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HILBERT MATRIX ON BERGMAN SPACES

E. DIAMANTOPOULOS

ABSTRACT. The Hilbert matrix acts on Bergman spaces by multiplication on Taylor coefficients. We find an upper bound for the norm of the induced operator.

1. Introduction

The Hilbert matrix H with entries $a_{i,j} = \frac{1}{i+j+1}$ for i and j positive integers induces an operator by multiplication on sequences,

$$H : (a_n)_{n\geq 0} \to \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right)_{n\geq 0}.$$

For $1 < p < \infty$, Hilbert's Inequality [HLP, p. 226]

$$\left(\sum_{n=0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{a_k}{n+k+1} \right|^p \right)^{1/p} \leq \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{1/p}, \tag{1}$$

implies that H induces a bounded operator on l^p spaces of $p-$summable sequences. Moreover, the constant $\pi/\sin(\pi/p)$ is best-possible and the norm of H is

$$\|H\|_{l^p\to l^p} = \frac{\pi}{\sin(\pi/p)}, \qquad 1 < p < \infty.$$

The Hilbert matrix also induces a transformation $\mathcal{H}$ on spaces of analytic functions by its action on Taylor coefficients defined by

$$\mathcal{H} : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} z^n,$$

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for those analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which the coefficients

$$A_n = \sum_{k=0}^{\infty} \frac{a_k}{n+k+1}, \qquad n = 0, 1, \ldots$$

converge.

The operator $\mathcal{H}$ has been studied on Hardy spaces. In [DS] we proved that $\mathcal{H}$ is a bounded operator on the Hardy spaces H^p, $p > 1$, and for $2 \le p < \infty$ we found the following upper bound for its norm (see [DS, Th. 1.1]):

$$\|\mathcal{H}\|_{H^p \to H^p} \le \frac{\pi}{\sin \pi/p}. \tag{2}$$

We also proved that for functions f such that $f(0) = 0$ the same estimate holds for $1 < p < 2$.

In this article we prove that $\mathcal{H}$ is a bounded operator on the Bergman spaces A^p, $2 < p < +\infty$, of analytic functions f on the unit disc for which

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dm(z) < +\infty,$$

where $dm(z) = (1/\pi)dxdy$ is the normalized Lebesgue measure on unit disc. We also provide norm estimates on those spaces. More precisely we show:

THEOREM 1. *The operator $\mathcal{H}$ is bounded on Bergman spaces A^p, $2 < p < +\infty$, and satisfies:*

(i) *If $4 \le p < \infty$ and $f \in A^p$, then*

$$\|\mathcal{H}(f)\|_{A^p} \le \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}.$$

(ii) *If $2 < p < 4$ and $f \in A^p$, then*

$$\|\mathcal{H}(f)\|_{A^p} \le \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p} \right)^{1/p} \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}.$$

(iii) *If $2 < p < 4$ and $f \in A^p$ with $f(0) = 0$, then*

$$\|\mathcal{H}(f)\|_{A^p} \le \left(\frac{p}{2} + 1 \right)^{1/p} \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}.$$

The proof of this result will be given in Section 3 and involves the representation of $\mathcal{H}$, used in [DS] to prove (2), in terms of weighted composition operators for which we can estimate the Bergman space norms. It uses a representation similar to one developed by A. G. Siskakis to prove that the Cesàro operator is bounded on the Hardy and Bergman spaces, respectively ([Sis1], [Sis2]). P. Galanopoulos [Ga] exploited the same representation to prove that the Cesàro operator is bounded on Dirichlet spaces.

1.1. Integral form of $\mathcal{H}$. We consider the operator

$$\mathcal{S}(f)(z) = \int_0^1 \frac{f(t)}{1-tz}\, dt. \tag{3}$$

This operator is well defined on Bergman spaces. Indeed, using [Vu, Corollary, p. 755] we have

$$|f(z)| \le \left(\frac{1}{1-|z|^2}\right)^{2/p} \|f\|_{A^p} \tag{4}$$

for $p > 2$ and $f \in A^p$, and hence

$$|\mathcal{S}(f)(z)| \le \frac{\int_0^1 \frac{1}{(1-t)^{2/p}}\, dt}{1-|z|} \|f\|_{A^p} < +\infty.$$

