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TRANSFERENCE OF BILINEAR MULTIPLIER OPERATORS ON LORENTZ SPACES

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Abstract. Let $m(\xi,\eta)$ be a bounded continuous function in $\mathbb{R}\times\mathbb{R}$, let $0 < p_i, q_i < \infty$ for $i = 1, 2$, and let $0 < p_3, q_3 \leq \infty$, be such that $1/p_1 + 1/p_2 = 1/p_3$. It is shown that

$$C_m(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi ix(\xi+\eta)}d\xi d\eta$$

is a bounded bilinear operator from $L^{p_1,q_1}(\mathbb{R})\times L^{p_2,q_2}(\mathbb{R})$ into $L^{p_3,q_3}(\mathbb{R})$ if and only if

$$P_{D_{\varepsilon^{-1}}m}(f,g)(\theta) = \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} \hat{f}(k)\hat{g}(k')m(\varepsilon k,\varepsilon k')e^{2\pi i\theta(k+k')}$$

are bounded bilinear operators from $L^{p_1,q_1}(\mathbb{T})\times L^{p_2,q_2}(\mathbb{T})$ into $L^{p_3,q_3}(\mathbb{T})$ with norm bounded by a uniform constant for all $\epsilon > 0$.

1. Introduction

Let $m(\xi_1,\xi_2,\ldots,\xi_n)$ be a bounded measurable function in $\mathbb{R}^n$ and define

$$C_m(f_1,f_2,\ldots,f_n)(x) = \int_{\mathbb{R}^n} \hat{f}_1(\xi_1)\ldots\hat{f}_n(\xi_n)m(\xi_1,\xi_2,\ldots,\xi_n)e^{2\pi ix(\xi_1+\xi_2+\cdots+\xi_n)}d\xi$$

for Schwartz test functions f_i in $\mathcal{S}$ for $i = 1,\ldots,n$.

Let now $0 < p_i \leq \infty$, $i = 1,\ldots,n$, be given, and let $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_n$. The function m is said to be a multilinear multiplier of strong type $(p_1,p_2,\ldots,p_n)$ (resp. weak type $(p_1,p_2,\ldots,p_n)$) if C_m extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R})\times\cdots\times L^{p_n}(\mathbb{R})$ into $L^q(\mathbb{R})$ (resp. to $L^{q,\infty}(\mathbb{R})$).

The study of such multilinear multipliers had been started by R. Coifman and Y. Meyer (see [4], [5], [6]) for smooth symbols. The interest in this area has increased in recent years following the work by M. Lacey and C. Thiele

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([20], [21], [22]) who showed that $m(\xi,\nu) = \operatorname{sign}(\xi + \alpha\nu)$ are multipliers of strong type (p_1, p_2) for $1 < p_1, p_2 \leq \infty$, $p_3 > 2/3$ and each $\alpha \in \mathbb{R} \setminus \{0, 1\}$.

New results for non-smooth symbols, extending those given by the bilinear Hilbert transform, have been achieved by J.E. Gilbert and A.R. Nahmod (see [10], [11], [12]) and by C. Muscalu, T. Tao and C. Thiele (see [25]).

We refer the reader to [13], [19], [9], and [14] for results on bilinear multipliers and related topics.

The first transference methods for linear multipliers were given by K. DeLeeuw. It is known that if m is continuous, then

$$T_m(f)(x) = \int_{\mathbb{R}} \hat{f}(\xi) m(\xi) e^{2\pi i x\xi} d\xi$$

(defined for $f \in S(\mathbb{R})$) is bounded on $L^p(\mathbb{R})$ if and only if the operators

$$\tilde{T}_{m_\varepsilon}(f)(\theta) = \sum_{k\in\mathbb{Z}} \hat{f}(k) m(\varepsilon k) e^{2\pi i\theta k}$$

(defined for trigonometric polynomials f) are uniformly bounded on $L^p(\mathbb{T})$ for all $\varepsilon > 0$ (see [8] and [29, p. 264]).

Although the results in this paper hold true for multilinear multipliers, for simplicity of the notation we restrict ourselves to bilinear multipliers and only state and prove the theorems in this situation.

Let $(m_{k,k'})$ be a bounded sequence. We use the notation

$$P_m(f,g)(\theta) = \sum_{k\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} a(k) b(k') m_{k,k'} e^{2\pi i\theta(k+k')}$$

for $f(t) = \sum_{n\in\mathbb{Z}} a(n) e^{2\pi i n t}$ and $g(t) = \sum_{n\in\mathbb{Z}} b(n) e^{2\pi i n t}$.

Let $0 < p_1, p_2 \leq \infty$ and let p_3 be such that $1/p_1 + 1/p_2 = 1/p_3$. We write $P_{D_{t^{-1}}m}$ when the symbol is $m(tk, tk')$ and call $m(tk, tk')$ a bounded multiplier of strong (resp. weak) type (p_1, p_2) on $\mathbb{Z} \times \mathbb{Z}$ if the corresponding operator $P_{D_{t^{-1}}m}$ is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_1}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ (resp. $L^{p_3,\infty}(\mathbb{T})$).

In a recent paper, D. Fan and S. Sato [9] obtained certain DeLeeuw type theorems for transferring multilinear operators on Lebesgue and Hardy spaces from $\mathbb{R}^n$ to $\mathbb{T}^n$. They showed that the multilinear version of the transference between $\mathbb{R}$ and $\mathbb{Z}$ holds, namely that for a continuous function $m(\xi, \eta)$ one has that m is a multiplier of strong (resp. weak) type (p_1, p_2) on $\mathbb{R} \times \mathbb{R}$ if and only if $(D_{\varepsilon^{-1}} m)_{k,k'} = (m(\varepsilon k, \varepsilon k'))_{k,k'}$ are uniformly bounded multipliers of strong (resp. weak) type (p_1, p_2) on $\mathbb{Z} \times \mathbb{Z}$.

The first author [3] proved a DeLeeuw type theorem for transferring bilinear multipliers from $L^p(\mathbb{R})$ to bilinear multipliers acting on $\ell_p(\mathbb{Z})$. The aim of this paper is to extend the results in [9] to bilinear multipliers acting on Lorentz spaces (see [9, Remark 3]).

