G-DIMENSION OVER LOCAL HOMOMORPHISMS. APPLICATIONS TO THE FROBENIUS ENDOMORPHISM

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ABSTRACT. We develop a theory of G-dimension over local homomorphisms which encompasses the classical theory of G-dimension for finitely generated modules over local rings. As an application, we prove that a local ring R of characteristic p is Gorenstein if and only if it possesses a nonzero finitely generated module of finite projective dimension that has finite G-dimension when considered as an R-module via some power of the Frobenius endomorphism of R. We also prove results that track the behavior of Gorenstein properties of local homomorphisms under composition and decomposition.

1. Introduction

The main goal of this article is to develop a theory of Gorenstein dimension over local homomorphisms. More precisely, given a local homomorphism $\varphi\colon R \to S$, to each finitely generated (in short: finite) S-module M, we attach an invariant $\text{G-dim}_\varphi(M)$, called the *G-dimension of M over φ*. This invariant is defined using the technology of Cohen factorizations, developed by Avramov, Foxby, and B. Herzog [8]. The reader can refer to Section 3 for the details. When M happens to be finite over R, for instance when $\varphi = \text{id}_R$, this coincides with the G-dimension of M over R as defined by Auslander and Bridger [2]; this is contained in Corollary 7.3.

One of the guiding examples for this work is the Frobenius map $\varphi\colon R \to R$, given by $x \mapsto x^p$, where R is a local ring of positive prime characteristic p. Since φ is a ring homomorphism, so is φ^n for each integer $n > 0$, and hence one can view R as a left module over itself via φ^n. Denote this R-module by $^{\varphi^n}R$. Like in the case of the residue field, it is known that certain homological properties of $^{\varphi^n}R$ determine and are determined by ring-theoretic properties

of R. Consider, for instance, regularity. The Auslander-Buchsbaum-Serre theorem says that a local ring is regular if and only if its residue field has finite projective dimension. Compare this with the fact that, when R has characteristic p, it is regular if and only if the flat dimension of $\varphi^n R$ is finite for some $n \geq 1$; this is proved by Kunz [26, (2.1)] and Rodicio [30, (2)]. This result may be reformulated as follows: the local ring R is regular if and only if $\mathrm{pd}(\varphi^n)$ is finite for some integer $n \geq 1$. Here, given any local homomorphism $\varphi \colon R \to S$, we write $\mathrm{pd}_\varphi(\text{-})$ for the *projective dimension over φ*, which is also defined via Cohen factorizations, and set $\mathrm{pd}(\varphi) = \mathrm{pd}_\varphi(S)$; see Section 4.

A key contribution of this paper, Theorem A below, is a similar characterization of the Gorenstein property for R. It is contained in Theorem 6.6 and is analogous to a classical result of Auslander and Bridger: for any local ring, the residue field has finite G-dimension if and only if the ring is Gorenstein.

THEOREM A.   *Let R be a local ring of positive prime characteristic p and φ its Frobenius endomorphism. The following conditions are equivalent.*

   (a) *The ring R is Gorenstein.*
   (b) *G-$\dim(\varphi^n)$ is finite for some integer $n \geq 1$.*
   (c) *There exists a nonzero finite R-module P of finite projective dimension and an integer $n \geq 1$ such that G-$\dim_{\varphi^n}(P)$ is finite.*

In the statement, G-$\dim(\varphi^n) = $ G-$\dim_{\varphi^n}(R)$. In the special case where φ is module-finite, the equivalence of conditions (a) and (b) coincides with a recent result of Takahashi and Yoshino [31, (3.1)]. These are related also to a theorem of Goto [19, (1.1)].

The bulk of the article is dedicated to a systematic investigation of the invariant G-$\dim_\varphi(\text{-})$. Some of the results obtained extend those concerning the classical invariant G-$\dim_R(\text{-})$. Others are new even when specialized to the absolute situation. The ensuing theorem is one such result. It is comparable to [17, (3.2)], which can be souped up to: if $\mathrm{pd}_\sigma(P)$ is finite, then $\mathrm{pd}_{\sigma\varphi}(P) = \mathrm{pd}(\varphi) + \mathrm{pd}_\sigma(P)$; see Theorem 5.7 for a further enhancement.

THEOREM B.   *Let $\varphi \colon R \to S$ and $\sigma \colon S \to T$ be local homomorphisms, and let P be a nonzero finite T-module. If $\mathrm{pd}_\sigma(P)$ is finite, then*

$$\text{G-}\dim_{\sigma\varphi}(P) = \text{G-}\dim(\varphi) + \mathrm{pd}_\sigma(P).$$

In particular, G-$\dim_{\sigma\varphi}(P)$ and G-$\dim(\varphi)$ are simultaneously finite.

This result is subsumed by Theorem 5.1. The special case $P = T$, spelled out in Theorem 5.2, may be viewed as a composition-decomposition theorem for maps of finite G-dimension. It is expected that the composition part of the result holds even when G-$\dim(\sigma)$ is finite [7, (4.8)]. However, as Example 5.3 demonstrates, the decomposition part cannot extend to that generality.

Theorem B and its counterpart for projective dimension are crucial ingredients in the following theorem that generalizes [5, (4.6.c)] and [7, (8.8)] proved by Avramov and Foxby.

THEOREM C. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $\sigma\colon (S, \mathfrak{n}) \to (T, \mathfrak{p})$ be local homomorphisms with $\mathrm{pd}(\sigma)$ finite. If $\sigma\varphi$ is (quasi-)Gorenstein at $\mathfrak{p}$, then φ is (quasi-)Gorenstein at $\mathfrak{n}$ and σ is Gorenstein at $\mathfrak{p}$.*

This result coincides with Theorem 5.5. Section 5 contains other results of this flavor. It is worth remarking that there is an analogue of Theorem C for complete intersection homomorphisms, due to Avramov [3, (5.7)].

It turns out that the *finiteness* of G-$\dim_\varphi(M)$ depends only on the R-module structure on M, although its value depends on φ; this is the content of Theorem 7.1 and Example 7.2. One way to understand this result would be to compare Gorenstein dimension over φ to various extensions of the classical G-dimension to R-modules that may not be finite. The last section deals with this problem, where Theorem 8.2 contains the following result; in it $\mathrm{Gfd}_R(M)$ is the Gorenstein flat dimension of M over R.

THEOREM D. *Assume R is a quotient of a Gorenstein ring and let $\varphi\colon R \to S$ be a local homomorphism. For each finite S-module M, one has*

$$\mathrm{Gfd}_R(M) - \mathrm{edim}(\varphi) \leq \text{G-}\dim_\varphi(M) \leq \mathrm{Gfd}_R(M)\,.$$

In particular, G-$\dim_\varphi(M)$ is finite if and only if $\mathrm{Gfd}_R(M)$ is finite.

Foxby, in an unpublished manuscript, has obtained the same conclusion assuming only that the formal fibres of R are Gorenstein. Specializing X to S yields that G-$\dim(\varphi)$ and $\mathrm{Gfd}_R S$ are simultaneously finite. This last result was proved also by Christensen, Frankild, and Holm [14, (5.2)], and our proof of Theorem D draws heavily on their work.

En route to the proof of Theorem D, we obtain results on G-flat dimension that are of independent interest, notably the following Auslander-Buchsbaum type formula for the depth of a module of finite G-flat dimension. It is contained in Theorem 8.7.

THEOREM E. *Let $(R, \mathfrak{m}, k)$ be a local ring and E the injective hull of k. If M is an R-module with $\mathrm{Gfd}_R(M)$ finite, then*

$$\mathrm{depth}_R(M) = \mathrm{depth}\, R - \sup(E \otimes_R^{\mathbf{L}} M)\,.$$

In the preceding discussion, we have focused on modules. However, most of our results are stated and proved for complexes of R-modules. This is often convenient and sometimes necessary, as is the case in Theorem 5.1. Section 2 is mainly a catalogue of standard notions and techniques from the homological algebra of complexes required in this work; most of them can be found in Foxby's notes [16] or Christensen's monograph [12].

2. Background

Let R be a commutative Noetherian ring. A complex of R-modules is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial_{i+1}} X_i \xrightarrow{\partial_i} X_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_i \partial_{i+1} = 0$ for all i. The *supremum*, the *infimum*, and the *amplitude* of a complex X are defined by the following formulas:

$$\sup(X) = \sup\{i \mid \mathrm{H}_i(X) \neq 0\},$$
$$\inf(X) = \inf\{i \mid \mathrm{H}_i(X) \neq 0\},$$
$$\mathrm{amp}(X) = \sup(X) - \inf(X).$$

Note that $\mathrm{amp}(X) = -\infty$ if and only if $\mathrm{H}(X) = 0$. The complex X is *homologically bounded* if $\mathrm{amp}(X) < \infty$, and it is *homologically degreewise finite* if $\mathrm{H}(X)$ is degreewise finite. When $\mathrm{H}(X)$ is both degreewise finite and bounded we say that X is *homologically finite*.

Let X and Y be complexes of R-modules. As is standard, we write $X \otimes_R^{\mathbf{L}} Y$ for the derived tensor product of X and Y, and $\mathbf{R}\operatorname{Hom}_R(X, Y)$ for the derived homomorphisms from X to Y. The symbol "$\simeq$" denotes an isomorphism in the derived category. For details on derived categories and derived functors, the reader may refer to the classics, Hartshorne [20] and Verdier [32], or, for a more recent treatment, to Gelfand and Manin [18].

Let X be a homologically bounded complex of R-modules. A *projective resolution* of X is a complex of projective modules P with $P_i = 0$ for $i \ll 0$ and equipped with an isomorphism $P \simeq X$. Such resolutions exist and can be chosen to be degreewise finite when X is homologically finite. The *projective dimension* of X is

$$\mathrm{pd}_R(X) := \inf\{\sup\{n \mid P_n \neq 0\} \mid P \text{ a projective resolution of } X\}.$$

Thus, if $\mathrm{H}(X) = 0$, then $\mathrm{pd}_R(X)$ is $-\infty$, and hence it is not finite. Flat resolutions and injective resolutions, and the corresponding dimensions $\mathrm{fd}_R(X)$ and $\mathrm{id}_R(X)$, are defined analogously.

The focus of this paper is G-dimension for complexes. In the next few paragraphs, we recall its definition and certain crucial results that allow one to come to grips with it.

2.1. A finite R-module G is *totally reflexive* if

(a) $\operatorname{Ext}_R^i(G, R) = 0$ for all $i > 0$;
(b) $\operatorname{Ext}_R^i(G^*, R) = 0$ for all $i > 0$, where $(\text{-})^*$ denotes $\operatorname{Hom}_R(\text{-}, R)$; and
(c) the canonical map $G \to G^{**}$ is bijective.

Let X be a homologically finite complex of R-modules. A *G-resolution* of X is an isomorphism $G \simeq X$ where G is complex of totally reflexive modules with $G_i = 0$ for $i \ll 0$. A degreewise finite projective resolution of X is also

a G-resolution, since every finite projective module is totally reflexive. The *G-dimension* of X is

$$\text{G-dim}_R(X) := \inf\{\sup\{n \mid G_n \neq 0\} \mid G \text{ is a G-resolution of } X\}.$$

The following paragraphs describe alternative, and often more convenient, ways to detect when a complex has finite G-dimension.

2.2.   A homologically finite complex X of R-modules is *reflexive* if
(a) $\mathbf{R}\operatorname{Hom}_R(X, R)$ is homologically bounded; and
(b) the canonical biduality morphism below is an isomorphism:

$$\delta_X^R \colon X \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X, R), R).$$

This notion is relevant to this article because of the next result, based on an unpublished work of Foxby; see [12, (2.3.8)] and [33, (2.7)].

2.3.   *The complex X is reflexive if and only if* $\text{G-dim}_R(X) < \infty$. *When X is reflexive,* $\text{G-dim}_R(X) = -\inf(\mathbf{R}\operatorname{Hom}_R(X, R))$.

Using this characterization, it is easy to verify the base change formula below; Christensen [11, (5.11)] has established a much stronger statement.

2.4.   *Let $R \to S$ be a flat local homomorphism and X a homologically finite complex of R-modules. Then* $\text{G-dim}_R(X) = \text{G-dim}_S(X \otimes_R S)$.

Henceforth, R is a local ring, where "local" means "local and Noetherian".

