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ELLIPSOIDAL TIGHT FRAMES AND PROJECTION DECOMPOSITIONS OF OPERATORS

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Abstract. We prove the existence of tight frames whose elements lie on an arbitrary ellipsoidal surface within a real or complex separable Hilbert space $\mathcal{H}$, and we analyze the set of attainable frame bounds. In the case where $\mathcal{H}$ is real and has finite dimension, we give an algorithmic proof. Our main tool in the infinite dimensional case is a result we have proven which concerns the decomposition of a positive invertible operator into a strongly converging sum of (not necessarily mutually orthogonal) self-adjoint projections. This decomposition result implies the existence of tight frames in the ellipsoidal surface determined by the positive operator. In the real or complex finite dimensional case, this provides an alternate (but not algorithmic) proof that every such surface contains tight frames with every prescribed length at least as large as $\dim \mathcal{H}$. A corollary in both finite and infinite dimensions is that every positive invertible operator is the frame operator for a spherical frame.

Introduction

Frames were first introduced by Dufflin and Schaeffer [6] in 1952 as a component in the development of non-harmonic Fourier series, and a paper by Daubechies, Grossmann, and Meyer [5] in 1986 initiated the use of frame theory in signal processing. A *frame* on a separable Hilbert space $\mathcal{H}$ is defined to be a complete collection of vectors $\{x_i\} \subset \mathcal{H}$ for which there exist constants $0 < A \leq B$ such that for any $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B\|x\|^2$$

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The constants A and B are known as the *frame bounds*. The collection is called a *tight frame* if $A = B$, and a *Parseval frame* if $A = B = 1$. (In some of the existing literature, Parseval frames have been called *normalized tight frames*; however it should be noted that other authors have used the term *normalized* to describe a frame consisting only of unit vectors.) The *length* of a frame is the number of vectors it contains, which cannot be less than the Hilbert space dimension. References in the study of frames include [4], [8], and [9].

Hilbert space frames are used in a variety of signal processing applications, often demanding additional structure. Tight frames may be constructed having specified length, components having a predetermined sequence of norms, or with properties making them resilient to erasures. For examples, see [1], [2], [7], and [10]. One area of rapidly advancing research lies in describing tight frames in which all the vectors are of equal norm, and thus are elements of a sphere, [1], [2].

Since frame theory is geometric in nature, it is natural to ask which other surfaces in a finite or infinite dimensional Hilbert space contain tight frames. By an *ellipsoidal surface* we mean the image of the unit sphere $\mathcal{S}_1 = \{x : \|x\| = 1\}$ under a bounded invertible operator $T \in \mathcal{B}(\mathcal{H})$. Let $\mathcal{E}_T$ denote the ellipsoidal surface $\mathcal{E}_T = T\mathcal{S}_1$. A frame contained in $\mathcal{E}_T$ is called an *ellipsoidal frame*, and if it is tight it is called an ellipsoidal tight frame (ETF) for that surface. We say that a frame bound K is *attainable* for $\mathcal{E}_T$ if there is an ETF for $\mathcal{E}_T$ with frame bound K. If an ellipsoid $\mathcal{E}$ is a sphere we will call a frame in $\mathcal{E}$ *spherical*.

Given an ellipsoid $\mathcal{E}$, we can assume $\mathcal{E} = \mathcal{E}_T$, where T is a positive invertible operator. Given A an invertible operator, let $A^* = U|A^*|$ be the polar decomposition where $|A^*| = (AA^*)^{1/2}$. Then $A = |A^*|U^*$. By taking $T = |A^*|$, we see that $T\mathcal{S}_1 = A\mathcal{S}_1$. Moreover it is easily seen that the positive operator T for which $\mathcal{E} = \mathcal{E}_T$ is unique.

Throughout the paper, $\mathcal{H}$ will be a separable real or complex Hilbert space and for $x, y, u \in \mathcal{H}$, we will use the notation $x \otimes y$ to denote the rank-one operator $u \mapsto \langle u, y\rangle x$. Note that $\|x\| = 1$ implies that $x \otimes x$ is a rank-1 projection.

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1. Theorems

There are three theorems in this paper. The first gives an elementary construction of ETF's when $\mathcal{H} = \mathbb{R}^n$, and is proved in Section 2.

THEOREM 1. *Let $n, k \in \mathbb{N}$ with $n \leq k$, let $a_1, \ldots, a_n \geq 0$ be such that $r := \sum_1^n a_j > 0$ and consider the (possibly degenerate) ellipsoid*

$$\mathcal{E} = \left\{ \mathbf{x} = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \mid \sum_1^n a_j x_j^2 = 1 \right\}.$$

Then there is a tight frame for $\mathbb{R}^n$ consisting of k vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathcal{E}$.