We shall show that if m is a bounded continuous function on $\mathbb{R}^2$, then C_m defines a bounded bilinear map from $L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R})$ into $L^{p_3,q_3}(\mathbb{R})$ if and only if the operators $P_{D_{t^{-1}}m}$, defined as the restriction to $m(tk, tk')$ for

$k, k' \in \mathbb{Z}$, define bilinear maps from $L^{p_1,q_1}(\mathbb{T}) \times L^{p_2,q_2}(\mathbb{T})$ into $L^{p_3,q_3}(\mathbb{T})$ that are uniformly bounded for $t > 0$.

Throughout the paper $|A|$ denotes the Lebesgue measure of A and we identify functions f on $\mathbb{T}$ and periodic functions on $\mathbb{R}$ with period 1 defined on $[-1/2, 1/2)$, that is, $f(x) = f(e^{2\pi ix})$ and $\int_{\mathbb{T}} f(z)dm(z) = \int_{-1/2}^{1/2} f(t)dt$. For $0 < p \leq \infty$, we write $D_t^p f(x) = t^{-1/p} f(t^{-1}x)$ (with the notation $D_t = D_t^\infty$), $M_y f(x) = f(x)e^{2\pi iyx}$ and $T_y f(x) = f(x-y)$ for the dilation, modulation and translation operators. In this way $(D_t^q f)\hat{} = D_{t^{-1}}^{q'} \hat{f}$, where, as usual, q' stands for the conjugate exponent of q.

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2. Preliminaries

Let (Ω, Σ, μ) be a σ-finite and complete measure space. Given a complex-valued measurable function f we shall denote the distribution function of f by $\mu_f(\lambda) = \mu(E_\lambda)$ for $\lambda > 0$, where $E_\lambda = \{w \in \Omega : |f(w)| > \lambda\}$. The nonincreasing rearrangement of f is denoted by $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$, and we set $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

Now the Lorentz space $L^{p,q}$ consists of those measurable functions f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} \left\{ \dfrac{q}{p} \displaystyle\int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t} \right\}^{1/q}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t) & 0 < p \leq \infty,\ q = \infty. \end{cases}$$

It is well known that

$$\|f\|_{p\infty} = \sup_{\lambda>0} \lambda \mu_f(\lambda)^{1/p}.$$

Here we shall use the following fact: If $0 < p, q < \infty$ and f is a measurable function, then

$$\|f\|_{pq}^* = \left(q \int_0^\infty \lambda^{q-1} \mu_f(\lambda)^{q/p} d\lambda \right)^{1/q}. \tag{1}$$

(This can be easily checked for simple functions.)

Let us recall some facts about these spaces. Simple functions are dense in $L^{p,q}$ for $q \neq \infty$, and we have $(L^{p,1})^* = L^{p',\infty}$ for $1 \leq p < \infty$, and $(L^{p,q})^* = L^{p',q'}$ for $1 < p, q < \infty$. Replacing f^* by f^{**} and putting $\|f\|_{pq} = \|f^{**}\|_{pq}^*$, we get a functional equivalent to $\|\cdot\|_{pq}^*$ (for $1 < p < \infty$) for which $L^{1,1}$ and $L^{p,q}$ for $1 < p \leq \infty$, $1 \leq q \leq \infty$, are Banach spaces.

The reader is referred to [17], [2], [29] or [24] for basic information on Lorentz spaces. We only consider the case when μ is either the Lebesgue measure on $\mathbb{R}$ or the normalized Lebesge measure on $\mathbb{T}$, and the distribution function will be denoted by m_f in both cases.

DEFINITION 2.1. Let m be a bounded measurable function on $\mathbb{R}^2$. Let $0 < p_i, q_i \leq \infty$ for $i = 1, 2, 3$. For $t > 0$ we define

$$C_{D_{t^{-1}}m}(f,g)(x) = C_t(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta) m(t\xi, t\eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

for $f, g \in S(\mathbb{R})$.

We say that m is a bilinear multipier in $(L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ if there exists $C > 0$ such that

$$\|C_1(f,g)\|_{L^{p_3,q_3}(\mathbb{R})} \leq C \|f\|_{L^{p_1,q_1}(\mathbb{R})} \|g\|_{L^{p_2,q_2}(\mathbb{R})}$$

for all $f, g \in S(\mathbb{R})$.

DEFINITION 2.2. Let $(m_{k_1,k_2})_{k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}}$ be a bounded sequence. Let $p_i, q_i > 0$ be such that $p_3^{-1} = p_1^{-1} + p_2^{-1}$. We define

$$P_m(f,g)(x) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} a_{k_1} b_{k_2} m_{k_1,k_2} e^{2\pi i(k_1+k_2)x}$$

for all trigonometric polynomials

$$f(x) = \sum_{|k| \leq N} a_k e^{2\pi i k x}, \quad g(x) = \sum_{|k| \leq M} b_k e^{2\pi i k x}$$

and $N, M \in \mathbb{N}$.

We say that $m_{k,k'}$ is a bilinear multipier in $(L^{p_1,q_1}(\mathbb{T}) \times L^{p_2,q_2}(\mathbb{T}), L^{p_3,q_3}(\mathbb{T}))$ if there exists $C > 0$ such that

$$\|P_m(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} \leq C \|f\|_{L^{p_1,q_1}(\mathbb{T})} \|g\|_{L^{p_2,q_2}(\mathbb{T})}$$

for all trigonometric polynomials f and g.

REMARK 2.1. A function m is a multiplier in $(L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ if and only if $D_{t^{-1}}m(\xi,\eta) = m(t\xi, t\eta)$ is also a multiplier for each $t > 0$.

Note that for each $t > 0$ we have $m_{D_t f}(\lambda) = t m_f(\lambda)$. Hence

$$\|D_t f\|_{L^{p,q}(\mathbb{R})} = t^{1/p} \|f\|_{L^{p,q}(\mathbb{R})} \tag{2}$$

for $0 < p, q \leq \infty$. The above claim now follows easily from the formula

$$C_t(f,g) = D_t C_1(D_{t^{-1}} f, D_{t^{-1}} g).$$

Actually we have $\|C_t\| = \|C_1\|$ for all $t > 0$.

Let us start by recalling some facts to be used in the sequel.