2.5.   A *dualizing complex* for R is a homologically finite complex of R-modules D of finite injective dimension such that the natural map $R \to \mathbf{R}\operatorname{Hom}_R(D, D)$ is an isomorphism. When R is a homomorphic image of a Gorenstein ring, for example, when R is complete, it possesses a dualizing complex.

Assume that R possesses a dualizing complex D. The *Auslander category of R*, denoted by $\mathcal{A}(R)$, is the full subcategory of the derived category of R whose objects are the homologically bounded complexes X such that
(a) $D \otimes_R^{\mathbf{L}} X$ is homologically bounded; and
(b) the canonical morphism below is an isomorphism

$$\gamma_X \colon X \to \mathbf{R}\operatorname{Hom}_R(D, D \otimes_R^{\mathbf{L}} X).$$

It should be emphasized that a complex can be in the Auslander category of R without being homologically finite. Those that are homologically finite are identified by the following result; see [12, (3.1.10)] for a proof.

2.6.   *Let X be a homologically finite complex. Then X is in $\mathcal{A}(R)$ if and only if* $\text{G-dim}_R(X) < \infty$.

The various homological dimensions are related to another invariant: depth.

2.7.   Let K be the Koszul complex on a generating sequence of length n for the maximal ideal of R. The *depth* of X is defined to be

$$\operatorname{depth}_R(X) = n - \sup(K \otimes_R X).$$

It is independent of the choice of generating sequence and may be calculated via the vanishing of appropriate local cohomology or Ext-modules [17, (2.1)].

For the basic properties of depth, we refer to [17]. However, there seems to be no available reference for the following result.

2.8. Lemma.   *Let $\varphi \colon R \to S$ be a local homomorphism and X a complex of S-modules. If $\operatorname{H}(X)$ is degreewise finite over R, then $\operatorname{depth}_S(X) = \operatorname{depth}_R(X)$.*

Proof. Let K denote the Koszul complex on a set of n generators for the maximal ideal of R. Note that $\operatorname{pd}_S(K \otimes_R S) = n$. Thus

$$\begin{aligned}
\operatorname{depth}_S(K \otimes_R X) &= \operatorname{depth}_S((K \otimes_R S) \otimes_S X) \\
&= \operatorname{depth}_S(X) - \operatorname{pd}_S(K \otimes_R S) \\
&= \operatorname{depth}_S(X) - n
\end{aligned}$$

where the second equality is by the Auslander-Buchsbaum formula [17, (2.4)]. Now, $\operatorname{H}(K \otimes_R X)$ is degreewise finite over R and is annihilated by the maximal ideal of R; see, for instance, [23, (1.2)]. Hence, each $\operatorname{H}_i(K \otimes_R X)$ has finite length over R, and, therefore, over S. In particular, by [17, (2.7)] one has $\operatorname{depth}_S(K \otimes_R X) = -\sup(K \otimes_R X)$. Combining this with the displayed formulas above yields that

$$\operatorname{depth}_S(X) = n + \operatorname{depth}_S(K \otimes_R X) = n - \sup(K \otimes_R X) = \operatorname{depth}_R(X).$$

This is the desired equality.   □

In [17, (3.1)], Foxby and Iyengar extend Iversen's Amplitude Inequality; we require a slight reformulation of their result.

2.9. Theorem.   *Let S be a local ring and let P be a homologically finite complex of S-modules with $\operatorname{pd}_S(P)$ finite. For each homologically degreewise finite complex X of S-modules, one has*

$$\operatorname{amp}(X) \leq \operatorname{amp}(X \otimes_S^{\mathbf{L}} P) \leq \operatorname{amp}(X) + \operatorname{pd}_S(P) - \inf(P).$$

In particular, $\operatorname{amp}(X)$ is finite if and only if $\operatorname{amp}(X \otimes_S^{\mathbf{L}} P)$ is finite.

Proof. The inequality on the left is contained in [17, (3.1)], while the one on the right is by [16, (7.28), (8.17)].   □

Here is a corollary; one can give a direct proof when the map α is between complexes that are homologically bounded to the right.

2.10. PROPOSITION. *Let S be a local ring, P a homologically finite complex of S-modules with $\mathrm{pd}_S(P)$ finite, and let α be a morphism of homologically degreewise finite complexes. Then α is an isomorphism if and only if the induced map $\alpha \otimes_S^{\mathbf{L}} P$ is an isomorphism.*

Proof. Let $\mathrm{C}(\alpha)$ and $\mathrm{C}(\alpha \otimes_S^{\mathbf{L}} P)$ denote the mapping cones of α and $\alpha \otimes_S^{\mathbf{L}} P$, respectively. The homology long exact sequence arising from mapping cones yields that $\mathrm{H}(\mathrm{C}(\alpha))$ is degreewise finite. Observe that $\mathrm{C}(\alpha \otimes_S^{\mathbf{L}} P) = \mathrm{C}(\alpha) \otimes_S^{\mathbf{L}} P$. By the previous theorem, $\mathrm{H}(\mathrm{C}(\alpha)) = 0$ if and only if $\mathrm{H}(\mathrm{C}(\alpha \otimes_S^{\mathbf{L}} P)) = 0$. $\square$

It is well known that the derived tensor product of two homologically finite complexes is homologically finite when one of them has finite projective dimension. In the sequel we require the following slightly more general result, contained in [4, (4.7.F)]. The proof is short and simple, and bears repetition.

2.11. LEMMA. *Let $\sigma \colon S \to T$ be a local homomorphism and let X and P be homologically finite complexes of modules over S and T, respectively. If $\mathrm{fd}_S(P)$ is finite, then the complex of T-modules $X \otimes_S^{\mathbf{L}} P$ is homologically finite.*

Proof. Replacing X by a soft truncation, one may assume that X is bounded; see, for example, [12, p. 165]. With F a bounded flat resolution of P over S, the complex $X \otimes_S^{\mathbf{L}} P$ is isomorphic to $X \otimes_S F$, which is bounded. Thus, $X \otimes_S^{\mathbf{L}} P$ is homologically bounded. As to its degreewise finiteness, let Y and Q be minimal free resolutions of X and P over S and T, respectively. Then $X \otimes_S^{\mathbf{L}} P$ is isomorphic to $Y \otimes_S Q$, which is a complex of finite T-modules. Therefore, the same is true of its homology, since T is Noetherian. $\square$

3. G-dimension over a local homomorphism

In this section we introduce the G-dimension over a local homomorphism and document some of its basic properties. We begin by recalling the construction of Cohen factorizations of local homomorphisms as introduced by Avramov, Foxby, and B. Herzog [8].

3.1. Given a local homomorphism $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$, the *embedding dimension* and *depth of φ* are

$$\mathrm{edim}(\varphi) := \mathrm{edim}(S/\mathfrak{m}S) \quad \text{and} \quad \mathrm{depth}(\varphi) := \mathrm{depth}(S) - \mathrm{depth}(R).$$

A *regular* (respectively, *Gorenstein*) *factorization* of φ is a diagram of local homomorphisms, $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$, where $\varphi = \varphi'\dot{\varphi}$, with $\dot{\varphi}$ flat, the closed fibre $R'/\mathfrak{m}R'$ regular (respectively, Gorenstein) and $\varphi' \colon R' \to S$ surjective.

Let $\widehat{S}$ denote the completion of S at its maximal ideal and $\iota\colon S \to \widehat{S}$ be the canonical inclusion. By [8, (1.1)] the composition $\grave{\varphi} = \iota\varphi$ admits a regular factorization $R \to R' \to \widehat{S}$ with R' complete. Such a regular factorization is said to be a *Cohen factorization* of $\grave{\varphi}$.

The result below is analogous to [7, (4.3)]. Here, and elsewhere, we write $\widehat{X}$ for $X \otimes_S \widehat{S}$ when X is a complex of S-modules.

3.2. THEOREM. *Let $\varphi\colon R \to S$ be a local homomorphism and X a homologically finite complex of S-modules. If $R \xrightarrow{\grave{\varphi}_1} R_1 \xrightarrow{\varphi'_1} \widehat{S}$ and $R \xrightarrow{\grave{\varphi}_2} R_2 \xrightarrow{\varphi'_2} \widehat{S}$ are Cohen factorizations of $\grave{\varphi}$, then*

$$\text{G-dim}_{R_1}(\widehat{X}) - \text{edim}(\grave{\varphi}_1) = \text{G-dim}_{R_2}(\widehat{X}) - \text{edim}(\grave{\varphi}_2).$$

Proof. Theorem [8, (1.2)] provides a commutative diagram
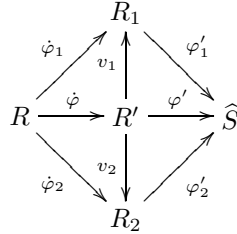
where $\varphi'\grave{\varphi}$ is a third Cohen factorization of $\grave{\varphi}$, and each v_i is surjective with kernel generated by an R'-regular sequence whose elements are linearly independent over $R'/\mathfrak{m}'$ in $\mathfrak{m}'/((\mathfrak{m}')^2 + \mathfrak{m}R')$. Here $\mathfrak{m}$ and $\mathfrak{m}'$ are the maximal ideals of R and R', respectively. Let c_i denote the length of a regular sequence generating $\ker(v_i)$. For $i = 1, 2$ one has that

$$\text{G-dim}_{R'}(\widehat{X}) - \text{edim}(\grave{\varphi}) = [\text{G-dim}_{R_i}(\widehat{X}) + c_i] - [\text{edim}(R_i/\mathfrak{m}R_i) + c_i]$$

$$= \text{G-dim}_{R_i}(\widehat{X}) - \text{edim}(\grave{\varphi}_i),$$

where [12, (2.3.12)] gives the first equality. This gives the desired result. $\square$

3.3. DEFINITION. Let $\varphi\colon R \to S$ be a local homomorphism and X a homologically finite complex of S-modules. Let $R \xrightarrow{\grave{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen factorization of $\grave{\varphi}$. The *G-dimension of X over φ* is the quantity

$$\text{G-dim}_\varphi(X) := \text{G-dim}_{R'}(\widehat{X}) - \text{edim}(\grave{\varphi}).$$

Theorem 3.2 shows that $\text{G-dim}_\varphi(X)$ does not depend on the choice of Cohen factorization. Note that $\text{G-dim}_\varphi(X) \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, and also that $\text{G-dim}_\varphi(X) = -\infty$ if and only if $\text{H}(X) = 0$.

The *G-dimension of φ* is defined to be

$$\text{G-dim}(\varphi) := \text{G-dim}_\varphi(S).$$

It is clear from the definitions that the corresponding notion of the finiteness of G-dim(φ) agrees with that in [7].

Here are some properties of the G-dim$_\varphi(\text{-})$.

3.4. PROPERTIES. Fix a local homomorphism $\varphi \colon R \to S$, a Cohen factorization $R \to R' \to \widehat{S}$ of $\grave{\varphi}$, and a homologically finite complex X of S-modules.

3.4.1. Let $\widehat{\varphi} \colon \widehat{R} \to \widehat{S}$ denote the map induced on completions. One has

$$\text{G-dim}_\varphi(X) = \text{G-dim}_{\grave{\varphi}}(\widehat{X}) = \text{G-dim}_{\widehat{\varphi}}(\widehat{X}).$$

More generally, let I and J be proper ideals of R and S, respectively, with $IS \subseteq J$, and let $\widetilde{R}$ and $\widetilde{S}$ denote the respective completions. With $\widetilde{\varphi} \colon \widetilde{R} \to \widetilde{S}$ the induced map, one has

$$\text{G-dim}_\varphi(X) = \text{G-dim}_{\widetilde{\varphi}}(\widetilde{S} \otimes_S X).$$

This is because the completion of $\widetilde{\varphi}$ at the maximal ideal of $\widetilde{S}$ is $\widehat{\varphi}$.

3.4.2. If $X \simeq X' \oplus X''$, then $\text{G-dim}_\varphi(X) = \max\{\text{G-dim}_\varphi(X'), \text{G-dim}_\varphi(X'')\}$; this follows from the corresponding property of the classical G-dimension.

3.4.3. If φ has a regular factorization $R \xrightarrow{\varphi_1} R_1 \xrightarrow{\varphi'} S$, then

$$\text{G-dim}_\varphi(X) = \text{G-dim}_{R_1}(X) - \text{edim}(\varphi_1),$$

because the diagram $R \xrightarrow{\grave{\varphi_1}} \widehat{R_1} \xrightarrow{\widehat{\varphi'}} \widehat{S}$ is a Cohen factorization of $\grave{\varphi}$.

