Illinois Journal of Mathematics
Volume 47, Number 4, Winter 2003, Pages 1345–1361
S 0019-2082

SPECTRAL PROPERTIES OF PARABOLIC LAYER POTENTIALS AND TRANSMISSION BOUNDARY PROBLEMS IN NONSMOOTH DOMAINS

STEVE HOFMANN, JOHN LEWIS, AND MARIUS MITREA

ABSTRACT. We study the invertibility of $\lambda I + K$ in $L^p(\partial\Omega \times \mathbf{R})$, for p near 2 and $\lambda \in \mathbf{R}$, $|\lambda| \geq 1/2$, where K is the caloric double layer potential operator and Ω is a Lipschitz domain. Applications to transmission boundary value problems are also presented.

1. Introduction

Recall the usual Gaussian in $\mathbf{R}^n \times \mathbf{R}$,

$$\Gamma(x,t) := (4\pi t)^{-n/2} \exp\Big(-|x|^2/4t\Big) \text{ if } t>0, \text{ and zero otherwise.} \tag{1.1}$$

Fix now a domain $\Omega \subset \mathbf{R}^n$ with outward unit normal ν, and denote by $d\sigma$ the surface measure on $\partial\Omega$. The classical caloric double layer potentials on the boundary of the cylinder $\Omega \times \mathbf{R}$ are then given by (cf. [9])

$$K'f(x,t) := \lim_{\epsilon \to 0+} \int_{-\infty}^{t-\epsilon} \int_{\partial\Omega} \frac{\partial}{\partial \nu_x} \Gamma(x-y, t-s) f(y,s)\, d\sigma_y ds, \quad x \in \partial\Omega,\ t \in \mathbf{R}, \tag{1.2}$$

$$Kf(x,t) := \lim_{\epsilon \to 0+} \int_{-\infty}^{t-\epsilon} \int_{\partial\Omega} \frac{\partial}{\partial \nu_y} \Gamma(x-y, t-s) f(y,s)\, d\sigma_y ds, \quad x \in \partial\Omega,\ t \in \mathbf{R}. \tag{1.3}$$

It is well-known that when Ω is a (bounded) smooth domain and $T > 0$ is finite then, much as in the elliptic case, the operator K' is compact on $L^p(\partial\Omega \times (0,T))$, for $1 < p < \infty$ (alternatively, its norm is small with T), and all its eigenvalues lie in the interval $(-\frac{1}{2}, \frac{1}{2})$. On the other hand, the nature of the operator in question changes fundamentally when $\partial\Omega$ is allowed to contain irregularities; in particular, the aforementioned compactness property is lost

Received January 6, 2003; received in final form June 9, 2003.
2000 *Mathematics Subject Classification.* Primary 35K20, 45B05. Secondary 42B20, 45P05.
All three authors were supported in part by NSF.

(although the mere boundedness of K' in the case of Lipschitz boundaries—going back to the results in [1], [7]—is preserved). Thus, in the light of the above discussion about the location of the point spectrum of the operator K', a natural question—posed to us by Luis Escauriaza—is whether the intersection of the entire spectrum of K' with the real axis (in the complex plane) remains a subset of the interval $(-\frac{1}{2}, \frac{1}{2})$ in the case when Ω is only Lipschitz smooth.

In this note, we address this issue and prove the following result.

THEOREM 1.1. *Let Ω be the domain above the graph of a Lipschitz function $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$. Then there exist $\varepsilon > 0$ and $\kappa > 0$, both depending only on n and $\|\nabla\varphi\|_{L^\infty(\mathbf{R}^{n-1})}$, with the following significance. For any $2-\varepsilon < p < 2+\varepsilon$ and any $\lambda \in \mathbf{R}$, $|\lambda| \geq 1/2$, the operator*

$$\lambda I + K' : L^p(\partial\Omega \times \mathbf{R}) \longrightarrow L^p(\partial\Omega \times \mathbf{R}) \tag{1.4}$$

is invertible and the norm estimate

$$\|(\lambda I + K')^{-1}\|_{\mathcal{L}(L^p(\partial\Omega\times\mathbf{R}))} \leq \kappa \tag{1.5}$$

holds.