Now, given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in A^p, let $f_N(z) = \sum_{n=0}^{N} a_n z^n$. We see that

$$\begin{aligned}\mathcal{H}(f_N)(z) &= \sum_{n=0}^{\infty}\sum_{k=0}^{N} \frac{a_k}{n+k+1} z^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{N} \int_0^1 t^{n+k}\, dt a_k z^n\\ &= \sum_{n=0}^{\infty} \int_0^1 f_N(t)(tz)^n\, dt\\ &= \mathcal{S}(f_N)(z),\end{aligned}$$

so $\mathcal{H}$ is well defined on polynomials. Also, for $z \in \mathbb{D}$ and $p > 2$ we see that

$$\begin{aligned}\left|\mathcal{S}(f)(z) - \sum_{n=0}^{\infty}\sum_{k=0}^{N} \frac{a_k}{n+k+1} z^n\right| &\le \frac{\int_0^1 |f(t) - f_N(t)|\, dt}{1-|z|}\\ &\le \frac{\int_0^1 \frac{1}{(1-t)^{2/p}}\, dt}{1-|z|} \|f - f_N\|_{A^p}.\end{aligned}$$

Thus, as $N \to \infty$, the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{N} \frac{a_k}{n+k+1} z^n$$

converges and defines an analytic function

$$\mathcal{H}(f)(z) = \mathcal{S}(f)(z) = \int_0^1 \frac{f(t)}{1-tz}\, dt,$$

which is in the Bergman spaces A^p, $p > 2$.

In the next section we derive the expression of $\mathcal{H}$ in terms of weighted composition operators mentioned above. In Section 3, we prove that $\mathcal{H}$ is bounded on Bergman spaces A^p for $p > 2$ and we give norm estimates. Finally, in Section 4, using the natural isometric isomorphism between A^2 and Dirichlet space $\mathcal{D}$, we prove that $\mathcal{H}$ is not bounded on A^2.

2. $\mathcal{H}$ in terms of composition operators

In this section we show how $\mathcal{H}$ can be written as an average of certain weighted composition operators.

Every analytic function $\phi : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator $C_\phi : f \to f \circ \phi$ on A^p for $1 \leq p \leq +\infty$; the norm of this operator satisfies [CM, p. 127]

$$\|C_\phi\|_{A^p} \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right)^{2/p}. \tag{5}$$

In addition, if $w(z)$ is a bounded analytic function, then the weighted composition operator

$$C_{w,\phi}(f)(z) = w(z) f(\phi(z))$$

is bounded on each A^p. This is the only property of this operator that we will use.

The connection between the Hilbert matrix and composition operators arises as follows. For $z \in \mathbb{D}$ and $0 < r < 1$ we define

$$C_r(f)(z) = \int_0^r f(t) \frac{1}{1 - tz}\, dt \tag{6}$$

and we see that

$$\mathcal{H}(f)(z) = \lim_{r \to 1} C_r(f)(z).$$

Given $z \in \mathbb{D}$ we choose the path of integration

$$t(s) = t_z(s) = \frac{rs}{r(s-1)z + 1}, \qquad 0 \leq s \leq 1,$$

and changing variables in (6) we obtain

$$\begin{aligned} C_r(f)(z) &= \int_0^r f(t) \frac{1}{1 - tz}\, dt \\ &= \int_0^1 f(t(s)) \frac{1}{1 - t(s)z} t'(s)\, ds \\ &= \int_0^1 \frac{r}{r(s-1)z + 1} f\left(\frac{rs}{r(s-1)z + 1}\right) ds. \end{aligned}$$

Now let $f \in A^p$, $p > 2$. For every $z \in \mathbb{D}$ and $0 \leq s \leq 1$ let

$$\begin{aligned} h_r(s) &= \frac{r}{r(s-1)z + 1} f\left(\frac{rs}{r(s-1)z + 1}\right) \\ &= \frac{r}{r(s-1)z + 1} f(\phi_{r,s}(z)), \end{aligned}$$

where $\phi_{r,s}(z) = rs/(r(s-1)z+1)$ is an analytic self-map of the unit disc.