DEFINITION 2.3. If f is a measurable function on $\mathbb{R}$ such that $\max\{|f(x)|, |\hat{f}(x)|\} \leq A/(1+|x|)^\alpha$ for some $A > 0$ and $\alpha > 1$, we define $\tilde{f}$ as the well-defined periodic function (see [29, pp. 250–253])

$$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.$$

LEMMA 2.4. *Let $0 < p < \infty$ and $0 < q \leq \infty$. If $f \in S(\mathbb{R})$ we have*

$$4^{-1/r}\|f\|_{L^{p,q}(\mathbb{R})} \leq \liminf_{t\to 0} t^{-1/p}\|\widetilde{D_t f}\|_{L^{p,q}(\mathbb{T})},$$
$$\limsup_{t\to 0} t^{-1/p}\|\widetilde{D_t f}\|_{L^{p,q}(\mathbb{T})} \leq 4^{1/r}\|f\|_{L^{p,q}(\mathbb{R})},$$

where $r = \log_2^{-1}(2^{1/p+1}\max(2^{(1/q)-1}, 1))$ and $\widetilde{D_t f}(x) = \sum_{k\in\mathbb{Z}} D_t f(x+k)$ is defined on $\mathbb{T}$.

Proof. Assume first that f has compact support. For $t > 0$ small enough we have $\operatorname{supp}(D_t f) \subset [-1/2, 1/2]$. This implies that

$$\widetilde{D_t f}\chi_{[-1/2,1/2]} = D_t f \chi_{[-1/2,1/2]} = D_t f.$$

In particular, for such t we have

$$\begin{aligned} m_{\widetilde{D_t f}}(\lambda) &= |\{x \in [-1/2, 1/2]/|D_t f(x)| > \lambda\}| \\ &= |\{x \in \mathbb{R}/|f(t^{-1}x)| > \lambda\}| = t m_f(\lambda) \end{aligned}$$

and

$$(\widetilde{D_t f})^*(x) = D_t(f^*)(x), \qquad x > 0.$$

Hence

$$\begin{aligned} \|\widetilde{D_t f}\|^q_{L^{p,q}(\mathbb{T})} &= \frac{q}{p}\int_0^1 \left(x^{1/p}(\widetilde{D_t f})^*(x)\right)^q \frac{dx}{x} \\ &= \frac{q}{p}\int_0^1 \left(x^{1/p} f^*(t^{-1}x)\right)^q \frac{dx}{x} \\ &= t^{q/p}\frac{q}{p}\int_0^{t^{-1}} \left(x^{1/p} f^*(x)\right)^q \frac{dx}{x} \end{aligned}$$

and therefore

$$\lim_{t\to 0} t^{-1/p}\|\widetilde{D_t f}\|_{L^{p,q}(\mathbb{T})} = \lim_{t\to 0}\left(\frac{q}{p}\int_0^{t^{-1}} \left(x^{1/p} f^*(x)\right)^q \frac{dx}{x}\right)^{1/q} = \|f\|_{L^{p,q}(\mathbb{R})}.$$

The case $q = \infty$ is simpler.

For the general case, set $f_n = f\chi_{[-n,n]}$ and observe that, for $|x| < 1/2$

$$\widetilde{D_t f}(x) - \widetilde{D_t f_n}(x) = \sum_{k\in\mathbb{Z}} f(t^{-1}(x+k)) - f_n(t^{-1}(x+k))$$
$$= \sum_{|k+x|>tn} f(t^{-1}(x+k)).$$

Hence, for any $m > 0$ we have

$$|\widetilde{D_t f}(x) - \widetilde{D_t f_n}(x)| \leq \sum_{|k+x|>tn} \frac{C_m}{(1+t^{-1}|x+k|)^m}$$
$$\leq t^m \sum_{|k+x|>tn} \frac{C_m}{|x+k|^m} \leq C_m t^m.$$

Selecting $m > 1/p$, we get

$$\lim_{t\to 0} t^{-1/p} \|\widetilde{D_t f_n} - \widetilde{D_t f}\|_{L^\infty(\mathbb{T})} \leq C_m \lim_{t\to 0} t^{m-1/p} = 0.$$

Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that

$$(1-\epsilon)\|f\|_{L^{p,q}(\mathbb{R})} \leq \|f_n\|_{L^{p,q}(\mathbb{R})} \leq \|f\|_{L^{p,q}(\mathbb{R})}.$$

Since $\|\cdot\|_{L^{p,q}(\mathbb{R})}$ is a quasi-norm with constant $C = 2^{1/p}\max(2^{(1/q)-1}, 1)$, by the Aoki-Rolewic theorem [26] it is equivalent to an r-norm, namely $|\cdot|$, for $r = \log_2^{-1}(2C)$. More precisely, we have

$$|f| \leq \|f\|_{L^{p,q}(\mathbb{R})} \leq 4^{1/r}|f|,$$

and thus we obtain the following triangle inequality for rth powers:

$$\|f+g\|^r_{L^{p,q}(\mathbb{R})} \leq 4(\|f\|^r_{L^{p,q}(\mathbb{R})} + \|g\|^r_{L^{p,q}(\mathbb{R})}).$$

Using this triangle inequality for $\|.\|^r_{L^{p,q}(\mathbb{T})}$ for the power $r \leq 1$ corresponding to the different values of p and q, and the previous case, we get the desired formula. $\square$

LEMMA 2.5. *Let $0 < p, q \leq \infty$, $\varphi = \chi_{[-1/2,1/2]}$, $f \in L^{p,q}(\mathbb{T})$ and $k \in \mathbb{N}$. Then*

$$\|f\|_{L^{p,q}(\mathbb{T})} = \|fD^p_k\varphi\|_{L^{p,q}(\mathbb{R})}.$$

Proof. Using the periodicity of f, we get

$$m_{fD^p_k\varphi}(\lambda) = \left|\left\{x \in \mathbb{R} : |f(x)k^{-1/p}\chi_{[-1/2,1/2]}(k^{-1}x)| > \lambda\right\}\right|$$
$$= \left|\left\{x \in [-\frac{k}{2}, \frac{k}{2}] : |f(x)| > k^{1/p}\lambda\right\}\right|$$
$$= k\left|\left\{x \in [-\frac{1}{2}, \frac{1}{2}] : |f(x)| > k^{1/p}\lambda\right\}\right| = km_f(k^{1/p}\lambda).$$

Hence

$$\begin{aligned}(fD_k^p\varphi)^*(t) &= \inf\{\lambda > 0 : km_f(k^{1/p}\lambda) < t\} \\ &= k^{-1/p}\inf\{\lambda > 0 : m_f(\lambda) < k^{-1}t\} \\ &= D_k^p f^*(t) = (D_k^p f)^*(t).\end{aligned}$$

