Illinois Journal of Mathematics
Volume 48, Number 2, Summer 2004, Pages 491–503
S 0019-2082

EXPLICIT FORMULAS FOR DIRICHLET AND HECKE L-FUNCTIONS

XIAN-JIN LI

ABSTRACT. In 1997, the author proved that the Riemann hypothesis holds if and only if $\lambda_n = \sum[1-(1-1/\rho)^n] > 0$ for all positive integers n, where the sum is over all complex zeros of the Riemann zeta function. In 1999, E. Bombieri and J. Lagarias generalized this result and obtained a remarkable general theorem about the location of zeros. They also gave an arithmetic interpretation for the numbers λ_n. In this note, the author extends Bombieri and Lagarias' arithmetic formula to Dirichlet L-functions and to L-series of elliptic curves over rational numbers.

1. Introduction

Let K be a finite field with q elements, and let E be an elliptic curve over K. In the 1930s, H. Hasse proved the inequality

$$|\#E(K) - q - 1| \le 2\sqrt{q},$$

where $\#E(K)$ is the number of K-rational points on E; see [12].

Let $a = 1 + q - \#E(K)$ and

$$L_E(s) = 1 - az + qz^2,$$

where $z = q^{-s}$. By Hasse's inequality we have

$$L_E(s) = (1 - \alpha z)(1 - \beta z)$$

with $|\alpha| = |\beta| = \sqrt{q}$. Hence

$$-\frac{d}{dz}\log L_E(s) = \sum_{n=0}^{\infty} \lambda_E(n+1) z^n,$$

where $\lambda_E(n) = \alpha^n + \beta^n$. It is clear that

$$|\lambda_E(n)| \le 2\sqrt{q^n}$$

Received April 24, 2003; received in final form October 17, 2003.
2000 *Mathematics Subject Classification.* Primary 11M26, 11M36.
Research supported by National Security Agency.

for $n = 1, 2, \ldots$. This estimate implies that all zeros of $L_E(s)$ lie on the line $\Re s = 1/2$.

Let

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function, and let $\lambda_\zeta(n)$, $n = 1, 2, \ldots$, be a sequence of numbers defined by

$$\frac{d}{dz} \ln \xi \left(\frac{1}{1-z} \right) = \sum_{n=0}^{\infty} \lambda_\zeta(n+1) z^n.$$

In 1997, the author obtained the following criterion for the Riemann hypothesis.

Theorem 1 ([9]). *All complex zeros of $\zeta(s)$ lie on the line $\Re s = 1/2$ if and only if $\lambda_\zeta(n) > 0$ for $n = 1, 2, \ldots$.*

In 1952, A. Weil [13] proved a famous criterion for the validity of the Riemann hypotheses for number fields. The following is Bombieri's refinement of Weil's criterion.

Bombieri's refinement ([2]). *All complex zeros of $\zeta(s)$ lie on the line $\Re s = 1/2$ if and only if*

$$\sum_{\rho} \widehat{f}(\rho)\widehat{\overline{f}}(1-\rho) \geq 0$$

for every complex-valued $f \in C_0^\infty(0, \infty)$ which is not identically 0, where the Mellin transform of f is given by

$$\widehat{f}(s) = \int_0^\infty f(x) x^{s-1} dx.$$

Let $f, g \in C_0^\infty(0, \infty)$. The multiplicative convolution of f and g is given by

$$(f * g)(x) = \int_0^\infty f(x/y) g(y) \frac{dy}{y}.$$

If $\widetilde{f}(x) = x^{-1} f(x^{-1})$, the Mellin transform of $f * \widetilde{\overline{f}}$ is $\widehat{f}(s)\widehat{\overline{f}}(1-s)$. Let $g_n(x)$ be the inverse Mellin transform of $1 - (1 - 1/s)^n$ for $n = 1, 2, \ldots$. E. Bombieri and J. Lagarias observed in [3] that

$$\left[1 - \left(1 - \frac{1}{s}\right)^n\right] + \left[1 - \left(1 - \frac{1}{1-s}\right)^n\right] = \left[1 - \left(1 - \frac{1}{s}\right)^n\right]\left[1 - \left(1 - \frac{1}{1-s}\right)^n\right],$$

and that

$$g_n(x) + \widetilde{g}_n(x) = (g_n * \widetilde{g}_n)(x).$$

Hence, the positivity in the author's criterion has the same meaning as that in Weil's criterion.

