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TRANSCENDENTAL MEROMORPHIC FUNCTIONS WITH THREE SINGULAR VALUES

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ABSTRACT. Every transcendental meromorphic function f in the plane which has only three critical values satisfies

$$\liminf_{r\to\infty} \frac{T(r,f)}{\log^2 r} \geq \frac{\sqrt{3}}{2\pi},$$

and this estimate is best possible.

A *singular value* of a meromorphic function f in the plane $\mathbf{C}$ is, by definition, a critical value or an asymptotic value. If we denote the closure of the set of singular values by S, then

$$f : \mathbf{C}\backslash f^{-1}(S) \to \overline{\mathbf{C}}\backslash S$$

is a covering map. Meromorphic functions with finitely many singular values play an important role in value distribution theory (see, for example, [8], [15], [16], and Jim Langley's papers on the distribution of values of derivatives), as well as in holomorphic dynamics [4], [6].

In this paper, "meromorphic function" will always mean a *transcendental meromorphic function in the plane*, unless some other region is specified. (For algebraic functions with three critical values see [3], [9].)

Langley [11], [12] discovered that there exists a relation between the number of singular values of a meromorphic function f and the growth of the Nevanlinna characteristic $T(r,f)$. In [12] he proved that all meromorphic functions f with finitely many singular values satisfy

$$\liminf_{r\to\infty} \frac{T(r,f)}{\log^2 r} > 0.$$

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On the other hand, he showed in [13] that for every $\epsilon > 0$ there exists a meromorphic function f with four singular values such that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\log^2 r} < \epsilon.$$

Concerning meromorphic functions with three singular values, Langley proved in [13] that they satisfy

$$\liminf_{r\to\infty} \frac{T(r,f)}{\log^2 r} \geq c, \tag{1}$$

where c is an absolute constant.

In this paper, the precise value of this constant is found. I thank Walter Bergweiler who brought [13] to my attention and suggested the extremal problem which the following theorem solves.

Theorem. *Let f be a meromorphic function with at most three singular values. Then* (1) *holds with $c = \sqrt{3}/(2\pi)$, and there exists a meromorphic function f_0 with three singular values, such that*

$$T(r, f_0)/\log^2 r \to \sqrt{3}/(2\pi) \quad \textit{as} \quad r \to \infty. \tag{2}$$

Remarks.

1. *If f has finitely many singular values and at least one of them is an asymptotic value, then*

$$\liminf_{r\to\infty} \frac{T(r,f)}{\sqrt{r}} > 0; \tag{3}$$

in particular, this holds for all entire functions with finitely many singular values. We sketch a proof of this, different from that mentioned in [13]. If a is an asymptotic value, and there are no other singular values in an ϵ-neighborhood of a, then one of the components of the set $\{z : |f(z) - a| < \epsilon\}$ is an unbounded region D whose boundary consists of one simple curve, and $f(z) \neq a$ in this region. Applying the standard growth estimates to the harmonic function $\log|f(z) - a|^{-1}$ in D we conclude that (3) holds.

2. *Let f be a meromorphic function with at most three critical values. Then the conclusion of the Theorem holds.* Indeed, we can assume that f has finite lower order (otherwise there is nothing to prove). For such functions with finitely many critical values, the set of asymptotic values is also finite. This was proved in [5] for functions of finite order and extended in [10] to functions of finite lower order. If there are asymptotic values, we apply Remark 1, if there are none, we apply the Theorem.

A similar improvement can be made in the results of Langley mentioned above.

3. All meromorphic functions with two singular values are of the form $L \circ \exp$, where L is a fractional-linear transformation.

4. Let μ be a probability measure on the Riemann sphere $\overline{\mathbf{C}}$, and $\nu = f^*\mu$ the pull-back of μ. We set

$$A_\mu(r) = \nu\left(\{z : |z| \le r\}\right), \quad r > 0.$$

If μ is the normalized spherical area, then $A_\mu(r) \equiv A(r)$ is the average number of sheets of the map $f : \{z : |z| \le r\} \to \overline{\mathbf{C}}$. The Nevanlinna characteristic satisfies

$$T(r, f) = \int_e^r \frac{A(t)}{t} dt + O(\log r), \quad r \to \infty.$$

For an arbitrary probability measure μ on the sphere we have

$$\int_e^r \frac{A_\mu(t)}{t} dt \le T(r, f) + O(\log r),$$

which is a consequence of the First Fundamental Theorem of Nevanlinna [14, VI, §4]. Thus, to prove our theorem, it is sufficient to show that

$$A_\mu(t) \ge \frac{\sqrt{3}}{\pi} \log t + O(1), \quad t \to \infty, \tag{4}$$

for some probability measure μ.