Therefore

$$\begin{aligned}\|fD_k^p\varphi\|_{L^{p,q}(\mathbb{R})}^q &= \frac{q}{p}\int_0^\infty t^{q/p}(fD_k^p\varphi)^*(t)^q\,\frac{dt}{t} \\ &= \frac{q}{p}\int_0^\infty t^{q/p}k^{-q/p}f^*(k^{-1}t)^q\,\frac{dt}{t} \\ &= \frac{q}{p}\int_0^\infty t^{q/p}f^*(t)^q\,\frac{dt}{t} = \|f\|_{L^{p,q}(\mathbb{T})}^q. \qquad \square\end{aligned}$$

LEMMA 2.6. *Let $0 < p < \infty$ and $f \in L^{p,\infty}(\mathbb{T})$. If $\varphi \in S(\mathbb{R})$ is radial and decreasing, then*

$$\limsup_{\epsilon\to 0}\|fD_{\epsilon^{-1}}^p\varphi\|_{L^{p,\infty}(\mathbb{R})} \leq \|\varphi\|_{L^p(\mathbb{R})}\|f\|_{L^{p,\infty}(\mathbb{T})}.$$

Proof. Note that for each $\epsilon > 0$ and $\lambda > 0$ we have

$$\begin{aligned}&|\{x\in\mathbb{R} : |f(x)\varphi(\epsilon x)| > t\}| = \left|\left\{|x| \leq 2^{-1}\lambda\epsilon^{-1} : |f(x)\varphi(\epsilon x)| > t\right\}\right| \\ &\qquad + \sum_{n=0}^\infty \left|\left\{2^{n-1}\lambda\epsilon^{-1} < |x| \leq 2^n\lambda\epsilon^{-1} : |f(x)\varphi(\epsilon x)| > t\right\}\right| \\ &\leq \left|\left\{|x| \leq 2^{-1}\lambda\epsilon^{-1} : |f(x)| > t\varphi(0)^{-1}\right\}\right| \\ &\qquad + \sum_{n=0}^\infty \left|\left\{2^{n-1}\lambda\epsilon^{-1} < |x| \leq 2^n\lambda\epsilon^{-1} : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\right\}\right| \\ &\leq \left|\left\{|x| \leq 2^{-1}([\lambda\epsilon^{-1}]+1) : |f(x)| > t\varphi(0)^{-1}\right\}\right| \\ &\qquad + \sum_{n=0}^\infty \left|\left\{2^{n-1}[\lambda\epsilon^{-1}] < |x| \leq 2^n([\lambda\epsilon^{-1}]+1) : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\right\}\right| \\ &= ([\lambda\epsilon^{-1}]+1)\left|\left\{x\in\mathbb{T} : |f(x)| > t\varphi(0)^{-1}\right\}\right| \\ &\qquad + \sum_{n=0}^\infty \left(2^{n+1}\left([\lambda\epsilon^{-1}]+1\right) - 2^n[\lambda\epsilon^{-1}]\right)\left|\left\{x\in\mathbb{T} : |f(x)| > t\varphi\left(\lambda 2^{n-1}\right)^{-1}\right\}\right| \\ &\leq (\lambda\epsilon^{-1}+1)\left|\left\{x\in\mathbb{T} : |f(x)| > t\varphi(0)^{-1}\right\}\right| \\ &\qquad + \sum_{n=0}^\infty 2^n(\lambda\epsilon^{-1}+2)\left|\left\{x\in\mathbb{T} : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\right\}\right|.\end{aligned}$$

Hence we get

$$m_{fD_{\epsilon^{-1}}\varphi}(t) \leq (\lambda\epsilon^{-1}+1)m_f(t\varphi(0)^{-1}) + (\lambda\epsilon^{-1}+2)\sum_{n=0}^{\infty} 2^n m_f(t\varphi(\lambda 2^{n-1})^{-1}). \tag{3}$$

Therefore, using that $m_f(t) \leq \|f\|_{p\infty}^p/t^p$, we get

$$\begin{aligned} m_{fD^p_{\epsilon^{-1}}\varphi}(s) &= m_{fD_{\epsilon^{-1}}\varphi}(s\epsilon^{-1/p}) \\ &\leq (\lambda\epsilon^{-1}+1)\epsilon s^{-p}\varphi(0)^p\|f\|^p_{L^{p,\infty}(\mathbb{T})} \\ &\quad + \sum_{n=0}^{\infty} 2^n(\lambda\epsilon^{-1}+2)\epsilon s^{-p}\varphi(\lambda 2^{n-1})^p\|f\|^p_{L^{p,\infty}(\mathbb{T})} \\ &\leq s^{-p}(\lambda+\epsilon)|\varphi(0)|^p\|f\|^p_{L^{p,\infty}(\mathbb{T})} \\ &\quad + s^{-p}\sum_{n=0}^{\infty} 2^n(\lambda+2\epsilon)\varphi(\lambda 2^{n-1})^p\|f\|^p_{L^{p,\infty}(\mathbb{T})}. \end{aligned}$$

Hence, if

$$\varphi_\lambda = \varphi(0)\chi_{[-\lambda 2^{-1},\lambda 2^{-1}]} + \sum_{n\geq 0}\varphi(\lambda 2^{n-1})\chi_{[-\lambda 2^n,\lambda 2^n]\setminus[-\lambda 2^{n-1},\lambda 2^{n-1}]},$$

we have

$$\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|^p_{L^{p,\infty}(\mathbb{R})} \leq \|\varphi_\lambda\|^p_{L^p(\mathbb{R})}\|f\|^p_{L^{p,\infty}(\mathbb{T})}.$$

Passing to the limit as λ goes to zero, we get the result. $\square$

LEMMA 2.7. *Let $0 < p, q < \infty$ and $f \in L^{p,q}(\mathbb{T})$. If $\varphi \in S(\mathbb{R})$ is radial and decreasing, then*

$$\begin{aligned} C_{p,s}\|\varphi\|_{L^{p,s}(\mathbb{R})}\|f\|_{L^{p,q}(\mathbb{T})} &\leq \liminf_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \\ &\leq \limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \\ &\leq C_{p,r}\|\varphi\|_{L^{p,r}(\mathbb{R})}\|f\|_{L^{p,q}(\mathbb{T})}, \end{aligned}$$

where $C_{p_1,p_2} = (2^{p_2/p_1}-1)^{-1/p_2}$, $r = \min(p,q)$ and $s = \max(p,q)$.