This result is valid for degenerate ellipsoids (in which some of the major axes are infinitely long). Our method of proof provides geometric insight to the problem, but does not extend to infinite dimensions.

We note that, in the non-degenerate case, the definition of an ellipsoidal surface $\mathcal{E}$ given in Theorem 1 is equivalent to the definition given in the introduction, specifying that the Hilbert space be $\mathbb{R}^n$. Indeed, if $a_i > 1$ for all $i = 1, \ldots, n$ and if $D = \text{diag}(a_1, a_2, \ldots, a_n)$, then $\sum_{i=1}^n a_i x_i^2 = 1$ iff $\langle Dx, x \rangle = 1$ iff $\|D^{1/2}x\| = 1$ iff $D^{1/2}x \in \mathcal{S}_1(\mathbb{R}^n)$ iff $x \in D^{-1/2}\mathcal{S}_1(\mathbb{R}^n)$. So $\mathcal{E} = \mathcal{E}_T$ for $T = D^{-1/2}$, and thus $\mathcal{E}$ has the requisite form. To reverse this argument for a non-diagonal positive operator T, first diagonalize it by an orthogonal transformation given by rotations. Reversing the steps will then show that $\mathcal{E}_T$ is equivalent to $\mathcal{E}$ for some choice of positive constants $\{a_1, \ldots, a_n\}$.

The second theorem is used to prove Theorem 3 in the infinite dimensional case. It has independent interest in operator theory, and to our knowledge is a new result. The proof, as well as the corresponding result in finite dimensions (Proposition 6), is contained in Section 3. Some preliminaries are required before we state Theorem 2.

It is well-known (see [12]) that a separably acting positive operator A decomposes as the direct sum of a positive operator A_1 with nonatomic spectral measure and a positive operator A_2 with purely atomic spectral measure (i.e., a diagonalizable operator). For $B \in \mathcal{B}(\mathcal{H})$, the *essential norm* of B is

$$\|B\|_{\text{ess}} := \inf \{ \|B - K\| \,:\, K \text{ is a compact operator in } \mathcal{B}(\mathcal{H}) \}.$$

In the proof of Proposition 11, we have the special case where A is a diagonal operator, $A = \text{diag}(a_1, a_2, \ldots)$, with respect to some orthonormal basis. In this case, it is clear that

$$\|A\|_{\text{ess}} = \sup \{ \alpha > 0 \,:\, |a_i| \geq \alpha \text{ for infinitely many } i \}.$$

For a positive operator A with spectrum $\sigma(A)$, we have $\|A\| = \sup\{\lambda : \lambda \in \sigma(A)\}$ and if A is invertible, then $\|A^{-1}\|^{-1} = \inf\{\lambda : \lambda \in \sigma(A)\}$. Similarly,

$\|A\|_{\text{ess}} = \sup\{\lambda : \lambda \in \sigma_{\text{ess}}(A)\}$ and $\|A^{-1}\|_{\text{ess}}^{-1} = \inf\{\lambda : \lambda \in \sigma_{\text{ess}}(A)\}$. In particular, $\|A^{-1}\|^{-1} \leq \|A^{-1}\|_{\text{ess}}^{-1} \leq \|A\|_{\text{ess}} \leq \|A\|$.

For A a positive operator, we say that A has a *projection decomposition* if A can be expressed as the sum of a finite or infinite sequence of (not necessarily mutually orthogonal) self-adjoint projections, with convergence in the strong operator topology.

THEOREM 2. *Let A be a positive operator in $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a real or complex Hilbert space with infinite dimension, and suppose $\|A\|_{\text{ess}} > 1$. Then A has a projection decomposition.*

Note that in this theorem A need not be invertible. There are theorems in the literature (e.g., [13]) expressing operators as linear combinations of projections and as sums of idempotents (non self-adjoint projections). The decomposition in Theorem 2 is different in that each term *is* a self-adjoint projection rather than a scalar multiple of a projection.

The next theorem states that every ellipsoidal surface contains a tight frame. We also include some detailed information about the nature of the set of attainable frame bounds.