In 1999, Bombieri and Lagarias obtained the following remarkable theorem.

THEOREM 2 (Bombieri-Lagarias [3]). *Let $\mathcal{R}$ be a set of complex numbers ρ whose elements have positive integral multiplicities assigned to them, such that $1 \notin \mathcal{R}$ and*

$$\sum_{\rho} \frac{1+|\Re\rho|}{(1+|\rho|)^2} < \infty.$$

Then the following conditions are equivalent:

(1) $\Re\rho \leq 1/2$ *for every* ρ *in* $\mathcal{R}$*;*

(2) $\sum_{\rho} \Re\left[1-\left(1-\frac{1}{\rho}\right)^{-n}\right] \geq 0$ *for* $n = 1, 2, \ldots$.

An arithmetic interpretation for the numbers $\lambda_\zeta(n)$ was given in [3].

THEOREM 3 (Bombieri-Lagarias [3]). *We have*

$$\lambda_\zeta(n) = \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^j}{j!} \lim_{N\to\infty} \left\{ j \sum_{k=1}^{N} \frac{\Lambda(k)}{k} (\ln k)^{j-1} - (\ln N)^j \right\}$$
$$+ 1 - \frac{n}{2}(\ln 4\pi + \gamma) + \sum_{j=2}^{n} \binom{n}{j} (-1)^j (1-2^{-j})\zeta(j)$$

for $n = 1, 2, \ldots$, *where* $\gamma = 0.5772\ldots$ *is Euler's constant and where* $\Lambda(k) = \ln p$ *when* k *is a power of a prime* p *and* $\Lambda(k) = 0$ *otherwise.*

Let χ be a primitive Dirichlet character of modulus $r > 1$, and $L(s,\chi)$ the Dirichlet L-function of character χ. If

$$\xi(s,\chi) = (\pi/r)^{-\frac{1}{2}(s+a)} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

where

$$a = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1, \end{cases}$$

then $\xi(s,\chi)$ is an entire function of order one and satisfies the functional equation

$$\xi(s,\chi) = \epsilon_\chi \xi(1-s,\bar{\chi}),$$

where ϵ_χ is a constant of absolute value one. By Theorem 2 of [1] we have

$$\xi(s,\chi) = \xi(0,\chi) \prod_{\rho} (1 - s/\rho),$$

where the product is over all the zeros of $\xi(s,\chi)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$.

For $n = 1, 2, \dots$ let

$$\lambda_\chi(n) = \sum_\rho \left[1 - (1 - 1/\rho)^n\right],$$

where the sum on ρ runs over all zeros of $\xi(s, \chi)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$. First, we give an arithmetic interpretation for the numbers $\lambda_\chi(n)$.

THEOREM 4. *Let χ be a primitive Dirichlet character of modulus $r > 1$. Then we have*

$$\lambda_\chi(n) = -\sum_{j=1}^n \binom{n}{j} \frac{(-1)^{j-1}}{(j-1)!} \sum_{k=1}^\infty \frac{\Lambda(k)}{k} \bar{\chi}(k)(\ln k)^{j-1} + \frac{n}{2}\left(\ln\frac{r}{\pi} - \gamma\right) + \tau_\chi(n),$$

where

$$\tau_\chi(n) = \begin{cases} \sum_{j=2}^n \binom{n}{j}(-1)^j\left(1 - \frac{1}{2^j}\right)\zeta(j) - \frac{n}{2}\sum_{l=1}^\infty \frac{1}{l(2l-1)}, & \text{if } \chi(-1) = 1, \\ \sum_{j=2}^n \binom{n}{j}(-1)^j 2^{-j}\zeta(j), & \text{if } \chi(-1) = -1. \end{cases}$$

Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. For each prime p, we denote by $\tilde{E}_p$ the reduction of E at p. Let

$$a_p = \begin{cases} p + 1 - \#\tilde{E}_p(\mathbb{F}_p), & \text{if } E \text{ has good reduction at } p, \\ 1, & \text{if } E \text{ has split multiplicative reduction at } p, \\ -1, & \text{if } E \text{ has non-split multiplicative reduction at } p, \\ 0, & \text{if } E \text{ has additive reduction at } p. \end{cases}$$

We define the L-series associated to E by the Euler product

$$L_E(s) = \prod_{p\nmid N}(1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{p|N}(1 - a_p p^{-s})^{-1}$$

for $\Re s > 3/2$; see [12].

Let k and N be positive integers, and let χ be a multiplicative character of modulus N with $\chi(1) = 1$ and $\chi(-1) = (-1)^k$. Let Γ be the Hecke congruence subgroup $\Gamma_0(N)$ of level N. We denote by $S_0(\Gamma, k, \chi)$ the space of all cusp forms of weight k and character χ for Γ. That is, f belongs to $S_0(\Gamma, k, \chi)$ if and only if f is holomorphic in the upper half-plane $\mathbb{H}$, satisfies

$$f\left(\frac{az + b}{cz + d}\right) = \chi(d)(cz + d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, satisfies the usual regularity conditions at the cusps of Γ, and vanishes at each cusp of Γ.

The Hecke operators T_n are defined by

$$(T_n f)(z) = \frac{1}{n}\sum_{ad=n} \chi(a)a^k \sum_{0\le b<d} f\left(\frac{az + b}{d}\right)$$

for any function f on $\mathbb{H}$. A function f in $S_0(\Gamma, k, \chi)$ is called a Hecke eigenform if

$$T_n f = \lambda(n) f$$

for all positive integers n with $(n, N) = 1$. The Fricke involution W is defined by

$$(Wf)(z) = N^{-k/2} z^{-k} F(-1/Nz),$$

and the complex conjugation operator K is defined by

$$(Kf)(z) = \bar{f}(-\bar{z}).$$

Set $\bar{W} = KW$. Then f is a newform if it is an eigenfunction of $\bar{W}$ and of all the Hecke operators T_n.

Let f be a newform in $S_0(\Gamma, k, \chi)$ normalized so that its Fourier coefficient is 1. Then it has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda(n) e^{2\pi i n z}$$

with the Fourier coefficients equal to the eigenvalues of Hecke operators. Since f is an eigenfunction of the involution $\bar{W}$, we can assume that

$$\bar{W} f = \eta f$$

for a constant η. Let

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$

for $\Re s > (k+1)/2$. This L-series has Euler product

$$L_f(s) = \prod_p (1 - \lambda(p) p^{-s} + \chi(p) p^{k-1-2s})^{-1}$$

and satisfies the functional identity

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s) = i^k \bar{\eta} \left(\frac{\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s) \bar{L}_f(k-\bar{s}).$$

When χ is primitive, we have $\eta = \tau(\bar{\chi}) \lambda(N) N^{-k/2}$ with $\tau(\chi)$ being the Gauss sum for χ. For the theory of Hecke L-functions see [8].

We denote by $S_2(N)$ the space of cusp forms of weight 2 with the principal character for $\Gamma_0(N)$.

Shimura-Taniyama Conjecture ([4], [14]). *There is a newform $f \in S_2(N)$ such that $L_f(s) = L_E(s)$.*

The Shimura-Taniyama conjecture has now been proved ([4], [14]).