Proof. Use (1) to write

$$\begin{aligned} \|fD^p_{\epsilon^{-1}}\varphi\|^q_{L^{p,q}(\mathbb{R})} &= \int_0^\infty qt^{q-1}(m_{fD_{\epsilon^{-1}}}(\epsilon^{-1/p}t))^{q/p}dt \\ &= \int_0^\infty qt^{q-1}(\epsilon m_{fD_{\epsilon^{-1}}}(t))^{q/p}dt. \end{aligned}$$

By the estimate in the previous lemma we have

$$\epsilon m_{fD_{\epsilon^{-1}}\varphi}(t) \leq (\lambda+\epsilon) m_f(t\varphi(0)^{-1}) + (\lambda+2\epsilon)\sum_{n=0}^{\infty} 2^n m_f(t\varphi(\lambda 2^{n-1})^{-1}).$$

Now we see that for $r = \min(p,q)$ we have

$$\text{(4)}\quad \limsup_{\epsilon\to 0} \|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \leq \left(\lambda^{r/p}\varphi(0)^r + \sum_{n=0}^{\infty}(\lambda 2^n)^{r/p}\varphi(\lambda 2^{n-1})^r\right)^{1/r} \|f\|_{L^{p,q}(\mathbb{T})}.$$

If $q \leq p$, then for all λ we have

$$\begin{aligned}
\|fD^p_{\epsilon^{-1}}\varphi\|^q_{L^{p,q}(\mathbb{R})} &= \int_0^\infty qt^{q-1}\left(\epsilon|\{x\in\mathbb{R}| f(x)\varphi(\epsilon x)| > t\}|\right)^{q/p} dt \\
&\leq \int_0^\infty qt^{q-1}\left((\lambda+\epsilon)m_f(t\varphi(0)^{-1}) + (\lambda+2\epsilon)\sum_{n=0}^{\infty}2^n m_f(t\varphi(\lambda 2^{n-1})^{-1})\right)^{q/p} dt \\
&\leq \int_0^\infty qt^{q-1}(\lambda+\epsilon)^{q/p} m_f(t\varphi(0)^{-1})^{q/p}dt \\
&\quad + \int_0^\infty qt^{q-1}(\lambda+2\epsilon)^{q/p}\sum_{n=0}^{\infty}2^{nq/p}m_f(t\varphi(\lambda 2^{n-1})^{-1})^{q/p}dt \\
&= (\lambda+\epsilon)^{q/p}\varphi(0)^q\int_0^\infty qt^{q-1}m_f(t)^{q/p}dt \\
&\quad + (\lambda+2\epsilon)^{q/p}\sum_{n=0}^{\infty}2^{nq/p}\varphi(\lambda 2^{n-1})^q\int_0^\infty qt^{q-1}m_f(t)^{q/p}dt \\
&= \left((\lambda+\epsilon)^{q/p}|\varphi(0)|^q + (\lambda+2\epsilon)^{q/p}\sum_{n=0}^{\infty}2^{nq/p}\varphi(\lambda 2^{n-1})^q\right)\|f\|^q_{L^{p,q}(\mathbb{T})}.
\end{aligned}$$

Therefore

$$\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \leq \left(\lambda^{q/p}\varphi(0)^q + \sum_{n=0}^{\infty}(\lambda 2^n)^{q/p}\varphi(\lambda 2^{n-1})^q\right)^{1/q}\|f\|_{L^{p,q}(\mathbb{T})},$$

which gives (4).

In the case $q > p$ we use Minkowski's inequality and get

$$\begin{aligned}
\|fD^p_{\epsilon^{-1}}\varphi\|^p_{L^{p,q}(\mathbb{R})} &= \left(\int_0^\infty \left(q^{p/q}t^{p(1-\frac{1}{q})}\epsilon\,|\{x\in\mathbb{R}: |f(x)\varphi(\epsilon x)|>t\}|\right)^{q/p} dt\right)^{p/q}\\
&\le \left(\int_0^\infty \left(q^{p/q}t^{p(1-\frac{1}{q})}(\lambda+\epsilon)m_f(t\varphi(0)^{-1})\right.\right.\\
&\qquad \left.\left. + (\lambda+2\epsilon)\sum_{n=0}^\infty 2^n q^{p/q}t^{p(1-\frac{1}{q})}m_f(t\varphi(\lambda 2^{n-1})^{-1})\right)^{q/p} dt\right)^{p/q}\\
&\le (\lambda+\epsilon)\left(\int_0^\infty \left(q^{p/q}t^{p(1-\frac{1}{q})}m_f(t\varphi(0)^{-1})\right)^{q/p} dt\right)^{p/q}\\
&\qquad + (\lambda+2\epsilon)\sum_{n=0}^\infty 2^n\left(\int_0^\infty \left(q^{p/q}t^{p(1-\frac{1}{q})}m_f(t|\varphi(\lambda 2^{n-1})|^{-1})\right)^{q/p} dt\right)^{p/q}\\
&= (\lambda+\epsilon)\varphi(0)^p\left(\int_0^\infty qt^{q-1}m_f(t)^{q/p}dt\right)^{p/q}\\
&\qquad + (\lambda+2\epsilon)\sum_{n=0}^\infty 2^n\varphi(\lambda 2^{n-1})^p\left(\int_0^\infty qt^{q-1}m_f(t)^{q/p}dt\right)^{p/q}\\
&= \left((\lambda+\epsilon)\varphi(0)^p + (\lambda+2\epsilon)\sum_{n=0}^\infty 2^n\varphi(\lambda 2^{n-1})^p\right)\|f\|^p_{L^{p,q}(\mathbb{T})}.
\end{aligned}$$

Therefore

$$\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \le \left(\lambda\varphi(0)^p + \sum_{n=0}^\infty \lambda 2^n\varphi(\lambda 2^{n-1})^p\right)^{1/p}\|f\|_{L^{p,q}(\mathbb{T})},$$

and (4) is proved.