Since

$$|r(s-1)z+1| \geq 1-|z|, \qquad 0 \leq s, r \leq 1,$$

we have

$$\frac{r}{|r(s-1)z+1|} \leq \frac{1}{1-|z|} \leq \frac{2}{1-|z|^2}.$$

By (4) we have

$$|f \circ \phi_{r,s}(z)| \leq \left(\frac{1}{1-|z|^2}\right)^{2/p} \|f \circ \phi_{r,s}(z)\|_{A^p},$$

and using (5) we obtain

$$\begin{aligned}\|f \circ \phi_{r,s}(z)\|_{A^p} &\leq \left(\frac{1+|\phi_{r,s}(0)|}{1-|\phi_{r,s}(0)|}\right)^{2/p} \|f\|_{A^p} \\ &= \left(\frac{1+rs}{1-rs}\right)^{2/p} \|f\|_{A^p} \\ &\leq \left(\frac{1+s}{1-s}\right)^{2/p} \|f\|_{A^p}.\end{aligned}$$

The above estimates give

$$|h_r(s)| \leq \frac{2}{(1-|z|^2)^{1+2/p}} \left(\frac{1+s}{1-s}\right)^{2/p} \|f\|_{A^p}.$$

For $p > 2$ the right-hand side of the latter inequality is an integrable function of s. By Lebesgue's dominated convergence theorem we conclude that

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(s-1)z+1} f\left(\frac{s}{(s-1)z+1}\right) ds,$$

that is, we can express $\mathcal{H}$ as an integral mean

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z)\, dt$$

of the family of weighted composition operators

$$T_t(f)(z) = \omega_t(z) f(\phi_t(z)),$$

where

$$\omega_t(z) = \frac{1}{(t-1)z+1}$$

and

$$\phi_t(z) = \frac{t}{(t-1)z+1}.$$

It is easy to see that ω_t is a bounded function for $0 < t < 1$, and that ϕ_t is a self-map of the disc. Thus, the operator $T_t : A^p \to A^p$, $1 \leq p < +\infty$, is bounded on A^p for every $0 < t < 1$.

3. Proof of the Theorem

We first obtain estimates for the norms of the weighted composition operators T_t.

LEMMA 2. *Let $2 < p < +\infty$. Then:*

(i) *If $4 \le p < +\infty$ and $f \in A^p$, then*

$$\|T_t(f)\|_{A^p} \le \frac{t^{2/p-1}}{(1-t)^{2/p}} \|f\|_{A^p}.$$

(ii) *If $2 < p < 4$ and $f \in A^p$, then*

$$\|T_t(f)\|_{A^p} \le \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p} \right)^{1/p} \frac{t^{2/p-1}}{(1-t)^{2/p}} \|f\|_{A^p}.$$

Proof. We can easily check that

$$\omega_t(z)^2 = \frac{1}{t(1-t)} \phi_t'(z)$$

Let $f \in A^p$, $p > 2$. Using the last equation we obtain

$$\begin{aligned} \|T_t(f)\|_{A^p}^p &= \int_{\mathbb{D}} |\omega_t(z)|^p |f(\phi_t(z))|^p \, dm(z) \\ &= \int_{\mathbb{D}} |\omega_t(z)|^{p-4} |\omega_t(z)|^4 |f(\phi_t(z))|^p \, dm(z) \\ &= \frac{1}{(t(1-t))^2} \int_{\mathbb{D}} |\omega_t(z)|^{p-4} |f(\phi_t(z))|^p |\phi_t'(z)|^2 \, dm(z) \\ &= \frac{1}{(t(1-t))^2} \int_{\phi_t(\mathbb{D})} |\omega_t(\phi_t^{-1}(z))|^{p-4} |f(z)|^p \, dm(z) \\ &= I. \end{aligned}$$

We now consider two cases.

