positive finite measure. Repeating the proof of Lemma 1.2 almost literally, we find $n > 0$ and $f_1, f_2 \in F_n$ such that that the n-cylinders $[f_1]_n$ and $[f_2]_n$ are 0.99-full of A_1 and A_2, respectively. Moreover, given $\epsilon > 0$, there exist $m > n$ and subsets $D_i \subset C_{n+1} + \cdots + C_m$ such that $(\kappa_{n+1} * \cdots * \kappa_m)(D_i) > 0.5$ and $[f_i + d]_m$ is $(1 - \epsilon)$-full of A_i for all $d \in D_i$, $i = 1, 2$. Recall that $*$ stands for the convolution of measures and κ_j is considered as a probability on G supported on C_j, $j = n+1, \ldots, m$. Hence $D_1 \cap D_2 \neq \emptyset$. Take any $d \in D_1 \cap D_2$ and apply the conditions of the lemma to $f_1 + d$ and $f_2 + d$ from F_m. Then there are a subset $A \subset [f_1 + d]_m$ and $l \in \mathbb{Z}$ such that $[f_1 + d]_m$ is $\delta(f_1 - f_2)$-full of A, $T_{lg}A \subset [f_2 + d]_m$ and

$$\frac{d\mu \circ T_{lg}}{d\mu}(x) \geq \beta(f_1 - f_2)\frac{\mu([f_2 + d]_m)}{\mu([f_1 + d]_m)}$$

for a.a $x \in A$. Hence $[f_1 + d]_m$ is $(\delta(f_1 - f_2) - \epsilon)$-full of $A \cap A_1$ and

$$\mu(T_{lg}(A \cap A_1)) \geq (\delta(f_1 - f_2) - \epsilon) \cdot \mu([f_1 + d]_m)\beta(f_1 - f_2) \cdot \frac{\mu([f_2 + d]_m)}{\mu([f_1 + d]_m)}.$$

Hence $[f_2 + d]_m$ is $(\delta(f_1 - f_2) - \epsilon) \cdot \beta(f_1 - f_2)$-full of $T_{lg}A_1$. It follows that $[f_2 + d]_m$ is $((\delta(f_1 - f_2) - \epsilon) \cdot \beta(f_1 - f_2) - \epsilon)$-full of $T_{lg}A_1 \cap A_2$. Thus if ϵ is small enough, then $\mu(T_{lg}A_1 \cap A_2) > 0$. □

In the remainder of this paper let $G = \mathbb{Z}$. We are going to distinguish a special subclass of 'modified' nonsingular (C, F)-actions of $\mathbb{Z}$. To this end we define a sequence $(h_n)_{n=1}^\infty$ of positive integers recursively by setting $h_1 := 1$, $h_{n+1} := 3h_n + 1$. It is easy to verify that the sequences of sets $F_{n-1} := \{0, 1, \ldots, h_n - 1\}$ and $C_n := \{0, h_n, 2h_n + 1\}$ satisfy (1.1)–(1.2) but do not satisfy (1.3). Nevertheless, it not difficult to see that $\bigcap_{g \in G} D_g = \bigcap_{g \in G} R_g = X \setminus D$, where

$$D := \bigcup_{n \geq 0} \{x = (x_k)_{k \geq n} \in X_n \mid \text{either } x_k = 0 \text{ eventually} \\ \text{or } x_k = 2h_k + 1 \text{ eventually}\}$$

(see Section 1 for the definition of D_g and R_g).

Next, given a sequence $(\kappa'_n)_{n=1}^\infty$ of distributions on $\{0, 1, 2\}$ with non-atomic product $\bigotimes_{n=1}^\infty \kappa'_n$ and $\kappa'_n(i) > 0$ for all i, we define measures κ_n on C_n as follows: $\kappa_n(0) := \kappa'_n(0)$, $\kappa_n(h_n) := \kappa'_n(1)$ and $\kappa_n(2h_n + 1) := \kappa'_n(2)$. Now take any sequence τ_n of measures on F_n satisfying (2.1).