Analogous results hold in the case of the operator (1.3). To state them, recall that, for $1 < p < \infty$, the parabolic Sobolev space $L^p_{1,1/2}(\partial\Omega \times \mathbf{R})$ is the collection of all functions f such that $|\nabla_{\tan} f|$ and $|D_t^{1/2} f|$ belong to $L^p(\partial\Omega \times \mathbf{R})$. Here $\nabla_{\tan}$ denotes the tangential gradient on $\partial\Omega$, and $D_t^{1/2}$ is the fractional derivative operator of order 1/2 in time.

COROLLARY 1.2. *With the notations and assumptions in Theorem 1.1, the operators*

$$\lambda I + K : L^p(\partial\Omega \times \mathbf{R}) \longrightarrow L^p(\partial\Omega \times \mathbf{R}), \tag{1.6}$$

$$\lambda I + K : L^p_{1,1/2}(\partial\Omega \times \mathbf{R}) \longrightarrow L^p_{1,1/2}(\partial\Omega \times \mathbf{R}) \tag{1.7}$$

are invertible for each $\lambda \in \mathbf{R}$ with $|\lambda| \geq 1/2$, and $p \in (2-\varepsilon, 2+\varepsilon)$. In particular, if Ω is convex, then the spectral radii of the operator K on the above spaces are $< 1/2$.

Results similar in spirit have been proved in the case of *harmonic* layer potentials in [5], [8]. The main idea there is to reconsider the Rellich identities associated with the Laplacian, originally used to prove the invertibility of operators like $\pm\frac{1}{2}I + K$ (cf. [24]), and carefully monitor the effect of replacing $\pm\frac{1}{2}$ by a more general parameter $\lambda \in \mathbf{R}$. While, in principle, this seems flexible enough a program to be worth pursuing in the case of the heat operator, the algebra associated with the problem at hand is different. In particular, there are several genuinely new terms in the parabolic case (cf. the discussion following (2.14)), and our main contribution is to indicate how these can be handled.

One significant application of Theorem 1.1 has to do with parabolic transmission boundary problems, i.e., when one is interested in finding two functions $u_\pm \in C^\infty(\Omega_\pm \times \mathbf{R})$ such that

$$(1.8) \qquad (TBVP) \begin{cases} (\partial_t - \Delta)u_\pm = 0 \text{ in } \Omega_\pm \times \mathbf{R}, \\ (\nabla u_\pm)^*,\ (D_t^{1/2} u_\pm)^* \in L^p(\partial\Omega \times \mathbf{R}), \\ (u_+)\big|_{\partial\Omega\times\mathbf{R}} - (u_-)\big|_{\partial\Omega\times\mathbf{R}} = F \in L^p_{1,1/2}(\partial\Omega \times \mathbf{R}), \\ (D_\nu u_+)\big|_{\partial\Omega\times\mathbf{R}} - \mu(D_\nu u_-)\big|_{\partial\Omega\times\mathbf{R}} = G \in L^p(\partial\Omega \times \mathbf{R}). \end{cases}$$

Hereafter, Δ is the usual Laplace operator in $\mathbf{R}^n$, and we set $\Omega_+ := \Omega$ and $\Omega_- := \mathbf{R}^n \setminus \bar{\Omega}$. Also, $\mu \in \mathbf{R}$, $\mu > 0$, $\mu \neq 1$, is the transmission parameter and $(\cdot)^*$ stands for the (parabolic) nontangential maximal function; more precise definitions are given in the body of the paper.

For related problems see [10] and the references therein (such as [4] which also contains an overview). A classical treaty on the heat equation, including transmission problems, is the monograph [17]. Other types of transmission boundary problems for parabolic PDE's on cylinders with Lipschitz interfaces have been considered in [6]. Here we prove the following theorem.