If

$$\xi_E(s) = c_E N^{s/2} (2\pi)^{-s} \Gamma\left(\frac{1}{2} + s\right) L_E\left(\frac{1}{2} + s\right),$$

where c_E is a constant chosen so that $\xi_E(1) = 1$, then $\xi_E(s)$ is an entire function and satisfies

$$\xi_E(s) = w\xi_E(1-s),$$

where $w = (-1)^r$ with r being the vanishing order of $\xi_E(s)$ at $s = 1/2$.

Let

$$\lambda_E(n) = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]$$

for $n = 1, 2, \ldots$, where the sum is over all the zeros of $\xi_E(s)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$. Since $\xi_E(s)$ is an entire function of order one, the conditions of Bombieri-Lagarias' theorem are satisfied, and hence all the zeros of $\xi_E(s)$ lie on the line $\Re s = 1/2$ if and only if

$$\lambda_E(n) > 0$$

for $n = 1, 2, \ldots$.

For each prime number p, we let α_p and β_p be the roots of $T^2 - a_pT + p$. Let $b(p^k) = a_p^k$ if $p|N$ and $b(p^k) = \alpha_p^k + \beta_p^k$ if $(p, N) = 1$. Next, we give an arithmetic interpretation for the numbers $\lambda_E(n)$.

Theorem 5. *We have*

$$\lambda_E(n) = n\left(\ln\frac{\sqrt{N}}{2\pi} - \gamma\right) - \sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{(j-1)!}\sum_{k=1}^{\infty}\frac{\Lambda(k)}{k^{3/2}}b(k)(\ln k)^{j-1}$$
$$+ n\left(-\frac{2}{3} + \sum_{l=1}^{\infty}\frac{3}{l(2l+3)}\right) + \sum_{j=2}^{n}\binom{n}{j}(-1)^j\sum_{l=1}^{\infty}\frac{1}{(l+1/2)^j}.$$

The author wishes to thank Brian Conrey for his help during the preparation of the manuscript, and William Duke for his valuable suggestions. He also wants to thank the referee for carefully reading the manuscript and for his/her valuable suggestions.

2. Proof of Theorem 4

Weil's explicit formula for $L(s,\chi)$ ([1], [13]). *Let $F(x)$ be a function defined on $\mathbb{R}$ such that*

$$2F(x) = F(x+0) + F(x-0)$$

for all $x \in \mathbb{R}$, such that $F(x)\exp((b+1/2)|x|)$ is of bounded variation on $\mathbb{R}$ for a constant $b > 0$, and such that

$$F(x) + F(-x) = 2F(0) + O(|x|^\ell)$$

as $x \to 0$ for a constant $\ell > 0$. Then

$$\sum_\rho \Phi(\rho) = F(0)\left(\ln\frac{r}{\pi} - \gamma\right) - \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\chi(n)F(\ln n) + \bar\chi(n)F(-\ln n)\right) + \int_{-\infty}^{\infty}\left(F(x)e^{(3/2-a)|x|} - F(0)\right)\frac{dx}{1-e^{2|x|}},$$

where the sum on ρ runs over all zeros of $\xi(s,\chi)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$, and

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)e^{(s-1/2)x}dx. \tag{2.1}$$

A multiset is a set whose elements have positive integral multiplicities assigned to them [3].

Lemma 2.1 ([3, (2,4)]). *Formally, if*

$$f(z) = \prod_\rho \left(1 - \frac{z}{\rho}\right)$$

and

$$\lambda_n = \sum_\rho \left[1 - (1-1/\rho)^n\right],$$

then we have

$$\frac{d}{dz}\ln f\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty}\lambda_{-n-1}z^n.$$

Lemma 2.2 ([3, Corollary 1]). *Let $\mathcal{R}$ be a multiset of complex numbers such that*

(1) $0, 1 \notin \mathcal{R}$;
(2) *if $\rho \in \mathcal{R}$, then $1-\rho$ and $\bar\rho$ are in $\mathcal{R}$ and have the same multiplicity as ρ;*
(3) $\sum_\rho (1+|\Re\rho|)/(1+|\rho|)^2 < \infty$.