3.4.4. If φ is surjective, then $R \xrightarrow{=} R \xrightarrow{\varphi} S$ is a regular factorization, so

$$\text{G-dim}_\varphi(X) = \text{G-dim}_R(X).$$

Corollary 7.3 below generalizes this to the case when $\text{H}(X)$ is finite over R.

The following theorem is an extension of the Auslander-Bridger formula, which is the special case $\varphi = \text{id}_R$.

3.5. THEOREM. *Let $\varphi \colon R \to S$ be a local homomorphism and X a homologically finite complex of S-modules. If $\text{G-dim}_\varphi(X) < \infty$, then*

$$\text{G-dim}_\varphi(X) = \text{depth}(R) - \text{depth}_S(X)$$

Proof. Let $R \to R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen factorization of $\grave{\varphi}$, and let $\mathfrak{m}$ be the maximal ideal of R. The classical Auslander-Bridger formula gives the first

of the following equalities; the flatness of $R \to R'$ and the surjectivity of φ' imply the second; the regularity of $R'/\mathfrak{m}R'$ yields the third.

$$\text{G-dim}_{R'}(\widehat{X}) = \text{depth}(R') - \text{depth}_{R'}(\widehat{X})$$
$$= [\text{depth}(R) + \text{depth}(R'/\mathfrak{m}R')] - \text{depth}_{\widehat{S}}(\widehat{X})$$
$$= \text{depth}(R) - \text{depth}_S(X) + \text{edim}(R'/\mathfrak{m}R').$$

This gives the desired equality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

As in the classical case, described in 2.6, when R has a dualizing complex one can detect finiteness of $\text{G-dim}_\varphi(\text{-})$ in terms of membership in the Auslander category of R.

3.6. PROPOSITION. *Let* $\varphi \colon R \to S$ *be a local homomorphism and* $R \xrightarrow{\dot{\varphi}}$ $R' \xrightarrow{\varphi'} \widehat{S}$ *a Cohen factorization of* $\dot{\varphi}$. *The following conditions are equivalent for each homologically finite complex* X *of* S-*modules.*

(a) $\text{G-dim}_\varphi(X) < \infty$.
(b) $\text{G-dim}_{R'}(\widehat{X}) < \infty$.
(c) $\widehat{X}$ *is in* $\mathcal{A}(R')$.
(d) $\widehat{X}$ *is in* $\mathcal{A}(\widehat{R})$.

When R possesses a dualizing complex, these conditions are equivalent to:

(e) X *is in* $\mathcal{A}(R)$.

Proof. Indeed, (a) $\iff$ (b) by definition, while (b) $\iff$ (c) by [12, (3.1.10)]. Moving on, (c) $\iff$ (d) is contained in [7, (3.7.b)], and, when R has a dualizing complex, the equivalence of (d) and (e) is [7, (3.7.a)]. $\quad\square$

Now we turn to the behavior of G-dimension with respect to localizations. Recall that, given a prime ideal $\mathfrak{p}$ and a totally reflexive R-module G, the $R_{\mathfrak{p}}$-module $G_{\mathfrak{p}}$ is totally reflexive. From this it is clear that for any homologically finite complex W, one has $\text{G-dim}_{R_{\mathfrak{p}}}(W_{\mathfrak{p}}) \leq \text{G-dim}_R(W)$; see [12, (2.3.11)]. For G-dimensions over φ, we know only the following weaker result; see also [9, (10.2)]. Its proof is omitted for it is verbatim that of [7, (4.5)], which is the special case $X = S$; only, one uses 3.6 instead of [7, (4.3)].

3.7. PROPOSITION. *Let* $\varphi \colon R \to S$ *be a local homomorphism, X a homologically finite complex of S-modules. Let $\mathfrak{q}$ be a prime ideal of S and $\varphi_{\mathfrak{q}}$ the local homomorphism* $R_{\mathfrak{q} \cap R} \to S_{\mathfrak{q}}$.

If $\text{G-dim}_\varphi(X) < \infty$, *then* $\text{G-dim}_{\varphi_{\mathfrak{q}}}(X_{\mathfrak{q}}) < \infty$ *under each of the conditions:*

(1) φ *is essentially of finite type; or*
(2) R *has Gorenstein formal fibres.* $\qquad\qquad\qquad\qquad$ $\square$

The next result shows that $\text{G-dim}_\varphi(X)$ can be computed via any *Gorenstein* factorization of φ, when such a factorization exists; see Definition 3.1.

When the factorization in the statement is regular, the equation becomes $\text{G-dim}_\varphi(X) = \text{G-dim}_{R'}(X) - \text{edim}(\dot\varphi)$; compare this with Definition 3.3.

3.8. PROPOSITION. *Let $\varphi\colon R \to S$ be a local homomorphism and X a homologically finite complex of S-modules. If φ possesses a Gorenstein factorization $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$, then*

$$\text{G-dim}_\varphi(X) = \text{G-dim}_{R'}(X) - \text{depth}(\dot\varphi).$$

Proof. One may assume that $\text{H}(X) \neq 0$. It is straightforward to verify that the diagram $R \to \widehat{R'} \to \widehat{S}$ is a Gorenstein factorization. It follows from [7, (3.7)] that $\widehat{X}$ is in $\mathcal{A}(\widehat{R})$ exactly when $\widehat{X}$ is in $\mathcal{A}(\widehat{R'})$, and, by Proposition 3.6, this implies that $\text{G-dim}_\varphi(X)$ is finite exactly when $\text{G-dim}_{R'}(X)$ is finite. So one may assume that both the numbers in question are finite. The Auslander-Bridger formula 3.5, and the fact that $\text{depth}_S(X) = \text{depth}_{R'}(X)$, give the first of the following equalities:

$$\begin{aligned}\text{G-dim}_{R'}(X) &= \text{G-dim}_\varphi(X) + [\text{depth}(R') - \text{depth}(R)]\\ &= \text{G-dim}_\varphi(X) + \text{depth}(\dot\varphi).\end{aligned}$$

The second equality is by definition. $\qquad\square$

4. Projective dimension

In this section we introduce a new invariant: projective dimension over a local homomorphism. To begin with, one has the following proposition. Its proof is similar to that of Theorem 3.2, and hence it is omitted.

4.1. PROPOSITION. *Let $\varphi\colon R \to S$ be a local homomorphism and X a homologically finite complex of S-modules. If $R \xrightarrow{\dot\varphi_1} R_1 \xrightarrow{\varphi_1'} \widehat{S}$ and $R \xrightarrow{\dot\varphi_2} R_2 \xrightarrow{\varphi_2'} \widehat{S}$ are Cohen factorizations of $\dot\varphi$, then*

$$\text{pd}_{R_1}(\widehat{X}) - \text{edim}(\dot\varphi_1) = \text{pd}_{R_2}(\widehat{X}) - \text{edim}(\dot\varphi_2). \qquad\square$$

This leads to the following definition:

4.2. DEFINITION. *Let $\varphi\colon R \to S$ be a local homomorphism and X a homologically finite complex of S-modules. The projective dimension of X over φ* is the quantity

$$\text{pd}_\varphi(X) := \text{pd}_{R'}(\widehat{X}) - \text{edim}(\dot\varphi)$$

for some Cohen factorization $R \to R' \to \widehat{S}$ of $\dot\varphi$. The *projective dimension of φ* is defined to be

$$\text{pd}(\varphi) := \text{pd}_\varphi(S).$$

The first remark concerning this invariant is that there is an "Auslander-Buchsbaum formula", which can be verified along the lines of its G-dimension counterpart, Theorem 3.5.

4.3. PROPERTY. If $\mathrm{pd}_\varphi(X) < \infty$, then

$$\mathrm{pd}_\varphi(X) = \mathrm{depth}(R) - \mathrm{depth}_S(X)$$

Other basic rules that govern the behavior of this invariant can be read from [8], although it was not defined there explicitly. For instance, [8, (3.2)] (rather, its extension to complexes; see [9, (2.5)]) translates to:

4.4. PROPERTY. There are inequalities

$$\mathrm{fd}_R(X) - \mathrm{edim}\,\varphi \le \mathrm{pd}_\varphi(X) \le \mathrm{fd}_R(X)\,.$$

In particular, the finiteness of $\mathrm{pd}_\varphi(X)$ is independent of S and φ.

One can interpret the difference between $\mathrm{fd}_R(X)$ and $\mathrm{pd}_\varphi(X)$ in terms of appropriate depths:

4.5. PROPOSITION. *Let* $\varphi\colon R \to S$ *be a local homomorphism and* X *a homologically finite complex of S-modules. Then*

$$\mathrm{pd}_\varphi(X) = \mathrm{fd}_R(X) + \mathrm{depth}_R(X) - \mathrm{depth}_S(X).$$

Proof. Indeed, by Property 4.4, we may assume that both $\mathrm{fd}_R(X)$ and $\mathrm{pd}_\varphi(X)$ are finite. Now, the first equality below is given by [4, (5.5)], and the second is due to [23, (2.1)].

$$\mathrm{fd}_R(X) = \sup(X \otimes_R^{\mathbf{L}} k) = \mathrm{depth}(R) - \mathrm{depth}_R(X).$$

The Auslander-Buchsbaum formula 4.3 gives the desired formula. $\square$

The G-dimension of a finite module, or a complex, is bounded above by its projective dimension. The same behavior carries over to modules and complexes over φ.

4.6. PROPOSITION. *Let* $\varphi\colon R \to S$ *be a local homomorphism. For each homologically finite complex X of S-modules, one has*

$$\mathrm{G\text{-}dim}_\varphi(X) \le \mathrm{pd}_\varphi(X);$$

equality holds when $\mathrm{pd}_\varphi(X) < \infty$.

Proof. Let $R \xrightarrow{\dot\varphi} R' \to \widehat{S}$ be a Cohen factorization of $\dot\varphi$. Then

$$\mathrm{G\text{-}dim}_\varphi(X) = \mathrm{G\text{-}dim}_{R'}(\widehat{X}) - \mathrm{edim}(\dot\varphi) \le \mathrm{pd}_{R'}(\widehat{X}) - \mathrm{edim}(\dot\varphi) = \mathrm{pd}_\varphi(X)$$

with equality if $\mathrm{pd}_{R'}(\widehat{X})$ is finite; see [12, (2.3.10)]. $\square$

Further results concerning $\mathrm{pd}_\varphi(\text{-})$ are given toward the end of the next section. One can introduce also Betti numbers and Poincaré series over local homomorphisms; an in-depth analysis of these and related invariants is carried out in [9].

5. Ascent and descent of G-dimension

The heart of this section, and indeed of this paper, is the following theorem. It is a vast generalization of a stability result of Yassemi [33, (2.15)], and contains Theorem B from the introduction.

5.1. THEOREM. *Let $\varphi\colon R \to S$ and $\sigma\colon S \to T$ be local homomorphisms. Let P be a complex of T-modules that is homologically finite with $\mathrm{pd}_\sigma(P)$ finite. For every homologically finite complex X of S-modules*

$$\mathrm{G\text{-}dim}_{\sigma\varphi}(X \otimes_S^{\mathbf{L}} P) = \mathrm{G\text{-}dim}_\varphi(X) + \mathrm{pd}_\sigma(P).$$

In particular, $\mathrm{G\text{-}dim}_{\sigma\varphi}(X \otimes_S^{\mathbf{L}} P)$ and $\mathrm{G\text{-}dim}_\varphi(X)$ are simultaneously finite.

The theorem is proved in 5.10, toward the end of the section. It is worth remarking that the displayed formula is *not* an immediate consequence of the finiteness of the G-dimensions in question and appropriate Auslander-Bridger formulas. What is missing is an extension of the Auslander-Buchsbaum formula 4.3; namely, under the hypotheses of the theorem above

$$\mathrm{pd}_\sigma(P) = \mathrm{depth}_S(X) - \mathrm{depth}_T(X \otimes_S^{\mathbf{L}} P).$$

It is not hard to deduce this equality from [17, (2.4)], using Cohen factorizations; see the argument in 5.10.

We draw a few corollaries that illustrate the power of Theorem 5.1. The first one is just the special case $X = S$ and $P = T$.