Definition 2.2. The corresponding dynamical system (X, μ, T), or simply T_1, is called the *nonsingular 2-cuts Chacon transformation associated with* $(C_n, F_n, \kappa_n)_n$. We call T

(1) *symmetric* if $\kappa'_n(0) = \kappa'_n(2)$,
(2) *stationary* if $\kappa'_1 = \kappa'_2 = \cdots$,
(3) *weakly stationary* if for any distribution κ on $\{0, 1, 2\}$ and every $n > 0$ there exists $m > n$ with $\kappa'_m = \cdots = \kappa'_{n+m} = \kappa$.

Since D is countable, $\mu(D) = 0$. Hence T is well defined on a μ-conull invariant subset $X \setminus D$. Being restricted to this subset, (X, μ, T) enjoys all the properties of 'usual' nonsingular (C, F)-actions.

Remark 2.3. It is worthwhile to give an alternative definition of T_1 via a more common inductive cutting-and-stacking process (see [JuS1], [HaS] and [AFS2]). Assume that at the n-th step we have a column $Y_n = \{I(i, n) \mid i \in F_n\}$ consisting of disjoint intervals $I(i, n) \subset \mathbb{R}$ such that $\bigsqcup_{i \in F_n} I(i, n) = [0, \tau_n(F_n))$. This partially defines an injective transformation T_1 by the affine maps

$$T_1 : I(j, n) \to I(j + 1, n),\ 0 \leq j < 3h_n.$$

Now cut each $I(j,n)$ into subintervals $I_k(j,n)$ for $k = 0,1,2$ (numbered from left to right) such that the proportions of their lengths are

$$\kappa'_{n+1}(0) : \kappa'_{n+1}(1) : \kappa'_{n+1}(2).$$

Let S_{n+1} be a new interval—*spacer*—which abuts with $[0, \nu_n(F_n))$ over the interval $I(3h_n, n)$ of length $\tau_{n+1}(2h_{n+1})$. We have 3 subcolumns $Y_{n,k} = \{I_k(i,n) \mid i \in F_n\}$ of Y_n, $k = 0,1,2$. Stack these from left to right and extend the transformation by the affine maps

$$\begin{aligned} T_1 &: I_0(3h_n, n) \to I_1(0,n), \\ T_1 &: I_1(3h_n, n) \to S_{n+1} \text{ and} \\ T_1 &: S_{n+1} \to I_2(0,n). \end{aligned}$$

Rename the new intervals by $I(0, n+1), \dots, I(3h_{n+1}, n+1)$. This defines the column Y_{n+1}. Notice that $\bigsqcup_{i \in F_{n+1}} I(i, n+1) = [0, \tau_{n+1}(F_{n+1}))$. In the limit we get a nonsingular transformation T_1 on the interval $[0, \lim_{n\to\infty} \tau_n(F_n))$ equipped with the Lebesgue measure. Our remark on $(\tau_n)_{n\geq 1}$ at the beginning of this section can be interpreted now as follows: *The lengths of spacers do not affect the isomorphism class of* T_1.

As far as we know, only the following two families of 2-cuts Chacon transformations were studied in the literature: stationary symmetric transformations ([JuS1], [JuS2], [AFS2]) and λ-weakly stationary symmetric transformations, where λ is the equi-distribution on $\{0,1,2\}$ [HaS]. In this paper we deal with the class of all weakly stationary 2-cuts Chacon maps, which includes these two families.

Proof of Theorem 0.2. Let T be κ-weakly stationary. Put

$$\delta := \min(\kappa(0), \kappa(1), \kappa(2)) \text{ and } \beta := \delta / \max(\kappa(0), \kappa(1), \kappa(2)).$$

We first prove that T_1 has infinite ergodic index. (We note that this proof will hold for the wider class of nonsingular (C,F)-transformations having the same sequences C_n, F_n, κ_n but with arbitrary $(h_n)_{n=1}^{\infty}$ satisfying $h_{n+1} \geq 3h_n + 1$.) Notice that if T is associated with $(C_n, F_n, \kappa_n)_{n\geq 1}$, then $T \times \cdots \times T$ (p times) is the nonsingular action of $\mathbb{Z}^p$ associated with $(C_n^p, F_n^p, \kappa_n^p)_{n\geq 1}$. Here and everywhere below the upper index p over a set or a measure means the p-fold direct product of this set or this measure, respectively. For $g = (g_1, \dots, g_p) \in \mathbb{Z}^p$, we set $\|g\| := \sum_{i=1}^{p} |g_i|$. Fix $n \in \mathbb{N}$ and find $m > n$ with $\kappa'_m = \cdots = \kappa'_{m+ph_{n+1}} = \kappa$.