THEOREM 3. *Let T be a bounded invertible operator on a real or complex Hilbert space. Then the ellipsoidal surface $\mathcal{E}_T$ contains a tight frame. If $\mathcal{H}$ is finite dimensional with $n = \dim \mathcal{H}$, then for any integer $k \geq n$, $\mathcal{E}_T$ contains a tight frame of length k, and every ETF on $\mathcal{E}_T$ of length k has frame bound $K = k\left[\text{trace}(T^{-2})\right]^{-1}$. If $\dim \mathcal{H} = \infty$, then for any constant $K > \|T^{-2}\|_{\text{ess}}^{-1}$, $\mathcal{E}_T$ contains a tight frame with frame bound K.*

2. A construction of ETF's in $\mathbb{R}^n$

We begin by showing that every ellipsoid can be scaled to contain an orthonormal basis.

LEMMA 4. *Let $n \in \mathbb{N}$, let $a_1, \ldots, a_n \geq 0$ be such that $\sum_1^n a_j = n$ and let*

$$\mathcal{E} = \left\{ \mathbf{x} = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \mid \sum_1^n a_j x_j^2 = 1 \right\}.$$

Then there is an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for $\mathbb{R}^n$ consisting of vectors $\mathbf{v}_j \in \mathcal{E}$.

Proof. Proceed by induction on n. The case $n = 1$ is trivial. Assume $n \geq 2$ and without loss of generality suppose $a_1 \geq 1$ and $a_2 \leq 1$. Let θ be such that $a_1(\cos\theta)^2 + a_2(\sin\theta)^2 = 1$ and let $b_2 = a_1(\sin\theta)^2 + a_2(\cos\theta)^2$. Consider the rotation matrix

$$R = \begin{pmatrix} \cos\theta & \sin\theta & & & \\ -\sin\theta & \cos\theta & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then

$$R^{-1}\mathcal{E} = \Bigg\{ (y_1, \ldots, y_n)^t \in \mathbb{R}^n \mid \\ y_1^2 + 2(a_1 - a_2) y_1 y_2 \cos\theta \sin\theta + b_2 y_2^2 + \sum_3^n a_j y_j^2 = 1 \Bigg\}.$$

We have $b_2 + \sum_3^n a_j = n - 1$. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^n$ consisting of all vectors of the form $(0, x_2, \ldots, x_n)^t$. By the induction hypothesis, there is an orthonormal basis $\mathbf{u}_2, \ldots, \mathbf{u}_n$ for $\mathcal{V}$ consisting of vectors $\mathbf{u}_j \in R^{-1}\mathcal{E}$. Let $\mathbf{u}_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^n$, and let $\mathbf{v}_j = R\mathbf{u}_j$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthonormal basis for $\mathbb{R}^n$ consisting of vectors $\mathbf{v}_j \in \mathcal{E}$. □

In the case of a general ellipsoid, where $\sum_{j=1}^n a_j = r > 0$, the lemma gives a constant multiple of an orthonormal basis on the ellipsoid.

Proof of Theorem 1. Consider the isometry $W : \mathbb{R}^n \to \mathbb{R}^k$ and the projection $P = W^* : \mathbb{R}^k \to \mathbb{R}^n$ given by

$$W(x_1, \ldots, x_n)^t = (x_1, \ldots, x_n, 0, \ldots, 0)^t,$$
$$P(x_1, \ldots, x_k)^t = (x_1, \ldots, x_n)^t.$$

Let $a_j = 0$ for $n + 1 \le j \le k$ and let

$$\mathcal{E}' = \left\{ \mathbf{y} = (y_1, \ldots, y_k)^t \in \mathbb{R}^k \mid \sum_1^k a_j y_j^2 = 1 \right\}.$$

By Lemma 4, there is a multiple of an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for $\mathbb{R}^k$ consisting of vectors $\mathbf{v}_j \in \mathcal{E}'$. Let $\mathbf{u}_j = P\mathbf{v}_j$. Then $\mathbf{u}_j \in \mathcal{E}$. Moreover, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is a tight frame for $\mathbb{R}^n$, because if $\mathbf{x} \in \mathbb{R}^n$, then

$$\sum_{j=1}^k |\langle \mathbf{x}, \mathbf{u}_j \rangle|^2 = \sum_{j=1}^k |\langle W\mathbf{x}, \mathbf{v}_j \rangle|^2 = \frac{k}{r} \|W\mathbf{x}\|^2 = \frac{k}{r} \|\mathbf{x}\|^2. \qquad \square$$