If

$$\varphi_\lambda = \varphi(0)\chi_{[-\lambda 2^{-1},\lambda 2^{-1}]} + \sum_{n\ge 0}\varphi(\lambda 2^{n-1})\chi_{[-\lambda 2^n,\lambda 2^n]\setminus[-\lambda 2^{n-1},\lambda 2^{n-1}]},$$

then clearly

$$\|\varphi_\lambda\|_p = \left(\lambda\varphi(0)^p + \sum_{n=0}^\infty \lambda 2^n\varphi(\lambda 2^{n-1})^p\right)^{1/p}.$$

Since φ and φ_λ are radial and decreasing, we have $\varphi^*_\lambda(t) = \varphi_\lambda(t/2)$ for $t>0$ and

$$\|\varphi_\lambda\|_{L^{pr}(\mathbb{R})} = \left(\lambda^{r/p}\varphi(0)^r + (2^{r/p}-1)\sum_{n=0}^\infty(\lambda 2^n)^{r/p}\varphi(\lambda 2^{n-1})^r\right)^{1/r}.$$

Hence, using $r \leq p$, we obtain

$$\left(\lambda^{r/p} \varphi(0)^r + \sum_{n=0}^{\infty} (\lambda 2^n)^{r/p} \varphi(\lambda 2^{n-1})^r \right)^{1/r} \leq (2^{r/p} - 1)^{-1/r} \|\varphi_\lambda\|_{L^{p,r}(\mathbb{R})}.$$

Finally, taking the limits as $\lambda \to 0$ gives

$$\begin{aligned} \limsup_{\epsilon \to 0} \|f D^p_{\epsilon^{-1}} \varphi\|_{L^{p,q}(\mathbb{R})} &\leq \lim_{\lambda \to 0} \left(\lambda^{r/p} \varphi(0)^r + \sum_{n=0}^{\infty} (\lambda 2^n)^{r/p} \varphi(\lambda 2^{n-1})^r \right)^{1/r} \|f\|_{L^{p,q}(\mathbb{T})} \\ &\leq (2^{r/p} - 1)^{-1/r} \limsup_{\lambda \to 0} \|\varphi_\lambda\|_{L^{p,r}(\mathbb{R})} \|f\|_{L^{p,q}(\mathbb{T})} \\ &= (2^{r/p} - 1)^{-1/r} \|\varphi\|_{L^{p,r}(\mathbb{R})} \|f\|_{L^{p,q}(\mathbb{T})}. \end{aligned}$$

This gives one of the inequalities of the lemma.

To get the other inequality, we use estimates from below to obtain

$$\liminf_{\epsilon \to 0} \|f D^p_{\epsilon^{-1}} \varphi\|_{L^{p,q}(\mathbb{R})} \geq \left(\lambda^{s/p} \varphi(\lambda 2^{-1})^s + \sum_{n=0}^{\infty} (\lambda 2^n)^{s/p} \varphi(\lambda 2^n)^s \right)^{1/s} \|f\|_{L^{p,q}(\mathbb{T})},$$

where $s = \max(p, q)$.

Using now that $s \geq p$, we get, arguing as above,

$$\left(\lambda^{s/p} \varphi(\lambda 2^{-1})^s + \sum_{n=0}^{\infty} (\lambda 2^n)^{s/p} \varphi(\lambda 2^n)^s \right)^{1/s} \geq (2^{s/p} - 1)^{-1/s} \|\varphi^\lambda\|_{L^{p,s}(\mathbb{R})},$$

where

$$\varphi^\lambda = \varphi(\lambda 2^{-1}) \chi_{[-\lambda 2^{-1}, \lambda 2^{-1}]} + \sum_{n \geq 0} \varphi(\lambda 2^n) \chi_{[-\lambda 2^n, \lambda 2^n] \setminus [-\lambda 2^{n-1}, \lambda 2^{n-1}]}.$$

Hence

$$\liminf_{\epsilon \to 0} \|f D^p_{\epsilon^{-1}} \varphi\|_{L^{p,q}(\mathbb{R})} \geq (2^{s/p} - 1)^{-1/s} \|\varphi\|_{L^{p,s}(\mathbb{R})} \|f\|_{L^{p,q}(\mathbb{T})}.$$

The proof is now complete. □

COROLLARY 2.8. *Let $0 < p < \infty$ and $f \in L^p(\mathbb{T})$. If $\varphi \in S(\mathbb{R})$ is radial and decreasing, then*

$$\|\varphi\|_{L^p(\mathbb{R})} \|f\|_{L^p(\mathbb{T})} = \lim_{\epsilon \to 0} \|f D^p_{\epsilon^{-1}} \varphi\|_{L^p(\mathbb{R})}.$$

In particular, for $p = 1$ and the periodic function f defined by $f = \widetilde{\chi_A}$ on $A \subset [-1/2, 1/2]$ we get

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x) D^1_{\epsilon^{-1}} \varphi(x) dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}} D_\epsilon f(x) \varphi(x) dx = m(A) \int_{\mathbb{R}} \varphi(x) dx.$$

We are now ready to prove our main result.

THEOREM 2.9. *Let m be a bounded continuous function on $\mathbb{R}^2$. Let $0 < p_i, q_i < \infty$, $i = 1, 2$, and let $0 < p_3, q_3 \leq \infty$ be such that $1/p_1 + 1/p_2 = 1/p_3$. Then m is a multiplier in $(L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ if and only if the functions $(D_{t^{-1}}m)_{t>0}$ restricted to $\mathbb{Z}^2$ are uniformly bounded multipliers in $(L^{p_1,q_1}(\mathbb{T}) \times L^{p_2,q_2}(\mathbb{T}), L^{p_3,q_3}(\mathbb{T}))$; i.e, setting $P_t = P_{(D_{t^{-1}}m)_{k,k'}}$, where $(D_{t^{-1}}m)_{k,k'} = m(tk, tk')$, there exists a constant $C > 0$ such that*

$$\|C_1(f,g)\|_{L^{p_3,q_3}(\mathbb{R})} \leq C\|f\|_{L^{p_1,q_1}(\mathbb{R})}\|g\|_{L^{p_2,q_2}(\mathbb{R})}$$

for $f, g \in S(\mathbb{R})$ if and only if there exists a constant $C' > 0$ such that

$$\|P_t(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} \leq C'\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|g\|_{L^{p_2,q_2}(\mathbb{T})}$$

uniformly in $t > 0$ for all trigonometric polynomials f, g.

Proof. "$\Rightarrow$": Let $\varphi = \chi_{[-1/2,1/2]}$ and $\psi(x) = \pi^{-1/2}e^{-x^2}$ Let $t > 0$ and let $f(x) = \sum_{k_1\in\mathbb{Z}} a_{k_1}e^{2\pi i k_1 x}$ and $g(x) = \sum_{k_2\in\mathbb{Z}} b_{k_2}e^{2\pi i k_2 x}$.