Claim A. *If $f, f' \in F_n^p$ and $\|f - f'\| = 1$, then for some $c, c' \in C_m^p$ and $\sigma \in \{-1, 1\}$, we have* $(\underbrace{T_1 \times \cdots \times T_1}_{p \text{ times}})^{\sigma h_m}[f + c]_m = [f' + c']_m$.

If $f'-f=(1,0,\dots,0)$, then put

$$c:=(2h_m+1,h_m,\dots,h_m),\ c':=(h_m,0,\dots,0)$$

and $\sigma:=-1$. It is straightforward to show that $\sigma(h_m,\dots,h_m)=f'-f+c'-c$. It remains to apply (P2). The other $2p-1$ cases are considered in a similar way. Thus Claim A is proved.

Now for arbitrary elements $f,f'\in F_n^p$, we have $s:=\|f-f'\|\le ph_{n+1}$. There exist $f_0,\dots,f_s\in F_m^p$ with $f_0=f,f_s=f'$ and $\|f_{i+1}-f_i'\|=1$, $i=0,\dots,s-1$. Then we apply Claim A s times to find $c_m,c_m'\in C_m^p$, $\dots$, $c_{m+s-1},c'_{m+s-1}\in C^p_{m+s-1}$ and $\sigma_1,\dots,\sigma_s\in\{-1,1\}$ with

$$(\underbrace{T_1\times\cdots\times T_1}_{p\text{ times}})^r[f+d+c_m+\cdots+c_{m+s-1}]_{m+s-1} \\ =[f'+d+c'_m+\cdots+c'_{m+s-1}]_{m+s-1}, \tag{2.3}$$

for all $d\in C_{n+1}^p+\cdots+C_{m-1}^p$, where $r:=\sigma_1h_m+\cdots+\sigma_sh_{m+s-1}$. We let

$$A:=\bigsqcup_{d\in C^p_{n+1}+\cdots+C^p_{m-1}}[f+d+c_m+\cdots+c_{m+s-1}]_{m+s-1}.$$

Clearly, $A\subset[f]_n$. By (2.3), $(T_1\times\cdots\times T_1)^rA\subset[f']_n$. Since $\mu^p\restriction X_n^p=\tau_n^p\otimes\bigotimes_{k>n}\kappa_k^p$ and $\kappa_k^p(c_k)\ge\delta^p$ for all $m\le k<m+s$, it follows that

$$\begin{aligned}\mu^p(A)&=\sum_{d\in C^p_{n+1}+\cdots+C^p_{m-1}}\mu([f+d+c_m+\cdots+c_{m+s-1}]_{m+s-1})\\ &=\sum_d\tau_n^p(f)\cdot(\kappa_{n+1}*\cdots*\kappa_{m-1})^p(d)\cdot\kappa_m^p(c_m)\cdots\kappa^p_{m+s-1}(c_{m+s-1})\\ &\ge\delta^{ps}\tau_n^p(f)\sum_d(\kappa_{n+1}*\cdots*\kappa_{m-1})^p(d)=\delta^{p\,\|f-f'\|}\mu^p([f]_n).\end{aligned}$$

Next, for a.a. $x\in[f+d+c_m+\cdots+c_{m+s-1}]_{m+s-1}$, we have by (2.2) that

$$\begin{aligned}&\frac{d\mu^p\circ(T_1\times\cdots\times T_1)^r}{d\mu^p}(x)\\ &\quad=\frac{\tau_n^p(f')\cdot(\kappa_{n+1}*\cdots*\kappa_{m-1})^p(d)\cdot\kappa_m^p(c'_m)\cdots\kappa^p_{m+s-1}(c'_{m+s-1})}{\tau_n^p(f)\cdot(\kappa_{n+1}*\cdots*\kappa_{m-1})^p(d)\cdot\kappa_m^p(c_m)\cdots\kappa^p_{m+s-1}(c_{m+s-1})}\\ &\quad\ge\beta^{ps}\cdot\frac{\mu^p([f']_n)}{\mu^p([f]_n)}.\end{aligned}$$