Remark 5. It is an elementary result in matrix theory [11, Thm. 1.3.4] that for any real $n \times n$ matrix B acting on $\mathbb{R}^n$ there is an orthonormal basis $\{u_1, \ldots, u_n\}$ for $\mathbb{R}^n$ so that the diagonal elements $\langle Bu_i, u_i \rangle$ of B with respect to $\{u_1, \ldots, u_n\}$ are all equal to $(1/n)\,[\mathrm{trace}(B)]$. If we let $D = \mathrm{diag}(a_1, \ldots, a_n)$, where the numbers a_i are as in Lemma 4, then the condition $\langle Dv, v \rangle = 1$ for a vector v is exactly the condition for v to be on the ellipsoid $\mathcal{E}$. Thus, letting $B = D$ and $v_i = u_i$ yields another proof of Lemma 4. The merit of the proof we give is that it is algorithmic and relates well to the paper. It was obtained by the second author in an undergraduate research (REU) program in which the other co-authors were mentors.

3. Projection decompositions for positive operators

The arguments in the remainder of this paper hold for $\mathcal{H}$ either a real or complex Hilbert space.

PROPOSITION 6. *Let $A \in \mathcal{B}(\mathcal{H})$ be a finite rank positive operator with integer trace k. If $k \geq \operatorname{rank}(A)$, then A is the sum of k projections of rank one.*

Proof. We will construct unit vectors $x_1, x_2, \ldots, x_k$ so that A is the sum of the projections $x_i \otimes x_i$. The proof uses induction on k. Let $n = \operatorname{rank}(A)$ and write $\mathcal{H}_n = \operatorname{range}(A)$. If $k = 1$, then A must itself be a rank-1 projection. Assume $k \geq 2$. Select an orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathcal{H}_n$ such that A can be written on $\mathcal{H}_n$ as a diagonal matrix with positive entries $a_1 \geq a_2 \cdots \geq a_n > 0$.

Case 1: $k > n$. In this case, we have $a_1 > 1$, so we can take $x_k = e_1$. The remainder on $\mathcal{H}_n$,

$$A - (x_k \otimes x_k) = \operatorname{diag}(a_1 - 1, a_2, \ldots, a_n),$$

has positive diagonal entries, still has rank n, and now has trace $k - 1 \geq n$. By the inductive hypothesis, the result holds.

Case 2: $k = n$. We now have that $a_1 \geq 1$ and $a_n \leq 1$. Given any finite rank, self-adjoint $R \in \mathcal{B}(\mathcal{H})$, let $\mu_n(R)$ denote the n-th largest eigenvalue of R counting multiplicity. Note that $\mu_n(A-(e_1 \otimes e_1)) \geq 0$, $\mu_n(A-(e_n \otimes e_n)) \leq 0$, and $\mu_n(A - (x \otimes x))$ is a continuous function of $x \in \mathcal{H}_n$. Hence, there exists $y \in \mathcal{H}_n$ such that $\mu_n(A - (y \otimes y)) = 0$. Choose $x_k = y$. Note the remainder $(A - (x_k \otimes x_k)) \geq 0$ and

$$\begin{aligned} \operatorname{trace}(A - (x_k \otimes x_k)) &= n - 1, \\ \operatorname{rank}(A - (x_k \otimes x_k)) &= n - 1 = k - 1. \end{aligned}$$

Again, by the inductive hypothesis, the result holds. □

LEMMA 7. *Let $P_1, P_2, \ldots, P_n$ be mutually orthogonal projections on a Hilbert space $\mathcal{H}$, all of the same nonzero rank k, where k can be finite or infinite. Let $r_1, r_2, \ldots, r_n$ be nonnegative real numbers, and let $r = \sum_1^n r_i$. Define the operator*

$$A = r_1 P_1 + r_2 P_2 + \cdots + r_n P_n.$$

If the sum r is an integer and $r \geq n$, then there exist rank-k projections $Q_1, \ldots, Q_r$ such that

$$A = Q_1 + Q_2 + \cdots + Q_r.$$

Proof. If $k = 1$, then $r = \text{trace}(A)$ and we have $\text{rank}(A) \le n \le r$, so the result follows from Proposition 6. If $k > 1$, each projection P_i can be written as a sum of k mutually orthogonal rank-1 projections:

$$P_i = P_{i1} + P_{i2} + \cdots + P_{ik}.$$

(Here and elsewhere in this proof, sums with indices running from 1 to k should be interpreted as infinite sums in the case where $k = \infty$.) All rank-1 projections P_{ij} are thus mutually orthogonal. Define operators $A_1, \ldots, A_k$ by

$$A_j = r_1 P_{1j} + r_2 P_{2j} + \cdots + r_n P_{nj}.$$

Now, $A = A_1 + \cdots + A_k$ and each A_j has rank n and trace r. By Proposition 6, each A_j can be written as a sum of r rank-1 projections:

$$A_j = T_{j1} + T_{j2} + \cdots + T_{jr}.$$

Note that projections T_{jl} and T_{mp} are orthogonal if $j \ne m$. Define the rank-k projections $Q_1, \ldots, Q_r$ by

$$Q_l = T_{1l} + T_{2l} + \cdots + T_{kl}.$$

This gives $A = Q_1 + Q_2 + \cdots + Q_r$. □

Lemma 8. *Let A be a positive operator with finite spectrum contained in the rationals $\mathbb{Q}$, such that all spectral projections are infinite dimensional, and also such that $\|A\| > 1$. Then A is a finite sum of self-adjoint projections.*

Proof. By hypothesis, there are mutually orthogonal infinite-rank projections $P_1, \ldots, P_n$ and positive rational numbers $r_1 \ge r_2 \ge \cdots \ge r_n$ such that

$$A = r_1 P_1 + \cdots + r_n P_n.$$

By hypothesis $\|A\| > 1$, hence $r_1 > 1$.

Write $r_i = s_i/t_i$ with s_i and t_i positive integers, and let $s = \sum_{i=1}^n s_i$, $t = \sum_{i=1}^n t_i$. We may assume $s \ge t$, for otherwise we can choose $m \in \mathbb{N}$ such that

$$ms_1 + s_2 + \cdots + s_n \ge mt_1 + t_2 + \cdots + t_n$$

and replace s_1 with ms_1 and t with mt_1.

Each P_i can be written as a sum of t_i mutually orthogonal infinite rank projections P_{ij}, $j = 1, \ldots, t_i$, which then allows us to write

$$A = \sum_{i=1}^{n} \sum_{j=1}^{t_i} r_i P_{ij}.$$

The operator is now a linear combination of $\sum t_i = t$ mutually orthogonal projections of infinite rank, and the sum of the coefficients is now an integer $\sum t_i r_i = \sum s_i = s$. Since $s \ge t$, Lemma 7 implies that A can be written as a sum of s projections. □

LEMMA 9. *Let A be a positive operator which has a projection-decomposition. Then either A is a projection or $\|A\| > 1$.*

Proof. Suppose, to obtain a contradiction, that $\|A\| \leq 1$ and that A is not a projection. By assumption, $A = \sum P_i$ with the series converging strongly. Thus $A - P_i \geq 0$ for all i. Then $P_i(A - P_i)P_i \geq 0$, so $P_iAP_i \geq P_i$.

Let $\mathcal{K}_i = P_i\mathcal{H}$ and $B = P_iA|_{\mathcal{K}_i}$. Then B_i is positive and $B_i \geq I_{\mathcal{K}_i}$ (the identity operator on $\mathcal{K}_i$). Since $\|B_i\| \leq 1$, this implies $B_i = I_{\mathcal{K}_i}$, and thus $P_iAP_i = P_i$.

Now, $P_i = P_i(\sum_j P_j)P_i = P_i + \sum_{j \neq i} P_iP_jP_i$, so $\sum_{j\neq i} P_iP_jP_i = 0$. Since each $P_iP_jP_i \geq 0$, this implies $P_iP_jP_i = 0$. Thus, $(P_jP_i)^*(P_jP_i) = 0$, so $P_jP_i = 0$. Since this is true for arbitrary i, j with $i \neq j$, this shows that A is the sum of mutually orthogonal projections, and hence is itself a projection. The contradiction proves the result. □

PROPOSITION 10. *Let A be a positive operator in $\mathcal{B}(\mathcal{H})$ with the property that all nonzero spectral projections for A are of infinite rank. If $\|A\| > 1$, then A admits a projection decomposition as a sum of infinite rank projections.*

Proof. We will show that A can be written as a sum $A = \sum_{i=1}^{\infty} A_i$ of positive operators, each satisfying the hypotheses of Lemma 8, where the sum converges in the strong operator topology. We can then decompose each of the operators A_i as a finite sum of projections A_{ij} and then re-enumerate with a single index to obtain a sequence Q_i of projections which sum to A in SOT. Indeed, the partial sums of $\sum Q_i$ are dominated by A, hence $\sum Q_i$ converges strongly to some operator C, and since the partial sums of $\sum A_i$ are also partial sums of $\sum Q_i$, the sequence of partial sums of $\sum Q_i$ has a subsequence which converges to A, and hence $C = A$.