Then $\Re\rho = 1/2$ for all $\rho \in \mathcal{R}$ if, and only if,

$$\lambda_n = \sum_{\rho\in\mathcal{R}}\left[1-(1-1/\rho)^n\right] \geq 0$$

for $n = 1, 2, 3, \ldots$.

Lemma 2.3 ([3, Lemma 2]). *For $n = 1, 2, \ldots$, let*

$$F_n(x) = \begin{cases} e^{x/2}\sum_{j=1}^{n}\binom{n}{j}\frac{x^{j-1}}{(j-1)!}, & \text{if } -\infty < x < 0,\\ n/2, & \text{if } x = 0,\\ 0, & \text{if } 0 < x.\end{cases}$$

Then

$$\Phi_n(s) = 1 - \left(1 - \frac{1}{s}\right)^n,$$

where Φ_n *is related to* F_n *as in* (2.1).

Proof of Theorem 4. For a sufficiently large positive number X that is not an integer let

$$F_{n,X}(x) = \begin{cases} F_n(x), & \text{if } -\ln X < x < \infty, \\ \frac{1}{2}F_n(-\ln X), & \text{if } x = -\ln X, \\ 0, & \text{if } -\infty < x < -\ln X. \end{cases}$$

Then $F_{n,X}(x)$ satisfies all conditions of Weil's explicit formula for $L(s,\chi)$. Let

$$\Phi_{n,X}(s) = \int_{-\infty}^{\infty} F_{n,X}(x) e^{(s-1/2)x} dx.$$

By using the Weil explicit formula, we obtain that

$$\begin{aligned}\sum_\rho \Phi_{n,X}(\rho) = F_{n,X}(0)\left(\ln\frac{r}{\pi} - \gamma\right)& \\ - \sum_{k=1}^{\infty} \frac{\Lambda(k)}{\sqrt{k}} \left(\chi(k)F_{n,X}(\ln k) + \bar{\chi}(k)F_{n,X}(-\ln k)\right)& \\ + \int_{-\infty}^{\infty}\left(F_{n,X}(x)e^{(3/2-a)|x|} - F_{n,X}(0)\right)\frac{dx}{1-e^{2|x|}},&\end{aligned}$$

where the sum on ρ runs over all zeros of $\xi(s,\chi)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$. It follows that

(2.2)
$$\begin{aligned}&\lim_{X\to\infty}\sum_\rho \Phi_{n,X}(\rho) \\ &= \frac{n}{2}\left(\ln\frac{r}{\pi} - \gamma\right) - \sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{(j-1)!}\sum_{k=1}^{\infty}\frac{\Lambda(k)}{k}\bar{\chi}(k)(\ln k)^{j-1} \\ &+ \begin{cases}\sum_{j=2}^{n}\binom{n}{j}(-1)^j\left(1-\frac{1}{2^j}\right)\zeta(j) - \frac{n}{2}\sum_{l=1}^{\infty}\frac{1}{l(2l-1)}, & \text{if } \chi(-1) = 1, \\ \sum_{j=2}^{n}\binom{n}{j}(-1)^j 2^{-j}\zeta(j), & \text{if } \chi(-1) = -1.\end{cases}\end{aligned}$$

Note that the infinite series in the second term on the right side of (2.2) converges by the prime number theorem for arithmetic progressions; see §19-20 of [5].