Proof of the Theorem. Let F be a connected oriented surface of finite topological type, possibly with boundary. A *triangular net* on F is a locally finite covering of F by closed sets T called *triangles*, such that:

(i) Each triangle T is a closed Jordan region (homeomorphic to a closed unit disc) with three marked distinct boundary points called *vertices.* A closed boundary arc between two adjacent vertices is called an *edge* of T.

(ii) The intersection of two triangles is either empty, or a union of common edges and common vertices.

(iii) Triangles are divided into two classes, white and black, so that any two triangles with a common edge are of different colors.

(iv) Vertices are labeled by the letters A, B and C, so that for each triangle all three labels are present on its boundary. Furthermore, the cyclic order on the oriented boundary of a white triangle is (A, B, C), and opposite on the oriented boundary of a black triangle.

Suppose that a covering of F by triangles satisfying (i) and (ii) is given, and there exists a labeling of vertices satisfying (iv). Then such a labeling is uniquely defined by a choice of labels on the three vertices of one triangle. The colors of triangles are evidently determined by the labeling of vertices.

Let f be a meromorphic function satisfying the assumptions of the Theorem. In view of Remarks 1–3, we assume without loss of generality that f has no asymptotic values and exactly three critical values, A, B and C. Composing f with a fractional linear transformation we may assume that A, B and C are real, and $A < B < C$.

Consider the triangular net N_0 on the Riemann sphere, which consists of two triangles, the closed upper half-plane (white) and the closed lower half-plane (black), and the vertices A, B and C.

We construct the f-preimage of this net, which will be called N. The vertices of N are preimages of the vertices of N_0, and they are labeled according to their images. The triangles of N are the closures of the components of the preimages of the open upper and lower half-planes, and the edges of N are defined in the evident way. Each triangle in N is assigned the same color as its image in N_0.

Our assumptions about f imply that N is a triangular net in the plane $\mathbf{C}$.

Choose one triangle $T_0 \in N$ and remove its interior $\operatorname{int} T_0$ from the plane. We obtain a bordered surface $D = \mathbf{C} \backslash \operatorname{int} T_0$ which is homeomorphic to a closed semi-infinite cylinder.

Let $G \subset D$ be a compact closed region homeomorphic to a closed ring, which separates ∂T_0 from ∞. We are going to estimate from below the *extremal length*[1] λ of the family of all closed non-contractible curves in G. For this purpose we construct a conformal metric ρ in D. In this metric, each triangle of the net $N \backslash T_0$ will be isometric to a Euclidean equilateral triangle Δ with sidelength 1.

First we define a flat conformal metric ρ_0 on $\overline{\mathbf{C}} \backslash \{A, B, C\}$. Let g be the conformal map of Δ onto the upper half-plane sending the vertices of Δ to A, B, C. (An explicit expression for g will be given at the end of the paper.) The metric ρ_0 is defined in the upper half-plane by the length element $ds = |(g^{-1})'(z)|\,|dz|$, so that g becomes an isometry from Δ with the Euclidean metric to the upper half-plane with the metric ρ_0. Using the Symmetry Principle, we extend ρ_0 to $\overline{\mathbf{C}} \backslash \{A, B, C\}$.

The Riemann sphere with the metric ρ_0 can be visualized as a "two-sided triangle Δ". The area of the sphere with respect to ρ_0 is $\sqrt{3}/2$.

Now we define $\rho = f^* \rho_0$, the pull-back of ρ_0 via f.

The metrics ρ and ρ_0 have isolated singularities at the vertices, but this does not cause any problems.

Now we estimate the ρ-length of non-contractible curves in D from below.

Lemma. *Let D' be a surface homeomorphic to $\{1 \le |z| \le 2\}$, equipped with a triangular net and an intrinsic metric*[2] *such that every triangle of the net is isometric to Δ. Then the length of every closed non-contractible curve in D' is at least $\sqrt{3}$.*

The idea of the following proof, which is simpler than the original one, was suggested by Mario Bonk.

[1] See, for example, [2] for a definition and simplest properties.

[2] A metric is called intrinsic if the distance between any two points equals the infimum of lengths of curves connecting these points.