5.2. THEOREM. *Let $\varphi\colon R \to S$ and $\sigma\colon S \to T$ be local homomorphisms with $\mathrm{pd}(\sigma)$ finite. Then*

$$\mathrm{G\text{-}dim}(\sigma\varphi) = \mathrm{G\text{-}dim}(\varphi) + \mathrm{pd}(\sigma).$$

In particular, $\mathrm{G\text{-}dim}(\sigma\varphi)$ is finite if and only if $\mathrm{G\text{-}dim}(\varphi)$ is finite. $\square$

The following example illustrates that the hypothesis on σ cannot be weakened to "$\mathrm{G\text{-}dim}(\sigma)$ finite". A similar example is constructed in [1, p. 931].

5.3. EXAMPLE. Let R be a local, Cohen-Macaulay ring with canonical module ω. Set $S = R \ltimes \omega$, the "idealization" of ω, and $\varphi\colon R \to S$ the canonical inclusion. Let $T = S/\omega \cong R$ with $\sigma\colon S \to T$ the natural surjection.

Now, $\sigma\varphi = \mathrm{id}^R$, hence $\mathrm{G\text{-}dim}(\sigma\varphi) = 0$, for example, by Proposition 4.6; also, S is Gorenstein [10, (3.3.6)], so $\mathrm{G\text{-}dim}(\sigma)$ is finite.

We claim that $\mathrm{G\text{-}dim}(\varphi)$ is finite if and only if R is Gorenstein. Indeed, $\mathrm{G\text{-}dim}(\varphi)$ and $\mathrm{G\text{-}dim}_R(S)$ are simultaneously finite, by Corollary 7.3.

From 3.4.2 it follows that G-$\dim_R(S) < \infty$ if and only if G-$\dim_R(\omega) < \infty$. The finiteness of G-$\dim_R(\omega)$ is equivalent to R being Gorenstein [12, (3.4.12)].

As noted in the introduction, Theorem 5.2 allows one to extend certain results of Avramov and Foxby on (quasi-)Gorenstein homomorphisms. In order to describe these, and because they are required in the sequel, we recall the relevant notions.

5.4. Let R be a local ring with residue field k. The *Bass series of R* is the formal power series $I_R(t) = \sum_i \mu_R^i(R)t^i$, where $\mu_R^i(R) = \operatorname{rank}_k \operatorname{Ext}_R^i(k, R)$. An important property of the Bass series is that R is Gorenstein if and only if $I_R(t)$ is a polynomial [28, (18.1)].

Let $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of finite G-dimension. Let $I_\varphi(t)$ denote the *Bass series of φ*, introduced in [7, Section 7]. The Bass series is a formal Laurent series with nonnegative integer coefficients and satisfies the equality

$$(\dagger) \qquad\qquad I_R(t)I_\varphi(t) = I_S(t).$$

Let $\sigma \colon (S, \mathfrak{n}) \to (T, \mathfrak{p})$ also be a homomorphism of finite G-dimension. Assuming G-$\dim(\sigma\varphi)$ is finite as well, it follows from ($\dagger$) that

$$(\ddagger) \qquad\qquad I_{\sigma\varphi}(t) = I_\sigma(t)I_\varphi(t).$$

The homomorphism φ is said to be *quasi-Gorenstein at $\mathfrak{n}$* if $I_\varphi(t)$ is a Laurent *polynomial*. When pd$(\varphi) < \infty$ and φ is quasi-Gorenstein at $\mathfrak{n}$, one says that φ is *Gorenstein at $\mathfrak{n}$*; see [7, (7.7.1)].

A noteworthy aspect of the class of such homomorphisms is that it is closed under compositions: if φ and σ are quasi-Gorenstein at $\mathfrak{n}$ and $\mathfrak{p}$, respectively, then $\sigma\varphi$ is quasi-Gorenstein at $\mathfrak{p}$. This follows from [7, (7.10)] and ($\ddagger$) above.

The result below is a decomposition theorem for Gorenstein and quasi-Gorenstein homomorphisms; it is Theorem C announced in the introduction.

5.5. THEOREM.   *Let $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $\sigma \colon (S, \mathfrak{n}) \to (T, \mathfrak{p})$ be local homomorphisms with* pd(σ) *finite. If $\sigma\varphi$ is (quasi-)Gorenstein at $\mathfrak{p}$, then φ is (quasi-)Gorenstein at $\mathfrak{n}$ and σ is Gorenstein at $\mathfrak{p}$.*

Proof. Assume that $\sigma\varphi$ is quasi-Gorenstein at $\mathfrak{p}$; so it has finite G-dimension, and $I_{\sigma\varphi}(t)$ is a Laurent polynomial. Now, G-$\dim(\varphi)$ is finite, by Theorem 5.2, as is G-$\dim(\sigma)$, by hypothesis, so equality ($\ddagger$) in 5.4 applies to yield an equality of formal Laurent series

$$I_{\sigma\varphi}(t) = I_\sigma(t)I_\varphi(t)$$

In particular, $I_\sigma(t)$ and $I_\varphi(t)$ are Laurent polynomials as well. Thus, both σ and φ are quasi-Gorenstein at the appropriate maximal ideals. Moreover, σ is Gorenstein because pd(σ) is finite.

Suppose that $\sigma\varphi$ is Gorenstein at $\mathfrak{p}$. Since $\mathrm{pd}(\sigma\varphi)$ and $\mathrm{pd}(\sigma)$ are both finite, [17, (3.2)], in conjunction with Proposition 4.5, yields that $\mathrm{pd}(\varphi)$ is finite. The already established part of the theorem gives the desired conclusion.   □

The next theorem generalizes another stability result of Yassemi [33, (2.14)].

5.6. THEOREM.   *Let $\varphi\colon R \to S$ be a local homomorphism and P a homologically finite complex of S-modules with $\mathrm{pd}_S(P)$ finite. For every homologically finite complex X of S-modules*

$$\mathrm{G\text{-}dim}_\varphi(\mathbf{R}\,\mathrm{Hom}_S(P,X)) = \mathrm{G\text{-}dim}_\varphi(X) - \inf(P).$$

Thus, $\mathrm{G\text{-}dim}_\varphi(X)$ and $\mathrm{G\text{-}dim}_\varphi(\mathbf{R}\,\mathrm{Hom}_S(P,X))$ are simultaneously finite.

Proof.   The tensor evaluation morphism $X \otimes_S^{\mathbf{L}} \mathbf{R}\,\mathrm{Hom}_S(P,S) \to \mathbf{R}\,\mathrm{Hom}_S(P,X)$ is an isomorphism, as P has finite projective dimension. Thus

$$\begin{aligned}
\mathrm{G\text{-}dim}_\varphi(\mathbf{R}\,\mathrm{Hom}_S(P,X)) &= \mathrm{G\text{-}dim}_\varphi(X \otimes_S^{\mathbf{L}} \mathbf{R}\,\mathrm{Hom}_S(P,S)) \\
&= \mathrm{G\text{-}dim}_\varphi(X) + \mathrm{pd}_S(\mathbf{R}\,\mathrm{Hom}_S(P,S)) \\
&= \mathrm{G\text{-}dim}_\varphi(X) - \inf(P),
\end{aligned}$$

where the second equality follows from Theorem 5.1 because $\mathbf{R}\,\mathrm{Hom}_S(P,S)$ has finite projective dimension over S.   □

Next we record the analogue of Theorem 5.1 for projective dimension; its proof is postponed to 5.11.

5.7. THEOREM.   *Let $\varphi\colon R \to S$ and $\sigma\colon S \to T$ be local homomorphisms. Let P be a complex of T-modules that is homologically finite with $\mathrm{pd}_\sigma(P)$ finite. For every homologically finite complex X of S-modules*

$$\mathrm{pd}_{\sigma\varphi}(X \otimes_S^{\mathbf{L}} P) = \mathrm{pd}_\varphi(X) + \mathrm{pd}_\sigma(P).$$

In particular, $\mathrm{pd}_{\sigma\varphi}(X \otimes_S^{\mathbf{L}} P)$ and $\mathrm{pd}_\varphi(X)$ are simultaneously finite.

Finally, here is the analogue of Theorem 5.6; it can be deduced from 5.7 in the same way that 5.6 was deduced from 5.1.

5.8. THEOREM.   *Let $\varphi\colon R \to S$ be a local homomorphism and P a homologically finite complex of S-modules with $\mathrm{pd}_S(P)$ finite. For every homologically finite complex X of S-modules*

$$\mathrm{pd}_\varphi(\mathbf{R}\,\mathrm{Hom}_S(P,X)) = \mathrm{pd}_\varphi(X) - \inf(P).$$

In particular, $\mathrm{pd}_\varphi(X)$ and $\mathrm{pd}_\varphi(\mathbf{R}\,\mathrm{Hom}_S(P,X))$ are simultaneously finite.   □

The proof of Theorem 5.1 uses a convenient construction, essentially given in [8], of Cohen factorizations of compositions of local homomorphisms.

5.9.   Let $R \xrightarrow{\varphi} S \xrightarrow{\sigma} T$ be local homomorphisms, and let

$$R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S \quad \text{and} \quad R' \xrightarrow{\dot{\rho}} R'' \xrightarrow{\rho'} T$$

be regular factorizations of φ and $\sigma\varphi'$, respectively.   The map ρ' factors through the tensor product $S' = R'' \otimes_{R'} S$ giving the commutative diagram
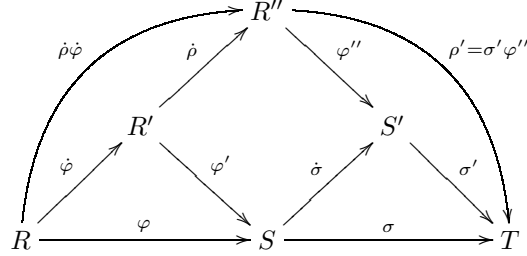
where $\dot{\sigma}$ and φ'' are the natural maps to the tensor products.   Then the diagrams $S \to S' \to T$, $R' \to R'' \to S'$, and $R \to R'' \to T$ are regular factorizations with

$$\operatorname{edim}(\dot{\sigma}) = \operatorname{edim}(\dot{\rho}) \quad \text{and} \quad \operatorname{edim}(\dot{\rho}\dot{\varphi}) = \operatorname{edim}(\dot{\rho}) + \operatorname{edim}(\dot{\varphi}).$$

Indeed, by flat base change, $\dot{\sigma}$ is flat and has closed fiber $S' \otimes_S l = R'' \otimes_{R'} l$, which is regular. Here l is the common residue field of R' and S. This tells us that $S \to S' \to T$ is a regular factorization and that $\operatorname{edim}(\dot{\sigma}) = \operatorname{edim}(\dot{\rho})$.

The diagram $R' \to R'' \to S'$ is a regular factorization because $\dot{\rho}$ is flat with a regular closed fibre, by hypothesis, and φ'' is surjective, by base change.

As to the diagram $R \to R'' \to T$, let $\mathfrak{m}$ and $\mathfrak{m}'$ denote the maximal ideals of R and R', respectively. The induced map $R'/\mathfrak{m}R' \to R''/\mathfrak{m}R''$ is flat with closed fibre $R''/\mathfrak{m}'R''$. Since $R'/\mathfrak{m}R'$ and $R''/\mathfrak{m}'R''$ are both regular, the same is true of $R''/\mathfrak{m}R''$, by [10, (2.2.12)]. Thus, $R \to R'' \to T$ is a regular factorization. Furthermore, it is stated explicitly in the proof of *loc. cit.* that $\operatorname{edim}(R''/\mathfrak{m}R'') = \operatorname{edim}(R'/\mathfrak{m}R') + \operatorname{edim}(R''/\mathfrak{m}'R'')$, which explains the second formula above.

5.10.   *Proof of Theorem 5.1.* Note that $X \otimes_S^{\mathbf{L}} P$ is homologically finite over T by Lemma 2.11, so one may speak of its G-dimension over $\sigma\varphi$. Passing to the completions of S and T at their respective maximal ideals, and replacing X and P by $\widehat{S} \otimes_S X$ and $\widehat{T} \otimes_T P$, respectively, one may assume that S and T are complete. In doing so, one uses the isomorphism

$$(\widehat{S} \otimes_S X) \otimes_{\widehat{S}}^{\mathbf{L}} (\widehat{T} \otimes_T P) \simeq \widehat{T} \otimes_T (X \otimes_S^{\mathbf{L}} P).$$

The next step is the reduction to the case where φ and σ are surjective. To achieve this, take Cohen factorizations $R \to R' \to S$ and $R' \to R'' \to T$, and expand to a commutative diagram as in 5.9.