We have

(2.3)

$$\begin{aligned}\Phi_n(s) - \Phi_{n,X}(s) &= X^{-s} \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} \sum_{k=0}^{j-1} \frac{(\ln X)^{j-k-1}}{(j-k-1)!} s^{-k-1} \\ &= \frac{X^{-s}}{s} \sum_{j=1}^{n} \binom{n}{j} \frac{(-\ln X)^{j-1}}{(j-1)!} + O\left(\frac{(\ln X)^{n-2}}{|s|^2} X^{-\Re s} \right).\end{aligned}$$

Let ρ be any zero of $\xi(s,\chi)$ which is not the Siegel zero when χ is a real nonprincipal character. By §14 of [5] we have

$$\frac{c}{\ln r(|\rho|+2)} \le \Re\rho \le 1 - \frac{c}{\ln r(|\rho|+2)}$$

for a positive constant c. An argument similar to that made in the proof of (3.9) of [3] shows that

$$\sum_{\rho} \frac{X^{-\Re\rho}}{|\rho|^2} \ll e^{-c'\sqrt{\ln X}} + \frac{X^{-\beta}}{\beta^2} \tag{2.4}$$

for a positive constant c', where the second term on the right side of the inequality exists only when β is the Siegel zero of $L(s,\chi)$.

Let

$$\psi_0(x,\chi) = \sum_{n\le x} \chi(n)\Lambda(n)$$

when x is not a prime power. By §19–20 of [5] we have

$$-\sum_{\rho} \frac{X^{\bar\rho}}{\bar\rho} = \psi_0(X,\bar\chi) + \frac{L'(0,\bar\chi)}{L(0,\bar\chi)} - \frac{1}{2}\ln\frac{X+1}{X-1}$$

when $\chi(-1) = -1$, and

$$-\sum_{\rho} \frac{X^{\bar\rho}}{\bar\rho} = \psi_0(X,\bar\chi) + b(\bar\chi) + \ln\sqrt{X^2-1}$$

when $\chi(-1) = 1$, where $b(\chi)$ is the constant term in the expansion of L'/L near $s = 0$,

$$\frac{L'(s,\chi)}{L(s,\chi)} = \frac{1}{s} + b(\chi) + \cdots,$$

and

$$\psi_0(X,\chi) = -\frac{X^{\beta}}{\beta} + O\left(Xe^{-c'\sqrt{\ln X}}\right)$$

for a positive constant c'. Since

$$\sum_\rho \frac{X^{-\rho}}{\rho} = \sum_\rho \frac{X^{-(1-\bar\rho)}}{1-\bar\rho}$$
$$= -\frac{1}{X}\sum_\rho \frac{X^{\bar\rho}}{\bar\rho} + O\left(\sum_\rho \frac{X^{-(1-\Re\rho)}}{|\rho|^2}\right)$$
$$= -\frac{1}{X}\sum_\rho \frac{X^{\bar\rho}}{\bar\rho} + O\left(e^{-c'\sqrt{\ln X}} + \frac{X^{\beta-1}}{\beta^2}\right),$$

we have

$$\sum_\rho \frac{X^{-\rho}}{\rho} \ll X^{\beta-1} + e^{-c'\sqrt{\ln X}} \tag{2.5}$$

for a positive constant c', where the term $X^{\beta-1}$ exists only when β is the Siegel zero of $L(s,\chi)$. It follows from (2.3), (2.4) and (2.5) that

$$\lim_{X\to\infty}\sum_\rho \Phi_{n,X}(\rho) = \sum_\rho \Phi_n(\rho).$$

This completes the proof of the theorem. □

3. Proof of Theorem 5

Explicit formula for $L_E(s)$ ([10]). *Let $F(x)$ be a function defined on $\mathbb{R}$ such that*

$$2F(x) = F(x+0) + F(x-0)$$

for all $x \in \mathbb{R}$, such that $F(x)\exp((\epsilon+1/2)|x|)$ is integrable and of bounded variation on $\mathbb{R}$ for a constant $\epsilon > 0$, and such that $(F(x)-F(0))/x$ is of bounded variation on $\mathbb{R}$. Then