First, suppose that $p \ge 4$. We compute

$$\phi_t^{-1}(z) = \frac{z-t}{(1-t)z}$$

and

$$\omega_t(\phi_t^{-1}(z)) = \frac{1}{(t-1)\phi_t^{-1}(z) + 1} = \frac{z}{t}.$$

Hence

$$I \le \frac{\|f\|_{A^p}^p}{t^{p-2}(1-t)^2}.$$

Next, assume that $2 < p < 4$. Then

$$\begin{aligned} I &= \frac{1}{t^2(1-t)^2} \int_{\phi_t(\mathbb{D})} |\omega_t(\phi_t^{-1}(w))|^{p-4} |f(w)|^p \, dm(w) \\ &= \frac{1}{t^2(1-t)^2} \int_{\phi_t(\mathbb{D})} \left| \frac{w}{t} \right|^{p-4} |f(w)|^p \, dm(w) \\ &= \frac{1}{t^{p-2}(1-t)^2} \int_{\phi_t(\mathbb{D})} |w|^{p-4} |f(w)|^p \, dm(w) \\ &\leq \frac{1}{t^{p-2}(1-t)^2} \int_{\mathbb{D}} |w|^{p-4} |f(w)|^p \, dm(w). \end{aligned}$$

The last integral is well defined near the origin, since

$$\int_{\mathbb{D}} |w|^{p-4} \, dm(w) = \frac{2}{p-2} < \infty, \qquad p > 2.$$

We write

$$\int_{\mathbb{D}} |w|^{p-4} |f(w)|^p \, dm(w) = \int_{|w|<1/2} + \int_{1/2 \leq |w| < 1} |w|^{p-4} |f(w)|^p \, dm(w),$$

and we estimate

$$\begin{aligned} \int_{|w|<1/2} |w|^{p-4} |f(w)|^p \, dm(w) &\leq \int_{|w|<1/2} \frac{|w|^{p-4}}{(1-|w|^2)^2} \, dm(w) \|f\|_{A^p}^p \\ &\leq \frac{1}{(1-(1/2)^2)^2} \int_{|w|<1/2} |w|^{p-4} \, dm(w) \|f\|_{A^p}^p \\ &= \frac{2^{7-p}}{9(p-2)} \|f\|_{A^p}^p, \end{aligned}$$

and

$$\begin{aligned} \int_{1/2 \leq |w| < 1} |w|^{p-4} |f(w)|^p \, dm(w) &\leq \left(\frac{1}{2} \right)^{p-4} \int_{1/2 \leq |w| < 1} |f(w)|^p \, dm(w) \\ &\leq 2^{4-p} \int_{\mathbb{D}} |f(w)|^p \, dm(w) \\ &= 2^{4-p} \|f\|_{A^p}^p. \end{aligned}$$

We conclude that for $2 < p < 4$,

$$I \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p} \right) \frac{t^{2-p}}{(1-t)^2} \|f\|_{A^p}^p,$$

which is the desired result. □

For the proof of the Theorem we need some classical identities for the Beta and Gamma functions; see, for example, [WW]. The Beta function is defined by

$$B(u,v)=\int_0^{+\infty}\frac{x^{u-1}}{(x+1)^{u+v}}\,dx=\int_0^1 s^{u-1}(1-s)^{v-1}\,ds,$$

for u,v such that $\Re(u)>0$, $\Re(v)>0$. The value $B(u,v)$ can be expressed in terms of Gamma function as

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Moreover, the Gamma function satisfies the functional equation

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},$$

for non-integral complex numbers z.

Now we can complete the proof of Theorem 1. Let $f\in A^p$. We have from the continuous version of Minkowski's inequality

$$\begin{aligned}\|\mathcal{H}(f)\|_{A^p}&=\left(\int_{\mathbb{D}}|\mathcal{H}(f)(z)|^p\,dm(z)\right)^{1/p}\\&=\left(\int_{\mathbb{D}}\left|\int_0^1 T_t(f)(z)\,dt\right|^p dm(z)\right)^{1/p}\\&\le\int_0^1\left(\int_{\mathbb{D}}|T_t(f)(z)|^p\,dm(z)\right)^{1/p}dt\\&=\int_0^1\|T_t(f)\|_{A^p}\,dt.\end{aligned}$$

Using Lemma 2 for $p\ge 4$ we conclude

$$\begin{aligned}\|\mathcal{H}(f)\|_{A^p}&\le\int_0^1 t^{2/p-1}(1-t)^{-2/p}\,dt\|f\|_{A^p}\\&=B\left(\frac{2}{p},1-\frac{2}{p}\right)\|f\|_{A^p}\\&=\Gamma\left(\frac{2}{p}\right)\Gamma\left(1-\frac{2}{p}\right)\|f\|_{A^p}\qquad(\Gamma(1)=1),\\&=\frac{\pi}{\sin(2\pi/p)}\|f\|_{A^p}.\end{aligned}$$