Hence

$$\frac{d\mu^p\circ(T_1\times\cdots\times T_1)^r}{d\mu^p}(x)\ge\beta^{p\,\|f'-f\|}\cdot\frac{\mu([f']_n)}{\mu([f]_n)}$$

for a.a. $x\in A$. By Lemma 2.1, the p-fold Cartesian power of T_1 is ergodic.

To prove that T_1 is power weakly mixing, we argue in a similar way. However, instead of Claim A we need another statement (Claim B below). Let

$g = (g_1, \dots, g_p) \in \mathbb{Z}^p$ and $g_i \neq 0$ for all i. Consider ternary expansions $g_i = \sum_{j=0}^{J_i} a_{i,j} 3^j$ with $a_{i,j} \in \{-1, 0, 1\}$ and $a_{i,J_i} \neq 0$, $i = 1, \dots, p$. We set $J := \max_{1 \le i \le p} J_i + 1$.

Claim B. *If $f \in \mathbb{Z}^p$ and $\|f\| = 1$, then there exists $\sigma \in \{-1, 1\}$ such that $\sigma 3^{n+1} g \in f + \sum_{j=n}^{n+J} (C_j^p - C_j^p)$ for any $n > 0$.*

Suppose that $f = (1, 0, \dots, 0)$ and $g_1 > 0$. The other cases are settled in a similar way. We put $\sigma := 1$. Notice that

$$C_i - C_i = \{-2h_i - 1, -h_i - 1, -h_i, 0, h_i, h_i + 1, 2h_i + 1\}. \tag{2.4}$$

Since $3^{n+1} g_i = \sum_{j=0}^{J_i} a_{i,j} 3^{j+n+1}$ and $3^i = 2h_{i-1} + 1 \in C_{i-1} - C_{i-1}$, we obtain

$$3^{n+1} g_i \in \sum_{j=n}^{n+J_i} (C_j - C_j),\ i = 1, \dots, p.$$

It remains to show that $3^{n+1} g_1 \in 1 + \sum_{j=n}^{n+J} (C_j - C_j)$. It follows from our assumptions on g_1 that $a_{1,J_1} = 1$. It is easy to check that

$$3^{J_1+n+1} = 1 + (-h_{J_1+n} - 1) + h_{J_1+n+1} \in 1 + \sum_{j=J_1+n}^{J_1+n+1} (C_j - C_j).$$

Hence

$$3^{n+1} g_1 = \sum_{j=0}^{J_1-1} a_{1,j} 3^{j+n+1} + 3^{J_1+n+1} \in \sum_{j=n}^{n+J_1-1} (C_j - C_j) + 1 + \sum_{j=n+J_1}^{n+J_1+1} (C_j - C_j),$$

and Claim B is proved.

Now fix n and find $m > n$ with $\kappa'_m = \dots = \kappa'_{m+ph_{n+1}J}$. The remainder of the proof is only a slight modification of the proof for the ergodicity of $T_1 \times \dots \times T_1$. We leave it to the reader. $\square$

We can easily modify the construction of Chacon maps to produce a family of non-power weakly mixing 2-cuts nonsingular transformations with infinite ergodic index. Recall that only *infinite measure preserving* transformations with these properties are known so far [AFS2], [G–W]. As for purely nonsingular case (type III of Krieger), all known examples of non-power weakly mixing maps with infinite ergodic index (see [Da]) have unbounded cuts.

Definition 2.4. We define nonsingular 2-cuts *Chacon** transformations almost exactly as in Definition 2.1 with the only difference that $h_{n+1} = 11h_n$, $n > 0$.

It is easy to see that T is well defined on a μ-conull invariant subset $X \setminus D^*$, where $D^* := \bigcup_{n \ge 0} \{x = (x_k)_{k \ge n} \mid x_k = 0 \text{ eventually}\}$.