$$\sum_\rho \Phi(\rho) = 2F(0)\ln\frac{\sqrt{N}}{2\pi} - \sum_{n=1}^\infty \frac{\Lambda(n)}{n} b(n)\left[F(\ln n) + F(-\ln n)\right]$$
$$- \int_0^\infty \left(\frac{F(x)+F(-x)}{e^x-1} - 2F(0)\frac{e^{-x}}{x}\right)dx,$$

where the sum on ρ runs over all zeros of $\xi_E(s)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$, and

$$\Phi(s) = \int_{-\infty}^\infty F(x)e^{(s-1/2)x}dx.$$

Lemma 3.1 ([6], [7], [11]). *Let f be a newform of weight 2 for $\Gamma_0(N)$. Then there an absolute effective constant $c > 0$ such that $L_f(s)$ has no zeros in the region*

$$\left\{s = \sigma + it : \sigma \geq 1 - \frac{c}{\ln(N+1+|t|)}\right\}.$$

Proof of Theorem 5. Since $\xi_E(s)$ is an entire function of order one satisfying $\xi_E(1) = 1$ and $\xi_E(s) = w\xi_E(1-s)$, we have

$$\xi_E(s) = w\prod_{\rho}(1 - s/\rho),$$

where the product is over all the zeros of $\xi_E(s)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$. If $\varphi_E(z) = \xi_E(1/(1-z))$, then

$$\frac{\varphi_E'(z)}{\varphi_E(z)} = \sum_{n=0}^{\infty}\lambda_E(n+1)z^n.$$

For a sufficiently large positive number X that is not an integer let

$$F_{n,X}(x) = \begin{cases} F_n(x), & \text{if } -\ln X < x < \infty, \\ \frac{1}{2}F_n(-\ln X), & \text{if } x = -\ln X, \\ 0, & \text{if } -\infty < x < -\ln X, \end{cases}$$

where $F_n(x)$ is given as in Lemma 2.3. Then $F_{n,X}(x)$ satisfies all conditions of the explicit formula for $L_E(s)$. Let

$$\Phi_{n,X}(s) = \int_{-\infty}^{\infty} F_{n,X}(x)e^{(s-1/2)x}dx.$$

By using the explicit formula, we obtain that

$$\begin{aligned}\sum_{\rho}\Phi_{n,X}(\rho) = 2F_{n,X}(0)\ln\frac{\sqrt{N}}{2\pi} - \sum_{k=1}^{\infty}\frac{\Lambda(k)}{k}b(k)\left[F_{n,X}(\ln k) + F_{n,X}(-\ln k)\right] \\ - \int_0^{\infty}\left(\frac{F_{n,X}(x) + F_{n,X}(-x)}{e^x - 1} - 2F_{n,X}(0)\frac{e^{-x}}{x}\right)dx,\end{aligned}$$

where the sum on ρ runs over all zeros of $\xi_E(s)$ in the order given by $|\Im\rho| < T$ for $T \to \infty$. It follows that

$$\begin{aligned}&\lim_{X\to\infty}\sum_{\rho}\Phi_{n,X}(\rho) \\ &\quad = n\left(\ln\frac{\sqrt{N}}{2\pi} - \gamma\right) - \sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{(j-1)!}\sum_{k=1}^{\infty}\frac{\Lambda(k)}{k^{3/2}}b(k)(\ln k)^{j-1} \\ &\qquad + n\left(-\frac{2}{3} + \sum_{l=1}^{\infty}\frac{3}{l(2l+3)}\right) + \sum_{j=2}^{n}\binom{n}{j}(-1)^j\sum_{l=1}^{\infty}\frac{1}{(l+1/2)^j}.\end{aligned}$$

We have
(3.1)

$$\begin{aligned}\Phi_n(s) - \Phi_{n,X}(s) &= X^{-s} \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} \sum_{k=0}^{j-1} \frac{(\ln X)^{j-k-1}}{(j-k-1)!} s^{-k-1} \\ &= \frac{X^{-s}}{s} \sum_{j=1}^{n} \binom{n}{j} \frac{(-\ln X)^{j-1}}{(j-1)!} + O\left(\frac{(\ln X)^{n-2}}{|s|^2} X^{-\Re s}\right).\end{aligned}$$