Analogously, for $2<p<4$ and $f\in A^p$ we have

$$\begin{aligned}\|\mathcal{H}(f)\|_{A^p}&\le\left(\frac{2^{7-p}}{9(p-2)}+2^{4-p}\right)^{1/p}\int_0^1\frac{t^{2/p-1}}{(1-t)^{2/p}}\,dt\|f\|_{A^p}\\&=\left(\frac{2^{7-p}}{9(p-2)}+2^{4-p}\right)^{1/p}\frac{\pi}{\sin(2\pi/p)}\|f\|_{A^p}.\end{aligned}$$

Now, consider $f \in A^p$, $2 < p < 4$ with $f(0) = 0$, and write $f(z) = zf_0(z)$. The function f_0 is a Bergman space function and satisfies

$$\|f_0\|_{A^p} \leq \left(\frac{p}{2} + 1\right)^{1/p} \|f\|_{A^p}.$$

Indeed, this estimate is a special case of a result on A^p-inner functions [HKZ, Corollary 3.23]. However, it is also possible to give an elementary proof.

LEMMA 3. *For every analytic function f,*

$$\int_{\mathbb{D}} |f(z)|^p \, dm(z) \leq \left(\frac{p}{2} + 1\right) \int_{\mathbb{D}} |zf(z)|^p \, dm(z).$$

Proof. Let $C > 1$. We compute

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^p \, dm(z) - C \int_{\mathbb{D}} |zf(z)|^p \, dm(z) &= \int_0^1 (r - Cr^{p+1}) \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, dr \\ &= \int_0^1 (r - Cr^{p+1}) M_p^p(f, r) \, dr \\ &= D.\end{aligned}$$

The real function $\sigma(r) = r - Cr^{p+1}$ is positive for $r \in (0, C^{-1/p})$ and negative for $r \in (C^{-1/p}, 1)$. In addition, it is well known that $M_p^p(f, r)$ is a nondecreasing function of r [Du, Theorem 1.6]. Hence, in order for D to be ≤ 0, it is enough to choose C such that the following inequality holds:

$$-\int_{C^{-1/p}}^1 (r - Cr^{p+1}) \, dr \geq \int_0^{C^{-1/p}} (r - Cr^{p+1}) \, dr$$

or, equivalently,

$$\int_0^1 (r - Cr^{p+1}) \, dr \leq 0.$$

From the last inequality we get the condition $C \geq p/2 + 1$. $\square$

Now we compute

$$\begin{aligned}\mathcal{H}(f)(z) &= \int_0^1 \frac{1}{(t-1)z + 1} f\left(\frac{t}{(t-1)z + 1}\right) dt \\ &= \int_0^1 \frac{t}{((t-1)z + 1)^2} f_0\left(\frac{t}{(t-1)z + 1}\right) dt \\ &= \int_0^1 \frac{1}{t} \phi_t(z)^2 f_0(\phi_t(z)) \, dt \\ &= \int_0^1 S_t(f_0)(z) \, dt,\end{aligned}$$

where

$$S_t(g)(z) = \frac{1}{t}\phi_t(z)^2 g(\phi_t(z)), \qquad g \in A^p,$$

and $\phi_t(z) = t/((t-1)z+1)$. An easy computation shows that

$$\phi_t(z)^2 = \frac{t}{1-t}\phi_t'(z), \qquad z \in \mathbb{D}, \quad 0 < t < 1.$$

It follows that

$$\begin{aligned}
\|S_t(g)\|_{A^p}^p &= \frac{1}{t^p}\int_{\mathbb{D}} |\phi_t(z)|^{2p} |g(\phi_t(z))|^p \, dm(z) \\
&= \frac{1}{t^p}\int_{\mathbb{D}} |\phi_t(z)|^{2p-4} |\phi_t(z)|^4 |g(\phi_t(z))|^p \, dm(z) \\
&\leq \frac{t^{2-p}}{(1-t)^2}\int_{\mathbb{D}} |\phi_t(z)|^{2p-4} |g(\phi_t(z))|^p |\phi_t'(z)|^2 \, dm(z) \\
&= \frac{t^{2-p}}{(1-t)^2}\int_{\phi_t(\mathbb{D})} |w|^{2p-4} |g(w)|^p \, dm(w) \\
&\leq \frac{t^{2-p}}{(1-t)^2}\int_{\phi_t(\mathbb{D})} |g(w)|^p \, dm(w) \\
&\leq \frac{t^{2-p}}{(1-t)^2}\int_{\mathbb{D}} |g(w)|^p \, dm(w) \\
&= \frac{t^{2-p}}{(1-t)^2}\|g\|_{A^p}^p.
\end{aligned}$$