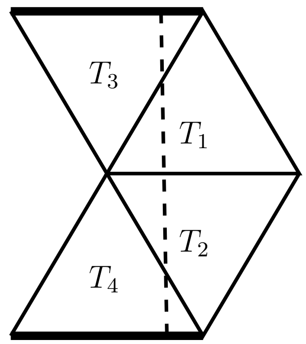

FIGURE 1. An extremal configuration. Bold segments are identified. An extremal curve is the broken line.

Proof. Let γ be a shortest non-contractible curve in D'; evidently such a curve exists. Then γ is homeomorphic to a circle, because otherwise we could remove extra loops and shorten γ.

Suppose first that γ passes through a vertex v of the net, that is $\gamma(t_0) = v$. Let F be the interior of the union of all triangles of the net that have v as a common vertex. Then F is a simply connected region, and to be non-contractible, our curve has to pass through a point $w \in \partial_D F = \partial F \cap \mathrm{int} D$. Then two arcs of γ from v to w have lengths at least $\sqrt{3}/2$ each, which proves the Lemma in this case.

From now on we assume that γ does not pass through the vertices.

As our metric ρ is flat away from the vertices, every point $w \in D' \backslash \{\text{vertices}\}$ has a neighborhood W which is isometric to a region V in the plane with the standard (intrinsic) metric. This isometry $\phi : W \to V$ is called the "developing map". It has an analytic continuation to every simply connected region in $D' \backslash \{\text{vertices}\}$ that contains w, and also an analytic continuation along any curve in D' which does not pass through the vertices. The image of any arc of our shortest curve γ under the developing map is an interval of a straight line in the plane.

Our curve γ evidently intersects some edge. Let v be a point of intersection with an edge e, and choose a parametrization $\gamma : [0, 1] \to D'$ such that $\gamma(0) = \gamma(1) = v$. Let ϕ be a germ of the developing map at v, and ϕ_1 the result of its analytic continuation along γ. Let $T_1, \ldots, T_n$ be the sequence of triangles in D' visited by γ, enumerated in the natural order. Then the branches of the developing map are defined in T_j by analytic continuation of the germ ϕ along γ. Let $\Delta_1, \ldots, \Delta_n$ be the images of the triangles $T_1, \ldots, T_n$ under these branches.

Reflections in the sides of Δ_1 generate a group Γ of isometries of the plane. It is clear that $\Delta_1, \ldots, \Delta_n$ can be obtained by applying some elements of this group to Δ_1. The full orbit of Δ_1 under Γ forms the hexagonal tiling of the plane by equilateral triangles of sidelength 1. We label the vertices of Δ_1 similarly to the corresponding labels of T_1. This uniquely defines the labeling of vertices of the hexagonal tiling of the plane, thus making it into a triangular net in $\mathbf{C}$. The branches of the developing map $T_j \mapsto \Delta_j$ preserve the labels of vertices.

The edges of our nets have orientations induced by the cyclic order on the labels of vertices. Let us consider the edges $e^* = \phi(e)$ and $e_1^* = \phi_1(e)$. We claim that the oriented edge e_1^* can be obtained from the oriented edge e^* by a translation from Γ which is not the identity map.

Let α be the angle, counted anti-clockwise from the positive direction of e to the tangent vector $\gamma'(0)$ at v. The image of γ under the analytic continuation of the developing map is a (non-degenerate) straight line segment which forms the same angle α with e^* and e_1^*. We conclude that e^* and e_1^* have the same direction but do not coincide. As e^* and e_1^* are edges of the hexagonal triangular net, we easily conclude that e_1^* can be obtained from e^* by a translation in Γ. This proves our claim.

Now, the shortest translation in Γ has magnitude $\sqrt{3}$, and this completes the proof of the Lemma. $\square$

As a corollary we obtain that the extremal length λ of the set of non-contractible curves in any compact ring $G \subset D$ separating ∂T_0 from ∞ in D satisfies

$$\lambda \geq \frac{3}{\operatorname{area}(G)}, \tag{5}$$

where the area corresponds to the metric ρ.