Since m is continuous we can write

$$\begin{aligned}
P_t(f,g)(x) &= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} a_{k_1}b_{k_2}m(tk_1,tk_2)e^{2\pi i(k_1+k_2)x}\\
&= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} a_{k_1}b_{k_2}\lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} D^1_\epsilon\psi(k_1-r)D^1_\epsilon\psi(k_2-s)m(tr,ts)e^{2\pi i(r+s)x}drds\\
&= \lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_{k_1\in\mathbb{Z}} a_{k_1}D^1_\epsilon\psi(r-k_1)\sum_{k_2\in\mathbb{Z}} b_{k_2}D^1_\epsilon\psi(s-k_2)m(tr,ts)e^{2\pi i(r+s)x}drds.
\end{aligned}$$

That is,

$$P_t(f,g)(x) = \lim_{\epsilon\to 0} C_t(f_\epsilon, g_\epsilon)(x), \tag{5}$$

where

$$\hat{f}_\epsilon = \sum_{k_1\in\mathbb{Z}} a_{k_1}T_{k_1}D^1_\epsilon\psi, \qquad \hat{g}_\epsilon = \sum_{k_2\in\mathbb{Z}} b_{k_2}T_{k_2}D^1_\epsilon\psi,$$

or, in other words,

$$f_\epsilon(x) = \sum_{k_1\in\mathbb{Z}} a_{k_1}M_{k_1}D^\infty_{\epsilon^{-1}}\check{\psi}(x) = \sum_{k_1\in\mathbb{Z}} a_{k_1}\check{\psi}(\epsilon x)e^{2\pi i k_1 x} = \check{\psi}(\epsilon x)f(x),$$

and a similar formula for g_ϵ. Moreover, the convergence is uniform since

$$\begin{aligned}
&|P_t(f,g)(x) - C_t(f_\epsilon,g_\epsilon)(x)|\\
&\quad\leq \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} |a_{k_1}||b_{k_2}|\times\\
&\qquad\times\int_{\mathbb{R}}\int_{\mathbb{R}} |m(tk_1,tk_2) - m(t(k_1-\epsilon r), t(k_2-\epsilon s))|\psi(r)\psi(s)drds,
\end{aligned}$$

which tends to zero uniformly in $x \in \mathbb{R}$ because the continuity of m. Thus

$$P_t(f,g) = \lim_{n\to\infty} C_t(f_n, g_n), \tag{6}$$

where $f_n(x) = \check{\psi}(n^{-1}x)f(x)$ and $g_n(x) = \check{\psi}(n^{-1}x)g(x)$ and the convergence is uniform. From Lemma 2.5 we also have for $k \in \mathbb{N}$

$$\|P_t(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} = \|P_t(f,g)D_k^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})}. \tag{7}$$

Combining these two facts we obtain

$$\begin{aligned}\|P_t(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} &= \|P_t(f,g)D_n^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})}\\ &\le C\Big(\|C_t(f_n,g_n)D_n^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})}\\ &\qquad + \|D_{n^{-1}}(P_t(f,g)-C_t(f_n,g_n))\varphi\|_{L^{p_3,q_3}(\mathbb{R})}\Big).\end{aligned}$$

For the first summand we use the estimate

$$\begin{aligned}\|C_t(f_n,g_n)D_n^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})} &= \|D_n^{p_3}(\varphi D_{n^{-1}}C_t(f_n,g_n))\|_{L^{p_3,q_3}(\mathbb{R})}\\ &= \|\varphi D_{n^{-1}}C_t(f_n,g_n)\|_{L^{p_3,q_3}(\mathbb{R})}\\ &\le \|D_{n^{-1}}C_t(f_n,g_n)\|_{L^{p_3,q_3}(\mathbb{R})}\|\varphi\|_{L^\infty(\mathbb{R})}\\ &= n^{-1/p_3}\|C_t(f_n,g_n)\|_{L^{p_3,q_3}(\mathbb{R})}\\ &\le n^{-1/p_3}C\|f_n\|_{L^{p_1,q_1}(\mathbb{R})}\|g_n\|_{L^{p_2,q_2}(\mathbb{R})}\\ &= Cn^{-1/p_1}\|f_n\|_{L^{p_1,q_1}(\mathbb{R})}n^{-1/p_2}\|g_n\|_{L^{p_2,q_2}(\mathbb{R})}.\end{aligned}$$

By Lemmas 2.6 and 2.7 we have

$$\lim_{n\to\infty} n^{-1/p_1}\|f_n\|_{L^{p_1,q_1}(\mathbb{R})} \le (2^{r_1/p_1}-1)^{-1/r_1}\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|\check{\psi}\|_{L^{p_1,r_1}(\mathbb{R})}$$

and

$$\lim_{n\to\infty} n^{-1/p_2}\|g_n\|_{L^{p_2,q_2}(\mathbb{R})} \le (2^{r_2/p_2}-1)^{-1/r_2}\|g\|_{L^{p_2,q_2}(\mathbb{T})}\|\check{\psi}\|_{L^{p_2,r_2}(\mathbb{R})}$$

with $r_i = \min(p_i, q_i)$ for $i = 1, 2$. Thus

$$\begin{aligned}\|P_t(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} &\le C\Big(\lim_{n\to\infty}\|C_t(f_n,g_n)D_n^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})}\\ &\qquad + \lim_{n\to\infty}\|P_t(f,g)-C_t(f_n,g_n)\|_{L^\infty(\mathbb{R})}\Big)\\ &= A(p_1,p_2)\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|g\|_{L^{p_2,q_2}(\mathbb{T})},\end{aligned}$$

and the implication "$\Rightarrow$" is proved.

"$\Leftarrow$": Assume that $D_{t^{-1}}m$ restricted to $\mathbb{Z}^2$ are uniformly bounded multipliers on $\mathbb{Z}^2$ and let $f, g \in S(\mathbb{R})$ be such that $\hat{f}$ and $\hat{g}$ have compact support contained in K.