Let $X' = S' \otimes_S X$. Since $S' = R'' \otimes_{R'} S$, by construction, $X' \cong R'' \otimes_{R'} X$ and hence $X' \otimes^{\mathbf{L}}_{S'} P \simeq X \otimes^{\mathbf{L}}_S P$. Since $R' \to R''$ is faithfully flat, 2.4 yields

$$\text{G-dim}_{R'}(X) = \text{G-dim}_{R''}(X').$$

The preceding equality, in conjunction with those in 5.9, yields

$$\text{pd}_\sigma(P) = \text{pd}_{S'}(P) - \text{edim}(\dot{\rho}),$$
$$\text{G-dim}_\varphi(X) = \text{G-dim}_{R''}(X') - \text{edim}(\dot{\varphi}),$$
$$\text{G-dim}_{\sigma\varphi}(X \otimes^{\mathbf{L}}_S P) = \text{G-dim}_{R''}(X' \otimes^{\mathbf{L}}_{S'} P) - \text{edim}(\dot{\rho}) - \text{edim}(\dot{\varphi}).$$

Therefore, it suffices to verify the identity for the diagram $R'' \to S' \to T$ and complexes X' and P. This places us in the situation where $R \to S$ is surjective, P is homologically finite over R, and then the equality we seek is

$$\text{G-dim}_R(X \otimes^{\mathbf{L}}_S P) = \text{G-dim}_R(X) + \text{pd}_S(P).$$

It suffices to prove that the G-dimensions over R of X and of $X \otimes^{\mathbf{L}}_S P$ are simultaneously finite. For, when they are both finite, one has

$$\begin{aligned}
\text{G-dim}_R(X \otimes^{\mathbf{L}}_S P) &= \text{depth}(R) - \text{depth}_S(X \otimes^{\mathbf{L}}_S P) \\
&= \text{depth}(R) - \text{depth}_S(X) + \text{pd}_S(P) \\
&= \text{G-dim}_R(X) + \text{pd}_S(P),
\end{aligned}$$

where the first and the third equalities are by the Auslander-Bridger formula, while the one in the middle is by [23, (2.2)].

The rest of the proof is dedicated to proving that X and $X \otimes^{\mathbf{L}}_S P$ have finite G-dimension over R simultaneously. In view of 2.3, this is tantamount to proving:

(a) $\mathbf{R}\text{Hom}_R(X, R)$ is homologically bounded if and only if the same is true of $\mathbf{R}\text{Hom}_R(X \otimes^{\mathbf{L}}_S P, R)$; and

(b) the biduality morphisms δ^R_X and $\delta^R_{X \otimes^{\mathbf{L}}_S P}$, defined as in 2.2, are isomorphisms simultaneously.

The proofs of (a) and (b) use the following observation: when U and V are complexes of S-modules such that V is homologically finite and $\text{pd}_S(V) < \infty$, the natural morphism

$$(*) \qquad \theta_{UV} : \mathbf{R}\text{Hom}_R(U, R) \otimes^{\mathbf{L}}_S \mathbf{R}\text{Hom}_S(V, S) \to \mathbf{R}\text{Hom}_R(U \otimes^{\mathbf{L}}_S V, R)$$

is an isomorphism. Indeed, it is the composition of tensor evaluation

$$\mathbf{R}\text{Hom}_R(U, R) \otimes^{\mathbf{L}}_S \mathbf{R}\text{Hom}_S(V, S) \to \mathbf{R}\text{Hom}_S(V, \mathbf{R}\text{Hom}_R(U, R)),$$

which is an isomorphism for V as above, followed by adjunction

$$\mathbf{R}\text{Hom}_S(V, \mathbf{R}\text{Hom}_R(U, R)) \xrightarrow{\cong} \mathbf{R}\text{Hom}_R(U \otimes^{\mathbf{L}}_S V, R).$$

Proof of (a). Since $\mathbf{R}\operatorname{Hom}_S(P, S)$ has finite projective dimension over S, one has the isomorphism

$$\theta_{XP} \colon \mathbf{R}\operatorname{Hom}_R(X, R) \otimes_S^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_S(P, S) \to \mathbf{R}\operatorname{Hom}_R(X \otimes_S^{\mathbf{L}} P, R).$$

Thus, Theorem 2.9 implies the desired equivalence.

Proof of (b). Consider the following commutative diagram of morphisms of complexes of S-modules:

$$
\begin{array}{ccc}
X \otimes_S^{\mathbf{L}} P & \xrightarrow{(\delta_X^R) \otimes_S^{\mathbf{L}} P} & \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X, R), R) \otimes_S^{\mathbf{L}} P \\
\Big\| & & \simeq \Big\downarrow \nu \\
& & \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X, R) \otimes_S^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_S(P, S), R) \\
\Big\| & & \simeq \Big\uparrow \mathbf{R}\operatorname{Hom}_R(\theta_{XP}, R) \\
X \otimes_S^{\mathbf{L}} P & \xrightarrow{\delta_{(X \otimes_S^{\mathbf{L}} P)}^R} & \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X \otimes_S^{\mathbf{L}} P, R), R)
\end{array}
$$

The morphism ν is the composition $\theta_{UV} \circ (1 \otimes_S^{\mathbf{L}} \delta_P^S)$, where $U = \mathbf{R}\operatorname{Hom}_R(X, R)$ and $V = \mathbf{R}\operatorname{Hom}_S(P, S)$. Note that δ_P^S, and hence $1 \otimes_S^{\mathbf{L}} \delta_P^S$, is an isomorphism because $\operatorname{pd}_S(P)$ is finite. Furthermore, θ_{UV} is an isomorphism since $\operatorname{pd}_S(V)$ is finite. This is why ν is an isomorphism.

From the diagram one obtains that $\delta_{X \otimes_S^{\mathbf{L}} P}^R$ is an isomorphism if and only if $(\delta_X^R) \otimes_S^{\mathbf{L}} P$ is. By Proposition 2.10, the morphisms $(\delta_X^R) \otimes_S^{\mathbf{L}} P$ and δ_X^R are isomorphisms simultaneously, as $\operatorname{pd}_S(P)$ is finite. $\qquad\square$

To wrap up this section, we give the

5.11. *Proof of Theorem 5.7.* Arguing as in the proof of Theorem 5.1 one reduces to the case where φ and σ are surjective and $\operatorname{pd}_S(P)$ is finite. In this situation, one has to verify that $\operatorname{pd}_R(X)$ and $\operatorname{pd}_R(X \otimes_S^{\mathbf{L}} P)$ are simultaneously finite. Let k be the residue field of R. It suffices to show that $\operatorname{amp}(k \otimes_R^{\mathbf{L}} X)$ and $\operatorname{amp}(k \otimes_R^{\mathbf{L}} (X \otimes_S^{\mathbf{L}} P))$ are simultaneously finite. By the isomorphism

$$k \otimes_R^{\mathbf{L}} (X \otimes_S^{\mathbf{L}} P) \simeq (k \otimes_R^{\mathbf{L}} X) \otimes_S^{\mathbf{L}} P,$$

this follows from Theorem 2.9. $\qquad\square$

6. Detecting the Gorenstein property

The theorem below extends the Auslander-Bridger characterization [2, (4.20)] of Gorenstein rings.

6.1. THEOREM. *Let R be a local ring. The following conditions are equivalent.*

(a) *R is Gorenstein.*

(b) *For every local homomorphism $\varphi\colon R \to S$ and for every homologically finite complex X of S-modules, $\mathrm{G\text{-}dim}_\varphi(X) < \infty$.*

(c) *There is a local homomorphism $\varphi\colon R \to S$ and an ideal I of S such that $I \supseteq \mathfrak{m}S$, where $\mathfrak{m}$ is the maximal ideal of R, and $\mathrm{G\text{-}dim}_\varphi(S/I) < \infty$.*

Proof. "(a) $\Longrightarrow$ (b)". Let $R \to R' \to \widehat{S}$ be a Cohen factorization of $\dot{\varphi}$. The R'-module $\mathrm{H}(\widehat{X})$ is finite, because the S-module $\mathrm{H}(X)$ is finite. Since R is Gorenstein, so is R' [10, (3.3.15)]. Thus, $\mathrm{G\text{-}dim}_{R'}(\widehat{X}) < \infty$, that is to say, $\mathrm{G\text{-}dim}_\varphi(X) < \infty$; see Proposition 3.6.

"(b) $\Longrightarrow$ (c)" is trivial.

"(c) $\Longrightarrow$ (a)". Let $R \to R' \to \widehat{S}$ be a Cohen factorization. Composing with the surjection $\widehat{S} \xrightarrow{\pi} \widehat{S}/I\widehat{S}$ gives a diagram $R \to R' \to \widehat{S}/I\widehat{S}$ that is also a Cohen factorization. Since $\mathrm{G\text{-}dim}_{R'}(\widehat{S}/I\widehat{S})$ is finite, so is $\mathrm{G\text{-}dim}(\pi\dot{\varphi})$. The composition $\pi\dot{\varphi}$ factors through the residue field k of R, giving the commutative diagram:

$$
\begin{array}{ccc}
R & \longrightarrow & \widehat{S}/I\widehat{S} \\
& \searrow \quad \nearrow & \\
& k &
\end{array}
$$

The map $k \to \widehat{S}/I\widehat{S}$ has finite projective dimension because k is a field. Therefore, Theorem 5.2 implies that the surjection $R \to k$ has finite G-dimension. Thus, R is Gorenstein by 3.4.4 and [2, (4.20)]. $\qquad\square$

When φ is finite and X is a module of finite projective dimension over both R and S, the implication "(c) $\Longrightarrow$ (a)" in the next result was proved by Apassov [1, Theorem G'].

6.2. THEOREM. *Let $\varphi\colon R \to S$ be a local homomorphism such that S is Gorenstein. The following conditions are equivalent.*

(a) *R is Gorenstein.*

(b) *$\mathrm{G\text{-}dim}(\varphi)$ is finite.*

(c) *There exists a homologically finite complex P of S-modules such that $\mathrm{pd}_S(P)$ is finite and $\mathrm{G\text{-}dim}_\varphi(P)$ is finite.*

Proof. The implication "(a) $\Longrightarrow$ (b)" is contained in Theorem 6.1, while "(b) $\Longleftrightarrow$ (c)" is given by Theorem 5.1.

"(b) $\Longrightarrow$ (a)". As S is Gorenstein, $I_S(t) = t^{\dim(S)}$, and hence

$$ I_R(t)I_\varphi(t) = I_S(t) = t^{\dim(S)}, $$

by equality (†) in 5.4. Now, both $I_R(t)$ and $I_\varphi(t)$ are Laurent series with nonnegative coefficients, so that $I_R(t)$ is a polynomial. This, as noted in 5.4, implies that R is Gorenstein. $\qquad\square$

The last theorem in this section is a characterization of the Gorenstein property of a local ring in terms of the finiteness of G-dimension of Frobenius-like endomorphisms. In order to describe this, we recall the definition of an invariant introduced by Koh and Lee [25, (1.1)].

6.3. For a finite module M over a local ring $(S, \mathfrak{n})$, set

$$\mathrm{s}(M) = \inf\{t \geq 1 \mid \mathrm{Soc}(M) \not\subseteq \mathfrak{n}^t M\},$$

where $\mathrm{Soc}(M)$ is the socle of M. Furthermore, let

$$\mathrm{crs}(S) = \inf\{\mathrm{s}(S/(\mathbf{x})) \mid \mathbf{x} = x_1, \ldots, x_r \text{ is a maximal } S\text{-sequence}\}.$$

The following is a complex version of Koh-Lee [24, (2.6)] (see also Miller [29, (2.2.8)]), which, in turn, generalizes a theorem of J. Herzog [21, (3.1)].

6.4. PROPOSITION.   *Let* $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a local homomorphism for which* $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^{\mathrm{crs}(S)}$, *and* X *a homologically finite complex of* R*-modules. If there is an integer* $t \geq \sup(X)$ *such that* $\mathrm{Tor}_{t+i}^R(X, S) = 0$ *for* $1 \leq i \leq \mathrm{depth}(S) + 2$, *then* $\mathrm{pd}_R(X) < \infty$.

Proof. Replace X with a minimal R-free resolution to assume that each X_i is a finite free R-module, and $\partial(X) \subseteq \mathfrak{m}X$. Set $Y = X \otimes_R S$. Then Y is a complex of finite free S-modules with $\partial(Y) \subseteq \mathfrak{n}^{\mathrm{crs}(S)}Y$ and $\mathrm{H}_i(Y) = \mathrm{Tor}_i^R(X, S)$.