To complete the proof, we consider the set

$$G(t) = \{z : |z| \leq t\} \backslash \operatorname{int} T_0.$$

This set is homeomorphic to a ring if t is large enough. The extremal length $1/\lambda$ of the family of curves connecting the boundary components of the ring $G(t)$ is $(2\pi)^{-1} \log t + O(1)$ as $t \to \infty$. According to (5), the area of this ring with respect to the metric ρ is at least

$$\frac{3}{\lambda} = \frac{3}{2\pi} \log t + O(1), \quad t \to \infty.$$

The ρ_0-area of the Riemann sphere is $\sqrt{3}/2$. Taking μ to be the ρ_0-area divided by $\sqrt{3}/2$, we obtain $A_\mu(t) \geq (\sqrt{3}/\pi) \log t + O(1)$, which is (4). $\square$

Example. Consider the equilateral triangle $\Delta \subset \mathbf{C}$ with vertices $0, i$ and $(\sqrt{3}+i)/2$. Let g be a conformal map of this triangle onto the right half-plane, sending the vertices to $\infty, ia, -ia$, where $a > 0$. By reflection, g extends to a meromorphic function in $\mathbf{C}$ with no asymptotic values and three critical values, $ia, -ia$ and ∞. All preimages of the critical values are critical points of order 3.

This function g is doubly periodic and its shortest period is $\sqrt{3}$. It can be expressed in terms of the Weierstrass function of an equiharmonic lattice; see [1], [7]. If we choose

$$a = k^3, \quad \text{where} \quad k = \frac{\Gamma^3(1/3)}{2\pi\sqrt{3}},$$

then $g = \wp'$ where $\wp$ is the Weierstrass function with periods $\sqrt{3}$ and $\sqrt{3}e^{2\pi i/3}$. The Riemannian metric ρ_0 in the proof of the Theorem corresponds to the length element $|(g^{-1})'(w)|\,|dw|$.

The function

$$f_1(z) = g\left(\frac{\sqrt{3}}{2\pi i}\log z\right)$$

is evidently meromorphic in $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ and has three critical values. Simple calculation shows that it satisfies

$$A(r, f_1) = \frac{\sqrt{3}}{\pi}\log r + O(1), \quad r \to \infty.$$

Now we modify f_1 to obtain a function meromorphic in $\mathbf{C}$. Consider the integral

$$I(z) = \int_0^z \frac{d\zeta}{\zeta^{2/3}(1-\zeta)^{1/3}}.$$

Using the positive value of the cubic root and integrating over $[0, 1]$, we obtain

$$I(1) = 2\pi/\sqrt{3}.$$

Using this, one can easily verify that

$$f_0(z) = g\left(\frac{\sqrt{3}}{2\pi i}I(z)\right)$$

is meromorphic in the plane. This function f_0 has three critical values, no asymptotic values, and satisfies (2), as a branch of $I(z)$ near infinity has the same asymptotic behavior as the logarithm.

We also sketch a purely geometric construction of f_0, based on the Uniformization Theorem.

Consider the closed region K_0 in $\mathbf{C}$ made of four equilateral triangles as in Figure 1, and put

$$K = \bigcup_{j=1}^{\infty}(K_0 + j),$$

the union of translates of K_0 by positive integers. Then K is a closed half-strip in the plane, and we define a Riemann surface P by identifying pairs of points with equal x-coordinates on the two horizontal boundary rays of

K. The surface P is homeomorphic to a semi-infinite closed cylinder, and its boundary consists of two edges. We "patch the hole" by adding a 2-gon consisting of two triangles with two common edges.

Thus we obtain a surface P' homeomorphic to the plane and covered by triangles. It is easy to see that one can label the vertices of triangles in P' to obtain a triangular net. So we obtain a triangular net N in the plane.

Every triangular net in the plane defines a ramified covering $h : \mathbf{C} \to \overline{\mathbf{C}}$ in the following way. Choose a triangular net N_0 of $\overline{\mathbf{C}}$ with two triangles and three vertices. First define h on the vertices, by sending each vertex of N to the similarly labeled vertex of N_0. Then extend h to the edges, so that the restriction to each edge is a homeomorphism, and the extended map is continuous on the 1-skeleton of N. Finally, extend h to the interiors of triangles, so that the extended map is a homeomorphism on each triangle.

By the Uniformization Theorem, there exists a homeomorphism $\psi : U \to \mathbf{C}$, where U is either a disc or the plane, such that $h \circ \phi$ is a meromorphic function.

To show that $U = \mathbf{C}$ and to prove (2) one can make extremal length estimates using the Euclidean metric on P' such that all triangles of the net are equilateral with side length 1. We omit the details which are similar to those in the proof of the Theorem.

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