By hypothesis, we have $\|A\| > 1$. We may choose a positive rational number $\alpha > 1$ and a nonzero spectral projection G for A such that $A \geq \alpha G$. Let $B = A - \alpha G$, so that $B \geq 0$. Using a standard argument, we can write $B = \sum_{i=1}^{\infty} B_i$, where each B_i is a positive rational multiple of a spectral projection for A, with convergence in the SOT.

We can write $G = \sum G_i$ as an infinite direct sum of nonzero infinite rank projections, with the requirement that G_i be a subprojection of G which commutes with all the spectral projections for A. (This can clearly be done when the spectral projections for A are all of infinite rank.) Now, let $A_i = \alpha G_i + B_i$. We have $\|A_i\| \geq \alpha > 1$.

By Lemma 8, it follows that A_i is a finite sum of projections. By the construction, we have the requisite form $A = \sum A_i$. □

PROPOSITION 11. *Let A be a positive operator in $\mathcal{B}(\mathcal{H})$ which is diagonal with respect to some orthonormal basis $\{e_i\}$ for the Hilbert space $\mathcal{H}$. Suppose*

$\|A\|_{\text{ess}} > 1$. *Then there is a sequence of rank-1 projections* $\{P_i\}_{i=1}^{\infty}$ *such that* $A = \sum P_i$, *where the sum converges in the strong operator topology.*

Proof. Write A as $\text{diag}(a_0, a_1, \ldots)$ and let $E_n = e_n \otimes e_n$. Since $\|A\|_{\text{ess}} > 1$, there is a constant $\alpha > 1$ such that $a_i \geq \alpha$ for infinitely many i. Let $k \geq 2$ be an integer such that $1 + 2/(k-1) \leq \alpha$. Permuting if necessary, we can without loss of generality assume that the indices n for which $a_n < \alpha$ are all multiples of k.

Let $B_0 = a_0 E_0 + \cdots + a_{k-1} E_{k-1}$. Therefore, we have $\text{rank}(B_0) \leq k$ and

$$\begin{aligned}\text{trace}(B_0) &= \sum_0^{k-1} a_i \\ &\geq a_0 + (k-1)\alpha \\ &\geq a_0 + (k-1)\left(1 + \frac{2}{k-1}\right) \\ &= a_0 + k + 1.\end{aligned}$$

Let L_0 be the greatest integer less than $\text{trace}(B_0)$. Then $L_0 \geq k+1$. Define a'_{k-1} to be the real number $0 \leq a'_{k-1} \leq a_{k-1}$ such that if

$$B'_0 = a_0 E_0 + \cdots + a_{k-2} E_{k-2} + a'_{k-1} E_{k-1},$$

then

$$\text{trace}(B'_0) = L_0 \geq k + 1 > \text{rank}(B'_0).$$

By Proposition 6, B'_0 can be written as a sum of L_0 rank-1 projections.

In the next step, let $a''_{k-1} = a_{k-1} - a'_{k-1}$ and let

$$B_1 = a''_{k-1} E_{k-1} + a_k E_k + a_{k+1} E_{k+1} + \cdots + a_{2k-1} E_{2k-1}.$$

Thus $\text{rank}(B_1) \leq k + 1$ and

$$\begin{aligned}\text{trace}(B_1) &= a''_{k-1} + a_k + (a_{k+1} + \cdots + a_{2k-1}) \\ &\geq a''_{k-1} + a_k + (k-1)\alpha \\ &\geq a''_{k-1} + a_k + (k-1)\left(1 + \frac{2}{k-1}\right) \\ &= a''_{k-1} + a_k + k + 1 \\ &\geq \text{rank}(B_1).\end{aligned}$$

Construct B'_1 in a similar manner, so that its trace is an integer greater than or equal to its rank. Then B'_1 can be written as a sum of rank-1 projections using Proposition 6.

Proceeding recursively in a like manner, we may write $A = \sum_{j=1}^{\infty} B'_j$ converging in SOT, where each B'_j is a positive operator supported in $E_{jk-1} + \cdots + E_{(j+1)k-1}$ and with $\text{trace}(B'_j)$ an integer that is greater than or equal to

$\operatorname{rank}(B_j')$. Invoking Proposition 6 again to write each B_j' as a sum of rank-1 projections, the proposition is proved. □