The desired conclusion is that $X_i = 0$ for $i \gg 0$. By the minimality of X, it suffices to prove that $X_i = 0$, equivalently, $Y_i = 0$, for some $i > \sup(X)$. Let $r = \mathrm{depth}(S)$ and $C = \mathrm{Coker}(\partial_{t+1}^Y)$. The truncated complex

$$Y_{t+r+2} \xrightarrow{\partial_{t+r+2}} Y_{t+r+1} \xrightarrow{\partial_{t+r+1}} \cdots \xrightarrow{\partial_{t+1}} Y_t \to 0$$

is the beginning of a minimal S-free resolution of C. If $\mathrm{pd}_S(C) = \infty$, then [29, (2.2.5),(2.2.6)] implies that each row of ∂_{t+r+2}^Y has an entry outside $\mathfrak{n}^{\mathrm{crs}(S)}$, a contradiction. Thus, $\mathrm{pd}_S(C) < \infty$, and the Auslander-Buchsbaum formula implies that the $\mathrm{pd}_S(C) \leq r$. The minimality of the complex above implies that $Y_{t+r+1} = 0$, completing the proof.   □

An arbitrary local homomorphism of finite G-dimension is far from being quasi-Gorenstein. Indeed, when R is Gorenstein, any local homomorphism $\varphi \colon R \to S$ has finite G-dimension by Theorem 6.1, whereas, by [7, (8.2)], such a φ is quasi-Gorenstein if and only if S is Gorenstein. Endomorphisms however are much better behaved in this regard.

6.5. PROPOSITION.   *Let* $\varphi \colon (R, \mathfrak{m}) \to (R, \mathfrak{m})$ *be a local homomorphism. If* G-$\dim(\varphi)$ *is finite, then* φ^n *is quasi-Gorenstein at* $\mathfrak{m}$, *for each integer* $n \geq 1$.

If the finiteness of G-dimension localizes—see Proposition 3.7—then one could draw the stronger conclusion that φ^n is quasi-Gorenstein at each prime ideal.

Proof of Proposition 6.5. Suppose that G-dim(φ) is finite. The equality (†) in 5.4 yields $I_\varphi(t) = 1$ so that φ is quasi-Gorenstein at $\mathfrak{m}$, the maximal ideal of R. In the light of the discussion in 5.4, the same is true of the n-fold composition φ^n, for all integers $n \geq 1$. $\qquad\square$

We are now ready to prove the following theorem that subsumes Theorem A in the introduction.

6.6. THEOREM. *Let* $\varphi\colon (R, \mathfrak{m}) \to (R, \mathfrak{m})$ *be a local homomorphism such that* $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ *for some integer* $i \geq 1$. *The following conditions are equivalent.*

(a) *The ring R is Gorenstein.*
(b) *G-dim(φ^n) is finite for some integer $n \geq 1$.*
(c) *There is a homologically finite complex P of R-modules with $\mathrm{pd}_R(P)$ finite and G-dim$_{\varphi^n}(P)$ finite, for some integer $n \geq 1$.*

When these conditions hold, G-dim$(\varphi^m) = 0$, *for all* $m \geq 1$.

Proof. "(a) $\Longrightarrow$ (b)" is contained in Theorem 6.1.

"(b) $\Longrightarrow$ (c)" is trivial.

"(c) $\Longrightarrow$ (a)". By Theorem 5.1, G-dim(φ^n) is finite. The completion $\widehat{\varphi}\colon \widehat{R} \to \widehat{R}$ is an endomorphism of $\widehat{R}$ such that $\widehat{\varphi}^i(\widehat{\mathfrak{m}}) \subseteq \widehat{\mathfrak{m}}^2$ and $\widehat{\varphi^n} = (\widehat{\varphi})^n$. Furthermore, R is Gorenstein if and only if $\widehat{R}$ is Gorenstein, and by 3.4.1, G-dim(φ^n) is finite if and only if G-dim$(\widehat{\varphi}^n)$ is finite. Thus, passing to $\widehat{R}$, one may assume that R is complete. Hence, R has a dualizing complex D.

Since G-dim(φ^n) is finite, Proposition 6.5 implies that the s-fold composition φ^{sn} of φ^n is also quasi-Gorenstein at $\mathfrak{m}$ for all integers $s \geq 1$. Thus, $D \otimes_R^{\mathbf{L}} {}^{\varphi^{sn}}R$ is a dualizing complex for R, for each $s \geq 1$, by [7, (7.8)]. This implies that $\mathrm{H}(D \otimes_R^{\mathbf{L}} {}^{\varphi^{sn}}R)$ is finite; that is to say, $\mathrm{Tor}_i^R(D, {}^{\varphi^{sn}}R) = 0$ for all $i \gg 0$. Therefore, $\mathrm{pd}_R(D)$ is finite, by Proposition 6.4. This is equivalent to R being Gorenstein; see, for example, [12, (3.4.12)]. This completes the proof that (c) implies (a).

When these conditions hold, the Auslander-Bridger formula 3.5 gives

$$\text{G-dim}(\varphi^m) = \mathrm{depth}(R) - \mathrm{depth}(R) = 0.$$

This is the desired formula. $\qquad\square$

The preceding theorem raises the following problem: given a local ring $(R, \mathfrak{m})$ construct endomorphisms of R that map $\mathfrak{m}$ into $\mathfrak{m}^2$. The prototype is the Frobenius endomorphism of a local ring of characteristic p. There are many such endomorphisms of power series rings over fields. The following

example gives a larger class of complete local rings with nontrivial endomorphisms.

6.7. EXAMPLE.   Let k be a field and $X_1, \ldots, X_n$ analytic indeterminates and $F_1, \ldots, F_m \in k[X_1, \ldots, X_n]$ homogeneous polynomials, and set

$$R = k[\![X_1, \ldots, X_n]\!]/(F_1, \ldots, F_m) = k[\![x_1, \ldots, x_n]\!].$$

Let g be an element in $(x_1, \ldots, x_n)R$. The assignment $x_i \mapsto x_i g$ gives rise to a well-defined ring endomorphism φ of R such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{m}^2$.

One property of the Frobenius endomorphism that is hard to mimic is the finiteness of the length of $R/\varphi(\mathfrak{m})R$. Again, over power series rings such endomorphisms abound. The desired property is satisfied by the ring R constructed above, when it is a one-dimensional domain and $g \neq 0$. Examples in dimension two or higher can be built from these by considering $R[\![Y_1, \ldots, Y_m]\!]$.

More interesting endomorphisms can be obtained as follows. Let

$$R = k[\![X_1, \ldots, X_n]\!]/(G_1 - H_1, \ldots, G_m - H_m),$$

where, for each i, the elements G_i and H_i are monomials of the same total degree. For each positive integer t, the assignment $x_i \mapsto x_i^t$ gives rise to a ring endomorphism φ_t of R such that $\varphi_t(\mathfrak{m}) \subseteq \mathfrak{m}^t$ and $R/\varphi_t(\mathfrak{m})R$ has finite length. This method allows one to construct Cohen-Macaulay normal domains of arbitrarily large dimension with nontrivial endomorphisms; consider, for example, the maximal minors of a $2 \times r$ matrix of variables.

7. Finiteness of G-dimension over φ

The import of the results of this section is that the finiteness of $\text{G-dim}_\varphi(X)$ is intrinsic to the R-module structure on X; this is exactly analogous to the behavior of $\text{pd}_\varphi(X)$; see 4.4. When R is complete, this is contained in Proposition 3.6; see also Theorem 8.2 ahead.

7.1. THEOREM.   *Let $\varphi \colon R \to S$ and $\psi \colon R \to T$ be local homomorphisms. Let X and Y be homologically finite complexes of S-modules and T-modules, respectively, that are isomorphic in the derived category of R. Then $\text{G-dim}_\varphi(X)$ is finite if and only if $\text{G-dim}_\psi(Y)$ is finite.*

Proof. One may assume that X and Y are homologically nonzero. First, we reduce to the case where $\mathfrak{m}$, the maximal ideal of R, annihilates $\text{H}(X)$ and $\text{H}(Y)$. To this end, let K be the Koszul complex on a finite generating sequence for $\mathfrak{m}$. Since $X \otimes_R K = X \otimes_S (S \otimes_R K)$ and $S \otimes_R K$ is a finite free complex of S-modules, Theorem 5.1 yields that $\text{G-dim}_\varphi(X \otimes_R K)$ and $\text{G-dim}_\varphi(X)$ are simultaneously finite. Similarly, $\text{G-dim}_\psi(Y \otimes_R K)$ and $\text{G-dim}_\psi(Y)$ are simultaneously finite. Moreover, $X \otimes_R K$ and $Y \otimes_R K$ are isomorphic in the derived category of R. As $\mathfrak{m}$ annihilates $\text{H}(X \otimes_R K)$ and

$H(Y \otimes_R K)$—see, for instance, [23, (1.2)]—replacing X and Y with $X \otimes_R K$ and $Y \otimes_R K$, respectively, gives the desired reduction.

Let $\alpha \colon X \to Y$ be an isomorphism. Let $\widetilde{\varphi} \colon \widehat{R} \to \widetilde{S}$ and $\widetilde{\psi} \colon \widehat{R} \to \widetilde{T}$ be the $\mathfrak{m}$-adic completions of φ and ψ, respectively, and $\iota \colon R \to \widehat{R}$ the completion map. In the derived category of R, one has a commutative diagram:

$$
\begin{array}{ccccc}
X = R \otimes_R X & \xrightarrow[\simeq]{\iota \otimes_R 1} & \widehat{R} \otimes_R X & \xrightarrow[\simeq]{\widetilde{\varphi} \otimes_\varphi 1} & \widetilde{S} \otimes_S X \\
{\scriptstyle\simeq}\downarrow{\scriptstyle\alpha} & & {\scriptstyle\simeq}\downarrow{\scriptstyle 1 \otimes_R \alpha} & & \\
Y = R \otimes_R Y & \xrightarrow[\simeq]{\iota \otimes_R 1} & \widehat{R} \otimes_R Y & \xrightarrow[\simeq]{\widetilde{\psi} \otimes_\psi 1} & \widetilde{T} \otimes_T Y
\end{array}
$$

Both $R \to \widehat{R}$ and $S \to \widetilde{S}$ are flat, so at the level of homology the top row of the diagram reads $H(X) \to \widehat{R} \otimes_R H(X) \to \widetilde{S} \otimes_S H(X)$. Since $H(X)$ is annihilated by $\mathfrak{m}$, these are both bijective, that is, $\iota \otimes_R 1$ and $\widetilde{\varphi} \otimes_\varphi 1$ are isomorphisms. A similar reasoning justifies the isomorphisms in the bottom row.

In the biimplications below, the first is by 3.4.1, the second is by Proposition 3.6, while the third is due to the fact that, by the diagram above, $\widetilde{S} \otimes_S X$ and $\widehat{R} \otimes_R X$ are isomorphic.

$$\text{G-dim}_\varphi(X) \text{ is finite} \iff \text{G-dim}_{\widetilde{\varphi}}(\widetilde{S} \otimes_S X) \text{ is finite}$$

$$\iff \widetilde{S} \otimes_S X \text{ is in } \mathcal{A}(\widehat{R})$$

$$\iff \widehat{R} \otimes_R X \text{ is in } \mathcal{A}(\widehat{R}).$$

By the same token, $\text{G-dim}_\psi(Y)$ is finite if and only if $\widehat{R} \otimes_R Y$ is in $\mathcal{A}(\widehat{R})$. This gives the desired conclusion, since $\widehat{R} \otimes_R X$ and $\widehat{R} \otimes_R Y$ are isomorphic. $\square$

Here is an example discovered by S. Paul Smith to illustrate that, in the set-up of the theorem above, $\text{G-dim}_\varphi(X)$ and $\text{G-dim}_\psi(Y)$ need not be equal; see, however, [9, (8.2.4)].