THEOREM 1.3. *With the notations and assumptions in Theorem 1.1, the transmission boundary problem* (1.8) *has a unique solution whenever* $2 - \varepsilon < p < 2 + \varepsilon$. *Furthermore, the estimate*

$$(1.9) \quad \|(\nabla u_\pm)^*\|_{L^p(\partial\Omega\times\mathbf{R})} + \|(D_t^{1/2} u_\pm)^*\|_{L^p(\partial\Omega\times\mathbf{R})} \\ \leq C\Big(\|\nabla_{\tan} F\|_{L^p(\partial\Omega\times\mathbf{R})} + \|D_t^{1/2} F\|_{L^p(\partial\Omega\times\mathbf{R})} + \|G\|_{L^p(\partial\Omega\times\mathbf{R})}\Big)$$

holds for some finite constant C, *depending only on* n, μ *and* $\|\nabla\varphi\|_{L^\infty(\mathbf{R}^{n-1})}$.

The proofs of the results stated here are collected in Section 2. All the above results remain true in the case when $\Omega \times \mathbf{R}$ is replaced by $\Omega \times (0, T)$, with Ω a *bounded* Lipschitz domain (i.e., a bounded domain whose boundary can be described locally by means of graphs of Lipschitz functions), and $T > 0$. See Section 3 for precise statements and proofs. Finally, in Section 4, we describe an adaptation of our results to the class of time-varying domains (in the sense of [18], [11]). Our approach from the cylindrical case continues to work, with the major difference that new error terms appear in this setting. Ultimately, they can be handled by invoking estimates originally proved in [12].

In closing, let us remark that starting with the (end-point) results contained in Corollary 1.2 and relying on real and complex and interpolation methods, one can produce further invertibility results for the operator $\lambda I + K$ in parabolic Sobolev-Besov spaces with fractional smoothness exponents. A thorough treatment of the classical caloric layer potential operators in this latter context can be found in [13]. Also, our methods seem flexible enough to be applied to certain types of parabolic systems (cf. [22], [20]).

2. The proofs of the main results

Proof of Theorem 1.1. To set the stage, we make several preliminary observations which will eventually allow us to reduce the proof of Theorem 1.1 to handling a technically simpler case. To begin, it suffices to show that there exists $\varepsilon > 0$ so that for any $2-\varepsilon < p < 2+\varepsilon$ the estimate

$$\|f\|_{L^p(\partial\Omega\times\mathbf{R})} \leq C\|(\lambda I + K')f\|_{L^p(\partial\Omega\times\mathbf{R})}, \qquad \forall\, f \in L^p(\partial\Omega\times\mathbf{R}), \tag{2.1}$$

holds uniformly in $\lambda \in \mathbf{R}$, $|\lambda| \geq \frac{1}{2}$. Indeed, this implies that each $\lambda I + K'$ is injective with closed range, on $L^p(\partial\Omega\times\mathbf{R})$. Consequently, $\lambda \mapsto \lambda I + K'$ is a continuous family of semi-Fredholm operators. Due to the homotopic invariance of the index and since, obviously, $\lambda I + K'$ is invertible when $|\lambda|$ is large, it follows that the index of each $\lambda I + K'$ is zero. Since all operators in question are one-to-one (as seen from (2.1)), we may finally conclude that $\lambda I + K'$ is an isomorphism of $L^p(\partial\Omega\times\mathbf{R})$ for each λ real with $|\lambda| \geq \frac{1}{2}$. (An alternative argument is as follows: If $\lambda I + K'$ failed to be invertible for $\lambda \in (-\infty, \frac{1}{2})$ or $\lambda \in (\frac{1}{2}, +\infty)$, then one of these intervals would intersect $\partial\sigma(K'; L^p(\partial\Omega\times\mathbf{R}))$, which is a subset of the approximate spectrum of K'—i.e., the collection of all complex z's such that $zI - K'$ is not an isomorphism onto its range. This scenario, in turn, is excluded by (2.1).)

Our next step in our series of reductions is to observe that, from the general perturbation theory in [14], it suffices to treat the case $p = 2$ only. In this context, the situation when $\lambda = \pm\frac{1}{2}$ is known—cf. [2]; thus, we shall henceforth assume that $|\lambda| > \frac{1}{2}$. In fact, since K' associated with Ω is the opposite of K' associated with the complementary domain, $\mathbf{R}^n \setminus \bar{\Omega}$, we may further reduce matters to the case when $\lambda > \frac{1}{2}$ (the non-trivial case being when λ lies in a compact subinterval of $(\frac{1}{2}, \infty)$; cf. the comments at the end of the proof). In summary, it suffices to prove (2.1) when $p = 2$ and $\lambda > \frac{1}{2}$.