Let ρ be any zero of $\xi_E(s)$. By Lemma 3.1 and the Shimura-Taniyama conjecture we have

$$\frac{c}{\ln(N+1+|\rho|)} \leq \Re\rho \leq 1 - \frac{c}{\ln(N+1+|\rho|)}$$

for a positive constant c. An argument similar to that made in the proof of (3.9) of [3] shows that

$$\sum_{\rho} \frac{X^{-\Re\rho}}{|\rho|^2} \ll e^{-c'\sqrt{\ln X}} \tag{3.2}$$

for a positive constant c'.

Since

$$\begin{aligned}\sum_{\rho} \frac{X^{-\rho}}{\rho} &= \sum_{\rho} \frac{X^{-(1-\rho)}}{1-\rho} \\ &= -\frac{1}{X} \sum_{\rho} \frac{X^{\rho}}{\rho} + O\left(\sum_{\rho} \frac{X^{-(1-\Re\rho)}}{|\rho|^2}\right) \\ &= -\frac{1}{X} \sum_{\rho} \frac{X^{\rho}}{\rho} + O\left(e^{-c'\sqrt{\ln X}}\right),\end{aligned}$$

and since

$$\lim_{X\to\infty} \frac{(\ln X)^{j-1}}{X} \sum_{\rho} \frac{X^{\rho}}{\rho} = 0$$

for $j = 1, 2, \ldots, n$ by Theorem 4.2 and Theorem 5.2 of [11], we have

$$\lim_{X\to\infty} (\ln X)^{j-1} \sum_{\rho} \frac{X^{-\rho}}{\rho} = 0 \tag{3.3}$$

for $j = 1, 2, \ldots, n$. It follows from (3.1), (3.2) and (3.3) that

$$\lim_{X\to\infty} \sum_{\rho} \Phi_{n,X}(\rho) = \sum_{\rho} \Phi_n(\rho).$$

This completes the proof of the theorem. □

References

[1] K. Barner, *On A. Weil's explicit formula*, J. Reine Angew. Math. **323** (1981), 139–152. MR **82i:**12014

[2] E. Bombieri, *Remarks on Weil's quadratic functional in the theory of prime numbers. I*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), 183–233 (2001). MR **2003c:**11100

[3] E. Bombieri and J. C. Lagarias, *Complements to Li's criterion for the Riemann hypothesis*, J. Number Theory **77** (1999), 274–287. MR **2000h:**11092

[4] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, *On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), 843–939. MR **2002d:**11058

[5] H. Davenport, *Multiplicative number theory*, Third Edition, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000. MR **2001f:**11001

[6] S. S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975. MR 52 #280

[7] J. Hoffstein and D. Ramakrishnan, *Siegel zeros and cusp forms*, Internat. Math. Res. Notices (1995), 279–308. MR **96h:**11040

[8] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997. MR **98e:**11051

[9] X.-J. Li, *The positivity of a sequence of numbers and the Riemann hypothesis*, J. Number Theory **65** (1997), 325–333. MR **98d:**11101

[10] J.-F. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Math. **58** (1986), 209–232. MR **87j:**11059

[11] C. J. Moreno, *Explicit formulas in the theory of automorphic forms*, Number Theory Day (Proc. Conf., Rockefeller Univ., New York, 1976), Lecture Notes in Math., vol. 626, Springer-Verlag, Berlin, 1977, pp. 73–216. MR 57 #16209

[12] J. T. Tate, *The arithmetic of elliptic curves*, Invent. Math. **23** (1974), 179–206. MR 54 #7380

[13] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Sém. Math. Univ. Lund (1952), 252–265. MR 14,727e

[14] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), 443–551. MR **96d:**11071

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
E-mail address: `xianjin@math.byu.edu`