7.2. EXAMPLE.   Let R be a field, S the localized polynomial ring $R[X]_{(X)}$, and let T be a field extension of R, with $\text{rank}_R T = \text{rank}_R S$; in particular, S and T are isomorphic as R-modules. Let $\varphi \colon R \to S$ and $\psi \colon R \to T$ be the canonical inclusions. Because R is a field, both $\text{G-dim}_\varphi(S)$ and $\text{G-dim}_\psi(T)$ are finite; see Theorem 6.1. By the Auslander-Bridger formula 3.5, one has

$$\text{G-dim}_\varphi(S) = -1 \quad \text{and} \quad \text{G-dim}_\psi(T) = 0.$$

The corollary below extends 3.4.4. It applies, for instance, when X is homologically finite over S and φ is module-finite.

7.3. COROLLARY.   *Let $\varphi \colon R \to S$ be a local homomorphism and X a complex of S-modules. If* $H(X)$ *is finite over R, then*

$$\text{G-dim}_\varphi(X) = \text{G-dim}_R(X).$$

Proof. Theorem 7.1 applied to the homomorphisms φ and id_R says that G-$\dim_\varphi(X)$ and G-$\dim_R(X)$ are simultaneously finite. When they are finite, the Auslander-Bridger formula and Lemma 2.8 yield the desired equality.   □

8. Comparison with Gorenstein flat dimension

Keeping in mind the conclusions of the preceding section, and Proposition 3.6, it is natural to ask how G-dimension over φ compares with other extensions of G-dimension to the non-finite arena. It turns out that the finiteness of G-$\dim_\varphi(X)$ is equivalent to the finiteness of $\mathrm{Gfd}_R(X)$, the *G-flat dimension* of X over R, at least when R has a dualizing complex. A more precise statement is contained in Theorem 8.2 below; it is analogous to Proposition 4.4 dealing with projective dimensions. We begin by recalling the relevant definitions.

8.1.   Let R be a commutative Noetherian ring. An R-module G is said to be *G-flat* if there exists an exact complex of flat R-modules

$$F = \cdots \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

with $\mathrm{Coker}(\partial_1) = G$ and $E \otimes_R F$ exact for each injective R-module E. Note that any flat module is G-flat. Thus, each homologically bounded complex of R-modules X admits a G-flat resolution, and one can introduce its G-flat dimension to be the number

$$\mathrm{Gfd}_R(X) \colon = \inf\{\sup\{n \mid G_n \neq 0\} \mid G \text{ a G-flat resolution of } X\}.$$

The reader may consult [12] or the book of Enochs and Jenda [15] for details.

Now we state one of the main theorems of this section; it implies Theorem D from the introduction because when R is a quotient of a Gorenstein ring, it has a dualizing complex. As noted in the introduction, Foxby has derived the inequalities below assuming only that the formal fibres of R are Gorenstein. Also, the simultaneous finiteness of G-$\dim_\varphi(S)$ and $\mathrm{Gfd}_R(S)$ is [14, (5.2)].

8.2. THEOREM.   *Suppose R has a dualizing complex. Let $\varphi \colon R \to S$ be a local homomorphism, and X a homologically finite complex of S-modules. Then*

$$\mathrm{Gfd}_R(X) - \mathrm{edim}(\varphi) \leq \text{G-}\dim_\varphi(X) \leq \mathrm{Gfd}_R(X).$$

In particular, G-$\dim_\varphi(X)$ is finite if and only if $\mathrm{Gfd}_R(X)$ is finite.

Observe that doing away with the hypothesis that R has a dualizing complex would provide us with another proof of Theorem 7.1. One can obtain useful bounds even when R has no dualizing complexes; this is explained in 8.12.

8.3.   The proof calls for considerable preparation and is given in 8.11. Here are the key steps in our argument:

Step 1. We verify that $\text{G-dim}_\varphi(X)$ and $\text{Gfd}_R(X)$ are simultaneously finite. This is an immediate consequence of [14, (4.3)] and Theorem 3.6.

Step 2. We prove, in Theorem 8.8, that if $\text{Gfd}_R(X)$ is finite, then it coincides with the number $\text{Rfd}_R(X)$, whose definition is recalled below. This step constitutes the bulk of work in this section and builds on recent work of Christensen, Frankild, and Holm [14]. They have informed us that they can prove the same result by using the methods in [22].

Step 3. The last step consists of verifying that when $\text{G-dim}_\varphi(X)$ is finite, it is sandwiched between $\text{Rfd}_R(X) - \text{edim}(\varphi)$ and $\text{Rfd}_R(X)$. The details of this step were worked out in conversations with Foxby, and we thank him for permitting us to present them here.

8.4. In the next few paragraphs, R denotes a commutative Noetherian ring, not necessarily local, and W a homologically bounded complex of R-modules; we do not assume that $\text{H}(W)$ is finite. The *large restricted flat dimension* of W over R, as introduced in [13], is the quantity

$$\text{Rfd}_R(W) = \sup\left\{\sup(F \otimes_R^{\mathbf{L}} W) \mid F \text{ an } R\text{-module with } \text{fd}_R(F) \text{ finite}\right\}$$

This number is finite, as long as $\text{H}(W)$ is nonzero and the Krull dimension of R is finite; see [13, (2.2)]. It is useful to keep in mind an alternative formula [13, (2.4)] for computing this invariant:

$$(1) \qquad \text{Rfd}_R(W) = \sup\{\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}}(W_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec } R\}\,.$$

We collect a few simple observations concerning this invariant.

8.5. Lemma. *Let $\psi\colon R \to T$ and $\kappa\colon T \to T'$ be homomorphisms of commutative Noetherian rings, and let W and Y be homologically bounded complexes of R-modules and of T-modules, respectively.*

(1) *If ψ is faithfully flat, then*

$$\text{Rfd}_T(T \otimes_R W) = \text{Rfd}_R(W) \quad \text{and} \quad \text{Rfd}_T(Y) \geq \text{Rfd}_R(Y)\,.$$

(2) *If κ is faithfully flat, then $\text{Rfd}_R(Y) = \text{Rfd}_R(T' \otimes_T Y)$.*

Proof. Let F be an R-module and let G be an T-module.
Proof of (1). The flatness of ψ implies:

(a) If $\text{fd}_R(F)$ is finite, then so is $\text{fd}_T(F \otimes_R T)$.
(b) If $\text{fd}_T(G)$ is finite, then so is $\text{fd}_R(G)$.

Remark (a), combined with the isomorphisms

$$(F \otimes_R T) \otimes_T^{\mathbf{L}} (T \otimes_R W) \simeq (F \otimes_R T) \otimes_R^{\mathbf{L}} W \simeq T \otimes_R (F \otimes_R^{\mathbf{L}} W)$$

and the faithful flatness of ψ, implies $\text{Rfd}_T(T \otimes_R W) \geq \text{Rfd}_R(W)$. The opposite inequality follows from (b) and the associativity isomorphism

$$G \otimes_T^{\mathbf{L}} (T \otimes_R W) \simeq G \otimes_R^{\mathbf{L}} W\,.$$

This justifies the equality. The inequality is a consequence of (a) and the isomorphism $(F \otimes_R T) \otimes_T^{\mathbf{L}} Y \simeq F \otimes_R^{\mathbf{L}} Y$.

As to (2): it is an immediate consequence of the isomorphism

$$F \otimes_R^{\mathbf{L}} (T' \otimes_T Y) \simeq (F \otimes_R^{\mathbf{L}} Y) \otimes_T T'$$

and the faithful flatness of κ. $\square$

The next lemma gives a lower bound for the large restricted flat dimension.

8.6. LEMMA.   *If $\psi \colon R \to T$ is a local homomorphism and Y is a complex T-modules, then*

$$\mathrm{Rfd}_R(Y) \geq \operatorname{depth} R - \operatorname{depth}_T(Y) \,;$$

equality holds if Y is homologically finite over R and G-$\dim_R(Y)$ is finite.

Proof. The inequality is a consequence of 8.4.1 and the (in)equalities

$$\operatorname{depth}_R(Y) = \operatorname{depth}_T(\mathfrak{m}T, Y) \leq \operatorname{depth}_T(Y),$$

where the first one is by [23, (5.2.1)] and the second is by [23, (5.2.2)]. If Y is homologically finite over R and G-$\dim_R(Y)$ is finite, then

$$
\begin{aligned}
\operatorname{depth} R - \operatorname{depth}_T Y &= \operatorname{depth} R - \operatorname{depth}_R Y \\
&= \text{G-}\dim_R(Y) \\
&\geq \text{G-}\dim_{R_{\mathfrak{p}}}(Y_{\mathfrak{p}}) \\
&= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(Y_{\mathfrak{p}}),
\end{aligned}
$$

where the first equality is by Lemma 2.8, the second and fourth are by the classical Auslander-Bridger formula, while the inequality is well known; see [12, (2.3.11)]. In view of 8.4.1, this justifies the claimed equality. $\square$

The next step towards Theorem 8.2 is the formula below. It may be viewed as an Auslander-Buchsbaum formula for complexes of finite G-flat dimension, for it is strikingly similar to one for complexes of finite *flat* dimension: $\operatorname{depth}_R(W) = \operatorname{depth} R - \sup(k \otimes_R^{\mathbf{L}} W)$ when $\mathrm{fd}_R(W)$ is finite; see [17, (2.4)]. What is more, $E \otimes_R^{\mathbf{L}} W \simeq E \otimes_R G$, where is G any G-flat resolution of W; this is contained in [14, (3.15)].

8.7. THEOREM.   *Let $(R, \mathfrak{m}, k)$ be a local ring and E the injective hull of k. If W is a complex of R-modules with $\mathrm{Gfd}_R(W)$ finite, then*

$$\operatorname{depth}_R(W) = \operatorname{depth} R - \sup(E \otimes_R^{\mathbf{L}} W)$$

In particular, $\sup(E \otimes_R^{\mathbf{L}} W)$ is finite if and only if $\operatorname{depth}_R(W)$ is finite.

Proof. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Faithful flatness of the completion homomorphism $R \to \widehat{R}$ implies that each injective $\widehat{R}$-module is injective also as an R-module. This remark and an elementary argument based

on the definition of G-flat dimension entail that $\mathrm{Gfd}_{\widehat{R}}(\widehat{R} \otimes_R W) \leq \mathrm{Gfd}_R(W)$; see also Holm [22, (3.10)]. Moreover,

$$\mathrm{depth}_{\widehat{R}}(\widehat{R} \otimes_R W) = \mathrm{depth}_R W \quad \text{and} \quad \mathrm{depth}\,\widehat{R} = \mathrm{depth}\,R.$$

Finally, $E \otimes_R^{\mathbf{L}} W \simeq E \otimes_{\widehat{R}}^{\mathbf{L}} (\widehat{R} \otimes_R W)$, since E has the structure of an $\widehat{R}$-module. Also, E is the injective hull of k over $\widehat{R}$. The upshot of this discussion is that one can replace R and W by $\widehat{R}$ and $\widehat{R} \otimes_R W$, respectively, and assume that R is complete. In particular, R has a dualizing complex D.