Hence

$$\|S_t(g)\|_{A^p} \leq \frac{t^{2/p-1}}{(1-t)^{2/p}}\|g\|_{A^p}.$$

For the norm of $\mathcal{H}$ we compute

$$\begin{aligned}
\|\mathcal{H}(f)\|_{A^p} &\leq \left(\int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, dt\right)\|f_0\|_{A^p} \\
&= \frac{\pi}{\sin(2\pi/p)}\|f_0\|_{A^p} \\
&\leq \left(\frac{p}{2}+1\right)^{1/p} \frac{\pi}{\sin(2\pi/p)}\|f\|_{A^p},
\end{aligned}$$

which is the desired result. The proof of Theorem 1 is complete.

4. $\mathcal{H}$ is not bounded on A^2

Let $\mathcal{D}$ be the usual Dirichlet space of analytic functions on the unit disc with square summable derivative. The following result is well known.

LEMMA 4. *Each bounded linear functional on the Bergman space A^2 can be associated to a function $g \in \mathcal{D}$ (by the pairing $\langle f, g\rangle = \sum_{n=0}^{\infty} a_n b_n$) and the association is an isometric isomorphism of the spaces.*

This yields the following result.

PROPOSITION 5. *There is no bounded linear operator $T : A^2 \to A^2$ satisfying*

$$T(\xi_n)(0) = \frac{1}{n+1}, \qquad n = 0, 1, 2, \ldots,$$

where $\xi_n(z) = z^n$.

Proof. Suppose, to the contrary, that there exists such an operator T. Using the pairing that defines an isometric isomorphism between $(A^2)^*$ and $\mathcal{D}$, we find that the adjoint operator $T^* : \mathcal{D} \to \mathcal{D}$ is bounded and satisfies

$$\langle T(f), g\rangle = \langle f, T^*(g)\rangle, \tag{7}$$

for every $f \in A^2$, $g \in \mathcal{D}$. We choose $g \equiv 1$ and write

$$T^*(1)(z) = \sum_{n=0}^{\infty} c_n z^n,$$

as the Taylor series of $T^*(1) \in \mathcal{D}$. Using (7) for $f = \xi_n$ and $g \equiv 1$ we have

$$\begin{aligned}\frac{1}{n+1} &= T(\xi_n)(0)\\ &= \langle T(\xi_n), 1\rangle\\ &= \langle \xi_n, T^*(1)\rangle\\ &= c_n,\end{aligned}$$

for every $n = 0, 1, 2, \ldots$. Hence

$$T^*(1)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n,$$

but this function is not in $\mathcal{D}$. □

Now we consider the integral

$$\mathcal{H}(f) = \int_0^1 f(t) \frac{1}{1-tz}\, dt.$$

This integral is well defined for polynomials, and polynomials are dense in A^2. It is not known if the last integral is well defined for all $f \in A^2$. In any case, from Proposition 5 we obtain:

COROLLARY 6. *$\mathcal{H}$ is not bounded on A^2.*

Proof. We apply Proposition 5 and note that

$$\mathcal{H}(\xi_n)(0) = \frac{1}{n+1}, \qquad n = 0, 1, 2, \ldots \qquad \square$$

Final remarks. We do not know if the inequalities in the theorem are sharp. In Hardy spaces H^p, $1 < p < \infty$, using the Hollenbeck-Verbitsky Theorem [HV], we can verify that the upper bound (2) for the norm of $\mathcal{H}$ is equal to the fraction $\pi/(\sin(\pi/p))$, without any additional constants. There is no evidence that the same is true for Bergman spaces.

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Taskou Papageorgiou 8, 54631 Thessaloniki, Greece
E-mail address: epdiamantopoulos@hotmail.com