By the Poisson formula we have

$$t\sum_{k_1}\hat{f}(tk_1)e^{2\pi ik_1x} = \sum_{k_1}(D_tf)\hat{\ }(k_1)e^{2\pi ik_1x} = \sum_{k_1}D_tf(x+k_1) = \widetilde{D_tf}(x).$$

Therefore, since m is continuous, we can write

$$\begin{aligned} C_1(f,g)(x) &= \iint_{K\times K} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \lim_{t\to 0} t^2 \sum_{k_1}\sum_{k_2} \hat{f}(tk_1)\hat{g}(tk_2)m(tk_1,tk_2)e^{2\pi i t(k_1+k_2)x} \\ &= \lim_{t\to 0} P_t(\widetilde{D_t f}, \widetilde{D_t g})(tx). \end{aligned}$$

Note that

$$\begin{aligned} &|\{x\in\mathbb{R} : |C_1(f,g)(x)| > \lambda\}| \\ &\qquad \le \liminf_{t\to 0} \left|\left\{|x| \le t^{-1}/2 : |P_t(\widetilde{D_t f}, \widetilde{D_t g})(tx)| > \lambda\right\}\right| \\ &\qquad \le \liminf_{t\to 0} t^{-1} \left|\left\{|x| \le 1/2 : |P_t(\widetilde{D_t f}, \widetilde{D_t g})(x)| > \lambda\right\}\right|. \end{aligned}$$

Therefore, formula (1) and Fatou's lemma give

$$\|C_1(f,g)\|^{p_3}_{L^{p_3,q_3}(\mathbb{R})} \le \liminf_{t\to 0} t^{-1} \|P_t(\widetilde{D_t f}, \widetilde{D_t g})\|^{p_3}_{L^{p_3,q_3}(\mathbb{T})}.$$

Using the assumption and Lemma 2.4 we obtain

$$\begin{aligned} \|C_1(f,g)\|_{L^{p_3,q_3}(\mathbb{R})} &\le \liminf_{t\to 0} t^{-1/p_3} \|\widetilde{D_t f}\|_{L^{p_1,q_1}(\mathbb{T})} \|\widetilde{D_t g}\|_{L^{p_2,q_2}(\mathbb{T})} \\ &\le C\|f\|_{L^{p_1,q_1}(\mathbb{R})}\|g\|_{L^{p_2,q_2}(\mathbb{R})}. \end{aligned}$$

This completes the proof. □

It is known that transference theorems can be extended to symbols that are more general than continuous symbols (see [8], [7], [9]). Actually, a bounded measurable function m_1 defined on $\mathbb{R}$ is called regulated if

$$\lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} m_1(x+t)dt = m_1(x)$$

for all $x \in \mathbb{R}$. As was pointed out in [8, Corollary 2.5], if m_1 is regulated and ϕ is non-negative, symmetric, smooth with compact support and $\int_{\mathbb{R}} \phi(t)dt = 1$, then

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}} m_1(x-\epsilon t)\phi(t)dt = \lim_{\epsilon\to 0} m_1 * D^1_\epsilon\phi(x) = m_1(x)$$

for all $x \in \mathbb{R}$. This actually implies that

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}} m_1(x-\epsilon t)\psi(t)dt = \lim_{\epsilon\to 0} m_1 * D^1_\epsilon\psi(x) = m_1(x), \tag{8}$$

where ψ is non-negative symmetric, smooth and satisfies $\int_{\mathbb{R}} \psi(t)dt = 1$. Indeed, given a function ψ, take non-negative, symmetric, smooth functions ϕ_n

with compact support and satisfying $\int_{\mathbb{R}} \phi_n(t)dt = 1$ such that $\lim_{n\to\infty} \|\psi - \phi_n\|_1 = 0$ and observe that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} (m_1(x-\epsilon t) - m_1(x))\psi(t)dt\right| \\
&\qquad \le 2\|m_1\|_\infty \int_{\mathbb{R}} |D^1_\epsilon \psi(t) - D^1_\epsilon \phi_n(t)|dt \\
&\qquad\qquad + \left|\int_{\mathbb{R}} (m_1(x-\epsilon t) - m_1(x))\phi_n(t)dt\right| \\
&\qquad = 2\|m_1\|_\infty \|\psi - \phi_n\|_1 + \left|\int_{\mathbb{R}} (m_1(x-\epsilon t) - m_1(x))\phi_n(t)dt\right|.
\end{aligned}$$

DEFINITION 2.10. Let $G(t,s) = \pi^{-1}e^{-(t^2+s^2)}$. A bounded measurable function m defined on $\mathbb{R}^2$ is G-regulated if

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} m(x-\epsilon t, y-\epsilon s)G(t,s)dtds = \lim_{\epsilon\to 0} m * D^1_\epsilon G(x,y) = m(x,y)$$

for all $(x,y) \in \mathbb{R}^2$.

An inspection of the proof of the preceding theorem shows that m need not be continuous but only G-regulated in order for the argument to work.

THEOREM 2.11. *Let m be a bounded G-regulated function on $\mathbb{R}^2$, $0 < p_i, q_i < \infty$, $i = 1,2$, and let $0 < p_3, q_3 \le \infty$ be such that $1/p_1 + 1/p_2 = 1/p_3$. If m is a multiplier in $(L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$, then m restricted to $\mathbb{Z}^2$ is a bounded multiplier in $(L^{p_1,q_1}(\mathbb{T}) \times L^{p_2,q_2}(\mathbb{T}), L^{p_3,q_3}(\mathbb{T}))$.*

This result can be applied to transfer results for the bilinear Hilbert transform in view of the following remark.

REMARK 2.2. If m_1 is a regulated function defined in $\mathbb{R}$, then $m_\alpha(x,y) = m_1(x+\alpha y)$ is G-regulated in $\mathbb{R}^2$. In particular, $m(x,y) = \mathrm{sign}(x+\alpha y)$ is G-regulated.

Indeed, observe that

$$\begin{aligned}
&\int_{\mathbb{R}^2} m_1(x - t + \alpha(y-s))D^1_\epsilon G(t,s)dtds \\
&\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} m_1(x+\alpha y - \epsilon(t+\alpha s))G(t,s)dtds \\
&\qquad = \int_{\mathbb{R}} m_1(x+\alpha y-\epsilon t)\left(\int_{\mathbb{R}} G(t-\alpha s, s)ds\right)dt \\
&\qquad = \int_{\mathbb{R}} m_1(x+\alpha y - \epsilon t)\psi_\alpha(t)dt,
\end{aligned}$$

where $\psi_\alpha(t) = \int_{\mathbb{R}} G(t - \alpha s, s)ds$. Hence we have, from (8), that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} m_\alpha(x-t, y-s) D^1_\epsilon G(t,s)dtds = m_\alpha(x,y).$$

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