The G-flat dimension of W is finite, so it follows from [14, (4.3)] that W belongs to $\mathcal{A}(R)$, the Auslander category of R; see 2.5. Thus, the canonical morphism $W \to \mathbf{R}\,\mathrm{Hom}_R(D, D \otimes_R^{\mathbf{L}} W)$ is an isomorphism, and this starts the chain of isomorphisms

$$\mathbf{R}\,\mathrm{Hom}_R(k, W) \simeq \mathbf{R}\,\mathrm{Hom}_R(k, \mathbf{R}\,\mathrm{Hom}_R(D, D \otimes_R^{\mathbf{L}} W))$$
$$\simeq \mathbf{R}\,\mathrm{Hom}_R(D \otimes_R^{\mathbf{L}} k, D \otimes_R^{\mathbf{L}} W)$$
$$\simeq \mathbf{R}\,\mathrm{Hom}_k(D \otimes_R^{\mathbf{L}} k, \mathbf{R}\,\mathrm{Hom}_R(k, D \otimes_R^{\mathbf{L}} W)).$$

The second isomorphism is adjunction, so is the last one, since $D \otimes_R^{\mathbf{L}} k$ is isomorphic to a complex of vector spaces over k; this latter fact is clear once we compute it with a free resolution of D. The complex $D \otimes_R^{\mathbf{L}} W$ is homologically bounded, since W is in $\mathcal{A}(R)$, so the isomorphisms above with [17, (1.5)] yield

$$\sup(\mathbf{R}\,\mathrm{Hom}_R(k, W)) = \sup(\mathbf{R}\,\mathrm{Hom}_R(k, D \otimes_R^{\mathbf{L}} W)) - \inf(D \otimes_R^{\mathbf{L}} k).$$

For each complex X of R-modules, $\sup(\mathbf{R}\,\mathrm{Hom}_R(k, X)) = -\mathrm{depth}_R(X)$ by [17, (2.1)], and $\inf(D \otimes_R^{\mathbf{L}} k) = \inf(D)$ since D is homologically finite, so the displayed equality translates to

$$\mathrm{depth}_R(W) = \mathrm{depth}_R(D \otimes_R^{\mathbf{L}} W) + \inf(D).$$

Here is a crucial swindle: since $\mathrm{Gfd}_R(W)$ is finite, so is $\mathrm{Gfd}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(W))$, where $\mathbf{R}\Gamma_{\mathfrak{m}}(W)$ is the derived local cohomology of W with respect to $\mathfrak{m}$; this is by [14, (5.9)]. Thus, the formula above applies to $\mathbf{R}\Gamma_{\mathfrak{m}}(W)$ as well, and reads

$$\mathrm{depth}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(W)) = \mathrm{depth}_R(D \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{m}}(W)) + \inf(D).$$

The homology modules of $\mathbf{R}\Gamma_{\mathfrak{m}}(W)$ are all $\mathfrak{m}$-torsion, so [17, (2.7)] yields the first the equality below, while [17, (2.1)] provides the second one,

$$\mathrm{depth}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(W)) = -\sup(\mathbf{R}\Gamma_{\mathfrak{m}}(W)) = \mathrm{depth}_R(W).$$

Now, $\mathbf{R}\Gamma_{\mathfrak{m}}(D) \simeq \Sigma^d E$ with $d = \inf(\mathbf{R}\Gamma_{\mathfrak{m}}(D))$, where $\Sigma^d(-)$ denotes a shift of d steps to the left, so

$$D \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{m}}(W) \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(D) \otimes_R^{\mathbf{L}} W \simeq \Sigma^d(E \otimes_R^{\mathbf{L}} W).$$

The first isomorphism may be justified by invoking [27, (3.1.2)]. The injective hull E is $\mathfrak{m}$-torsion, so the homology modules of $E \otimes_R^{\mathbf{L}} W$ are $\mathfrak{m}$-torsion:

compute via a free resolution of W. Thus, $\operatorname{depth}_R(E \otimes_R^{\mathbf{L}} W) = -\sup(E \otimes_R^{\mathbf{L}} W)$ by [17, (2.7)]. Combining the preceding equalities gets us

$$\operatorname{depth}_R(W) = -\sup(E \otimes_R^{\mathbf{L}} W) + \inf(\mathbf{R}\Gamma_{\mathfrak{m}}(D)) + \inf(D).$$

Since R itself has finite G-flat dimension, this formula with R substituted for W reads: $\operatorname{depth} R = \inf(\mathbf{R}\Gamma_{\mathfrak{m}}(D)) + \inf(D)$. Feeding this back into the formula above completes the proof.   □

In the case when $\mathrm{H}(W)$ is concentrated in a single degree, the next theorem is part of [22, (3.19)].

8.8. THEOREM.   *Let R be a commutative Noetherian ring, and W a complex of R-modules. If $\operatorname{Gfd}_R(W)$ is finite, then $\operatorname{Gfd}_R(W) = \operatorname{Rfd}_R(W)$.*

Proof. The proof is the following sequence of equalities:

$$\begin{aligned}
\operatorname{Gfd}_R(W) &= \sup\{\sup(I \otimes_R^{\mathbf{L}} W) \mid I \text{ an injective } R\text{-module}\} \\
&= \sup\{\sup(E(R/\mathfrak{p}) \otimes_R^{\mathbf{L}} W) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \sup\{\sup(E(R/\mathfrak{p}) \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} W_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \operatorname{Rfd}_R(W).
\end{aligned}$$

The first one is [14, (2.8)]; the second follows from this, given the structure of injective modules over commutative Noetherian rings; the third equality is due to the isomorphism $E(R/\mathfrak{p}) \otimes_R^{\mathbf{L}} W \simeq E(R/\mathfrak{p}) \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} W_{\mathfrak{p}}$, the fourth is by Theorem 8.7, as $\operatorname{Gfd}_{R_{\mathfrak{p}}}(W_{\mathfrak{p}}) \leq \operatorname{Gfd}_R(W)$ by [12, (5.2.7)], whilst the last equality is 8.4.1.   □

We pause to record a corollary.

8.9. COROLLARY.   *Let $\psi\colon R \to T$ be a faithfully flat homomorphism of commutative, Noetherian rings, and let W be a complex of R-modules. If $\operatorname{Gfd}_R(W)$ is finite, then*

$$\operatorname{Gfd}_T(T \otimes_R W) = \operatorname{Gfd}_R(T \otimes_R W) = \operatorname{Gfd}_R(W).$$

Proof. When $\operatorname{Gfd}_R(W)$ is finite, so are $\operatorname{Gfd}_R(T \otimes_R W)$ and $\operatorname{Gfd}_T(T \otimes_R W)$; the second by [22, (3.10)] and the first by the easily verifiable remark that if G is a G-flat R-module, then so is $F \otimes_R G$ for any flat R-module F. The desired equalities are now a consequence of Theorem 8.8 and Lemma 8.5.   □

This result prompts us to raise the following question:

8.10. QUESTION.   Does the conclusion of Corollary 8.9 remain true without assuming *a priori* that $\operatorname{Gfd}_R(W)$ is finite?

Here, at last, is the proof of Theorem 8.2; before jumping into it, the reader may wish to glance at 8.3, which outlines the basic argument.

8.11. *Proof of Theorem 8.2.* To begin with,

$$\text{G-dim}_\varphi(X) < \infty \iff X \in A(R) \iff \text{Gfd}_R(X) < \infty,$$

where the first biimplication is by Proposition 3.6, while the second one is contained in [14, (4.3)]. Thus, one may assume that both $\text{G-dim}_\varphi(X)$ and $\text{Gfd}_R(X)$ are finite. In this case, thanks to Theorems 3.5 and 8.8, what we need to prove is that

$$(\dagger) \qquad \text{Rfd}_R(X) - \text{edim}(\varphi) \leq \text{depth}\, R - \text{depth}_S X \leq \text{Rfd}_R(X)$$

when $\text{G-dim}_\varphi(X)$ is finite. The inequality on the right is contained in Lemma 8.6. That leaves us with the one on the left.

Let $\widehat{S}$ be the completion of S at its maximal ideal and set $\widehat{X} = \widehat{S} \otimes_S X$. By Lemma 8.5.2, the faithful flatness of the homomorphism $S \to \widehat{S}$ implies $\text{Rfd}_R(\widehat{X}) = \text{Rfd}_R(X)$. The other quantities involved in $(\dagger)$ also remain unchanged if we substitute $\widehat{S}$ for S and $\widehat{X}$ for X, so we may do so and thereby assume that S is complete. With $R \to R' \to S$ a minimal Cohen factorization of φ, Lemma 8.5 provides the inequality below.

$$\begin{aligned}
\text{Rfd}_R(X) &\leq \text{Rfd}_{R'}(X) \\
&= \text{depth}\, R' - \text{depth}_{R'}(X) \\
&= \text{depth}\, R + \text{edim}(\varphi) - \text{depth}_{R'}(X) \\
&= \text{depth}\, R + \text{edim}(\varphi) - \text{depth}_S(X).
\end{aligned}$$

Lemma 8.6 explains the first equality; the second holds as $R \to R'$ is flat and $R'/\mathfrak{m}R'$ is regular, and the last holds because $R' \to S$ is surjective. $\square$

8.12. Let $\varphi \colon (R, \mathfrak{m}, k) \to S$ be a local homomorphism and X a homologically finite complex of S-modules. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R, and $\widetilde{S}$ the $\mathfrak{m}S$-adic completion of S. Since $\widehat{R}$ has dualizing complex, it follows from 3.4.1 and Theorem 8.2 that

$$\text{Gfd}_{\widehat{R}}(\widetilde{S} \otimes_S X) - \text{edim}(\varphi) \leq \text{G-dim}_\varphi(X) \leq \text{Gfd}_{\widehat{R}}(\widetilde{S} \otimes_S X).$$

At any rate, one has the consolation of knowing a partial result:

8.13. PROPOSITION. *Let $\varphi \colon R \to S$ be a local homomorphism. Each homologically finite complex of S-modules X satisfies $\text{G-dim}_\varphi(X) \leq \text{Gfd}_R(X)$.*

Proof. The plan is to reduce to the case where R is complete and then apply Theorem 8.2; see the proof of Theorem 7.1. Let K be the Koszul complex on a minimal set of generators for $\mathfrak{m}$, the maximal ideal of R. Thus,

$\operatorname{pd}_R(K) = \operatorname{edim} R = \operatorname{pd}_S(K \otimes_R S)$. Now, if G is a G-flat resolution of X over R, then $K \otimes_R G$ is a G-flat resolution of $K \otimes_R X$. This implies that

$$\operatorname{Gfd}_R(K \otimes_R X) \le \operatorname{Gfd}_R(X) + \operatorname{edim}(R).$$

Moreover, since $K \otimes_R X \cong (K \otimes_R S) \otimes_S X$, Theorem 5.1 applied to the diagram $R \to S \xrightarrow{=} S$, and with $P = (K \otimes_R S)$, yields

$$\text{G-dim}_\varphi(K \otimes_R X) = \text{G-dim}_\varphi(X) + \operatorname{edim}(R).$$

Thus, it suffices to prove the desired inequality for the complex of S-modules $K \otimes_R X$; in particular, one may pass to $K \otimes_R X$ and assume $\mathfrak{m} \cdot \text{H}(X) = 0$. Now, we adopt the notation of 8.12, where we noted that

$$\text{G-dim}_\varphi(X) \le \operatorname{Gfd}_{\widehat{R}}(\widetilde{S} \otimes_S X).$$

It is elementary to verify that the canonical homomorphism of complexes of $\widehat{R}$-modules $\widehat{R} \otimes_R X \to \widetilde{S} \otimes_S X$ is a homology isomorphism, since $\mathfrak{m} \cdot \text{H}(X) = 0$. This gives us the equality below:

$$\operatorname{Gfd}_{\widehat{R}}(\widetilde{S} \otimes_S X) = \operatorname{Gfd}_{\widehat{R}}(\widehat{R} \otimes_R X) \le \operatorname{Gfd}_R(X);$$

the inequality is the version for complexes of [22, (3.10)], and may be deduced directly from the definitions. To complete the proof, put together the composed inequality above with the penultimate one. □

This proposition leads to analogues of the theorems in Section 6, with $\operatorname{Gfd}_R(-)$ playing the role of $\text{G-dim}_\varphi(-)$. The result below, which parallels Theorem 6.6, is one such analogue; in it, for any complex of R-modules, we write $^{\varphi^n} X$ to indicate that R acts on X via φ^n.

8.14. THEOREM.   *Let $\varphi \colon (R, \mathfrak{m}) \to (R, \mathfrak{m})$ be a local homomorphism such that $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some integer $i \ge 1$. The following conditions are equivalent.*

(a) *The ring R is Gorenstein.*
(b) $\operatorname{Gfd}_R(^{\varphi^n} R)$ *is finite for each integer $n \ge 1$.*
(c) *There is a homologically finite complex P of R-modules with $\operatorname{pd}_R(P)$ finite and $\operatorname{Gfd}_R(^{\varphi^n} P)$ finite, for some integer $n \ge 1$.*

Proof. Over a Gorenstein ring, any module has finite G-flat dimension; see [12, (5.2.10)]. This justifies "(a) $\implies$ (b)", while "(b) $\implies$ (c)" is trivial.

"(c) $\implies$ (a)". The preceding proposition yields that $\text{G-dim}_{\varphi^n}(P)$ is finite, so it remains to invoke the corresponding implication in Theorem 6.6.   □

8.15.   The hypotheses of the preceding result are satisfied when φ is the Frobenius endomorphism of a local ring of R of positive prime characteristic. In this case, one can add a fourth equivalent condition to those given above:

(b′) the R-module $^{\varphi^n} R$ is G-flat for one integer $n \ge 1$.

Indeed, it is clear from 8.4.1 that $\mathrm{Rfd}_R(\varphi^n R) = 0$ for each integer $n \geq 0$. Therefore, by [22, (3.19)], or by its successor, Theorem 8.8, one obtains that the R-module $\varphi^n R$ has finite G-flat dimension if and only if it is already G-flat.

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