Proof of Theorem 0.3. Let $T = (T_n)_{n\in\mathbb{Z}}$ be a weakly stationary nonsingular Chacon* map with 2-cuts. That T_1 has infinite ergodic index, we already demonstrated in the first part of the proof of Theorem 0.2 (this part is independent on the choice of h_n). We show that $T_1 \times T_3$ is not conservative (and hence T_1 is not power weakly mixing) by contradiction. Suppose that

$$(T_m \times T_{3m})([0]_0 \times [0]_0) \cap ([0]_0 \times [0]_0) \neq 0$$

for a sufficiently large $m > 0$. Arguing as in the proof of Theorem 0.1(ii), we deduce from (P3) that there exist n, n' with

$$\begin{cases} m = \sum_{i=1}^{n} d_i, \\ 3m = \sum_{i=1}^{n'} d_i', \end{cases} \tag{2.5}$$

where $d_i, d_i' \in C_i - C_i$ and $d_n, d'_{n'} \neq 0$. Since

$$\sum_{i=1}^{k} |d_i| \leq \sum_{i=1}^{k} (2h_i + 1) = 0.2(11^{k+1} - 1) + k - 1 < 0.21 h_{k+1}$$

for all sufficiently large k, we deduce from (2.4) and (2.5) that

$$\begin{cases} \text{either } m = \alpha + h_n \text{ or } m = \alpha + 2h_n \text{ with } |\alpha| < 0.21 h_n \text{ and} \\ \text{either } 3m = \beta + h_{n'} \text{ or } 3m = \beta + 2h_{n'} \text{ with } |\beta| < 0.21 h_n. \end{cases}$$

It is easy to verify that neither of the four possible alternatives is true. For instance, if $m = (1 \pm 0.21)h_n$ and $3m = (1 \pm 0.21)h_{n'}$, then

$$3 = 11^{n'-n} \frac{1 \pm 0.21}{1 \pm 0.21}$$

and hence $1.6 \cdot 11^{n'-n} > 3 > 0.6 \cdot 11^{n'-n}$. The left inequality implies $n' > n$, while the right one implies $n \geq n'$, a contradiction. The other three cases are handled in a similar way. $\square$

We conclude this paper with a short discussion on how 'representative' the above two classes of Chacon maps with 2-cuts are.

Recall that two nonsingular transformations are *orbit equivalent* if there exists a nonsingular isomorphism of the underlying measure spaces which maps bijectively the orbits of the first transformation onto the orbits of the second one. By the Dye theorem any two ergodic probability preserving transformations are orbit equivalent. As for nonsingular ergodic maps without an equivalent probability measure, there is a one-to-one correspondence between the orbit equivalent classes of such maps and the conjugacy classes of ergodic nonsingular flows (see [HO1] for a detailed exposition of these results). Next, an ergodic flow corresponds to a (C, F)-transformation if and only if it is an AT-flow (see [CW] and [Da]). Unfortunately, there is no satisfactory characterization of the ergodic flows corresponding to (C, F)-maps with bounded cuts (i.e., with the sequence $\#C_n$ bounded). We list some facts about the orbit equivalence of (C, F)-transformations.

- There are ergodic transformations that are not orbit equivalent to any nonsingular (C, F)-transformation (this follows from [Kr] and [CW]).
- There are nonsingular (C, F)-transformations that are not orbit equivalent to any (C, F)-map with bounded cuts (this follows from [Kr] and [GS2]).
- Any nonsingular (C, F)-transformation with bounded cuts is orbit equivalent to a weakly stationary symmetric Chacon (and Chacon*) map with 2-cuts (this can be deduced from [GS1]).
- The class of flows associated with weakly stationary symmetric 2-cuts Chacon (and Chacon*) maps includes the flows with pure point spectrum θD for any $\theta \in \mathbb{R}$ and a subgroup D of $\mathbb{Q}$ (this can be deduced from [GS2] or [HO2]). In particular, there exist weakly stationary symmetric 2-cuts Chacon (and Chacon*) maps of any Krieger type III_λ, $\lambda \in (0, 1]$, and even a continuum of pairwise non-orbit equivalent such maps of type III_0.

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