Proof of Theorem 2. Write $A = A_1 + A_2$, where A_1 and A_2 respectively denote the nonatomic and purely atomic parts of A. Then $\|A_1\|_{\text{ess}} = \|A_1\|$, and $\|A\|_{\text{ess}} = \max\{\|A_1\|, \|A_2\|_{\text{ess}}\}$. So $\|A\|_{\text{ess}} > 1$ implies $\|A_1\| > 1$ or $\|A_2\|_{\text{ess}} > 1$. Suppose first that $\|A_1\| > 1$. Then there is a nonzero spectral projection P for A_1 and a constant $\alpha > 1$ such that $A_1 P \geq \alpha P$. Let Q be a nonzero spectral projection for A_1 dominated by P such that $P - Q \neq 0$. Then $A_1 - \alpha Q$ satisfies the hypotheses of Proposition 10, so is projection decomposable. Also, $QA_2 = A_2 Q = 0$, so $A_2 + \alpha Q$ is a diagonal operator with essential norm greater than or equal to α, and so it is projection decomposable by Proposition 11. The result follows by decomposing $A_1 - \alpha Q$ and $A_2 + \alpha Q$ as sums of projections and combining the series.

For the case $\|A_1\| \leq 1$ and $\|A_2\|_{\text{ess}} > 1$, we use a similar argument. There is a constant $\alpha > 1$ and an infinite rank spectral projection P for A_2 such that $A_2 - \alpha P \geq 0$. Then P dominates a projection Q that commutes with A_2 such that both Q and $P - Q$ are of infinite rank. Then $A_2 - \alpha Q$ satisfies Proposition 11 and hence has a projection decomposition. The operator $A_1 + \alpha Q$ has norm greater than or equal to α and all of its nonzero spectral projections have infinite rank, so it satisfies the hypotheses of Proposition 10. Thus, $A_1 + \alpha Q$ has a projection decomposition, and we combine this decomposition with the decomposition of $A_2 - \alpha Q$ to get a projection decomposition for A. □

4. Ellipsoidal tight frames

Let $\mathcal{H}$ be a finite or countably infinite dimensional Hilbert space. Let $\{x_j\}_{j \in \mathbb{J}}$ be a frame for $\mathcal{H}$, where $\mathbb{J}$ is some index set. Consider the standard frame operator defined by

$$Sw = \sum_{j \in \mathbb{J}} \langle w, x_j \rangle x_j = \sum_{j \in \mathbb{J}} (x_j \otimes x_j)\, w.$$

Thus, $S = \sum_{\mathbb{J}} x_j \otimes x_j$, where this series of positive rank-1 operators converges in the strong operator topology (i.e., the topology of pointwise convergence). In the special case where each $\|x_j\| = 1$, S is the sum of the rank-1 projections $P_j = x_j \otimes x_j$. If we let $y_j = S^{-1/2} x_j$, then it is well-known that $\{y_j\}_{j \in \mathbb{J}}$ is a Parseval frame (i.e., tight with frame bound 1). If each $\|x_j\| = 1$, then $\{y_j\}_{j \in \mathbb{J}}$ is an ellipsoidal tight frame for the ellipsoidal surface $\mathcal{E}_{S^{-1/2}} = S^{-1/2} \mathcal{S}_1$. Moreover, it is well-known (see [8]) that a sequence $\{x_j\}_{j \in \mathbb{J}} \subseteq \mathcal{H}$ is a tight frame for $\mathcal{H}$ if and only if the frame operator S is a positive scalar multiple of the identity, i.e., $S = KI$, and in this case K is the frame bound.

Remark 12. From the above paragraph, it is clear that a positive invertible operator is the frame operator for a frame of unit vectors if and only if it admits a projection decomposition. (Each projection can be further decomposed into rank-1 projections, as needed.)

The link between Theorem 2 and Theorem 3 is the following:

PROPOSITION 13. *Let T be a positive invertible operator in $\mathcal{B}(\mathcal{H})$, and let $K > 0$ be a positive constant. The ellipsoidal surface $\mathcal{E}_T = T\mathcal{S}_1$ contains a tight frame $\{y_j\}$ with frame bound K if and only if the operator $R = KT^{-2}$ admits a projection decomposition. In this case, R is the frame operator for the spherical frame $\{T^{-1}y_j\}$.*

Proof. We present the proof in the infinite dimensional setting, and note that the calculations in the finite dimensional case are identical but do not require discussion of convergence. Let $\mathbb{J}$ be a finite or infinite index set. Assume $\mathcal{E}_T$ contains a tight frame $\{y_j\}_{j\in\mathbb{J}}$ with frame bound K. Then $\sum_{j\in\mathbb{J}} y_j \otimes y_j = KI$, with the series converging in the strong operator topology. Let $x_j := T^{-1}y_j \in \mathcal{S}_1$, so $x_j \otimes x_j$ are projections. We can then compute:

$$\begin{aligned} R = KT^{-2} &= T^{-1}\left(\sum_{j\in\mathbb{J}} y_j \otimes y_j\right)T^{-1} \\ &= \sum_{j\in\mathbb{J}} T^{-1}y_j \otimes T^{-1}y_j = \sum_{j\in\mathbb{J}} x_j \otimes x_j. \end{aligned}$$

This shows that R can be decomposed as required. Conversely, suppose R admits a projection decomposition $R = \sum P_j$, where $\{P_j\}$ are self-adjoint projections and convergence is in the strong operator topology. We can assume that the P_j have rank-1, for otherwise we can decompose each P_j as a strongly convergent sum of rank-1 projections, and re-enumerate appropriately. Since $P_j \geq 0$, the convergence is independent of the enumeration used. Write $P_j = x_j \otimes x_j$ for some unit vector x_j. Letting $y_j = Tx_j$, we have $y_j \in \mathcal{E}_T$, and we also have

$$\begin{aligned} KI = TRT &= T\left(\sum_{j\in\mathbb{J}} x_j \otimes x_j\right)T \\ &= \sum_{j\in\mathbb{J}} Tx_j \otimes Tx_j = \sum_{j\in\mathbb{J}} y_j \otimes y_j. \end{aligned}$$

This shows that $\sum y_j \otimes y_j$ converges in the strong operator topology to KI. Thus, $\{y_j\}_{j\in\mathbb{J}}$ is a tight frame on $\mathcal{E}_T$, as required. □

Proof of Theorem 3. Let $\mathcal{E}$ be an ellipsoid. Then $\mathcal{E} = \mathcal{E}_T = T\mathcal{S}_1$ for some positive invertible $T \in \mathcal{B}(\mathcal{H})$. Let K be a positive constant, and let $R = KT^{-2}$.

The condition $K > \|T^{-2}\|_{\text{ess}}^{-1}$ implies $\|R\|_{\text{ess}} > 1$. So, by Theorem 2, R admits a projection decomposition, and thus Proposition 13 implies that $\mathcal{E}$ contains a tight frame with frame bound K.

In the finite dimensional case, let $n = \dim \mathcal{H}$. Proposition 13 states that $\mathcal{E}$ will contain a tight frame with frame bound D if and only if KT^{-2} admits a projection decomposition, and by Proposition 6 this happens if and only if $\text{trace}(KT^{-2})$ is an integer $k \geq n$, and in this case k is the length of the frame. Thus, we have $K = k[\text{trace}(T^{-2})]^{-1}$. Therefore, every ellipsoid $\mathcal{E} = \mathcal{E}_T$ contains a tight frame of every length $k \geq n$, and every such tight frame has frame bound $k[\text{trace}(T^{-2})]^{-1}$. □

Corollary 14. *Every positive invertible operator S on a separable Hilbert space $\mathcal{H}$ is the frame operator for a spherical frame. If $\mathcal{H}$ has finite dimension n, then for every integer $k \geq n$, S is the frame operator for a spherical frame of length k, and the radius of the sphere is $\sqrt{\text{trace}(S)/k}$. If $\mathcal{H}$ is infinite dimensional, the radius of the sphere can be taken to be any positive number $r < \|S\|_{\text{ess}}^{1/2}$.*

Proof. In the finite dimensional case, let $c = k/\text{trace}(S)$ and $A = cS$, so that $\text{trace}(A) = k$. Then, by Proposition 6, A has a projection decomposition into k rank-1 projections, making A the frame operator for the frame of unit vectors $\{x_i\}_{i=1}^k$. Thus, S is the frame operator for $\{x_i/\sqrt{c}\}_{i=1}^k$.

When $\mathcal{H}$ has infinite dimension, let c be any constant greater than $\|S\|_{\text{ess}}^{-1}$, and let $A = cS$. The hypotheses of Theorem 2 are satisfied, so A admits a projection decomposition. Then A is the frame operator for a frame $\{x_i\}$ of unit vectors, so S is the frame operator for the spherical frame $\{x_i/\sqrt{c}\}$. □

Remark 15. We know of at least two groups who have independently and simultaneously proved our finite dimensional ellipsoidal tight frame results. Holmes and Paulsen [10] have a proof similar to the discussion in Remark 5. Casazza and Leon [3] have shown the existence of "spherical frames for $\mathbb{R}^n$ with a given frame operator", which is an equivalent problem.

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