Denote by $\langle\cdot,\cdot\rangle$ the usual inner product of vectors in $\mathbf{R}^n$, set $e_o := (1, 0, ..., 0) \in \mathbf{R}^n$ and, for arbitrary $x \in \mathbf{R}^n$, define $x_o := \langle x, e_o\rangle$. In particular, a simple calculation reveals that

$$\nu_o = \langle \nu, e_o\rangle \leq \kappa < 0 \tag{2.2}$$

for some κ depending only on n and $\|\nabla\varphi\|_{L^\infty(\mathbf{R}^{n-1})}$. Finally, set $D_v := \langle v, \nabla\rangle$, i.e., the directional derivative operator in the direction of the vector $v \in \mathbf{R}^n$.

Fix now a caloric function (i.e., a null-solution of the heat operator) u in $(\mathbf{R}^n \times \mathbf{R}) \setminus (\partial\Omega\times\mathbf{R})$ which decays at infinity and which has a reasonable behavior near the boundary (so that all subsequent integration by parts formulas are justified). A classical identity, originally due to Rellich (and rediscovered several times since—cf. [21], [16], and the references therein) reads

$$-\Delta u\, D_{e_o} u = \operatorname{div}\Big(\tfrac{1}{2} e_o |\nabla u|^2 - \nabla u\, D_{e_o} u\Big). \tag{2.3}$$

Recall that ν stands for the outward unit normal to Ω. Next, set $\nabla_{\tan} u := \nabla u - (D_\nu u)\nu$ for the tangential gradient on $\partial\Omega$ so that

$$|\nabla u|^2 = |D_\nu u|^2 + |\nabla_{\tan} u|^2 \quad \text{and} \quad D_{e_o} u = \nu_o D_\nu u + (\nabla_{\tan} u)_o. \tag{2.4}$$

On account of (2.3)–(2.4) and the fact that $\Delta u = \partial_t u$, the Divergence Theorem eventually gives

$$\begin{aligned} -\tfrac{1}{2}\int_{\partial\Omega} \nu_o |D_\nu u|^2\, d\sigma + \tfrac{1}{2}\int_{\partial\Omega} \nu_o |\nabla_{\tan} u|^2\, d\sigma \\ = \int_{\partial\Omega} (\nabla_{\tan} u)_o\, D_\nu u\, d\sigma - \iint_{\Omega} (\partial_t u)(D_{e_o} u)\, dx. \end{aligned} \tag{2.5}$$

Hence, after integrating in the time variable, this equality becomes

$$\begin{aligned} -\tfrac{1}{2}\int_{\mathbf{R}}\int_{\partial\Omega} \nu_o |D_\nu u|^2\, d\sigma dt + \tfrac{1}{2}\int_{\mathbf{R}}\int_{\partial\Omega} \nu_o |\nabla_{\tan} u|^2\, d\sigma dt \\ = \int_{\mathbf{R}}\int_{\partial\Omega} (\nabla_{\tan} u)_o\, D_\nu u\, d\sigma dt - \int_{\mathbf{R}}\iint_{\Omega} (\partial_t u)(D_{e_o} u)\, dx dt. \end{aligned} \tag{2.6}$$

Going further, recall the caloric single layer potential operator

$$\mathcal{S}f(x,t) := \int_{-\infty}^{t}\int_{\partial\Omega} \Gamma(x-y, t-s) f(y,s)\, d\sigma_y ds, \quad x \notin \partial\Omega,\ t \in \mathbf{R}, \tag{2.7}$$

and specialize (2.6) to the case when $u := \mathcal{S}f$, for some arbitrary $f \in C^\infty_{\text{comp}}(\partial\Omega \times \mathbf{R})$. (This is just for technical convenience, to ensure the validity of the various formal manipulations we shall make, such as integrations by parts; that we can eventually return to $L^2(\partial\Omega \times \mathbf{R})$ is guaranteed by the boundedness of the operators we are dealing with.)

The arguments in [5] for the Laplacian make essential use of duality, an ingredient which is desirable to avoid in the case of the non-selfadjoint heat operator considered here. Instead, we rely on the good algebraic interactions between the versions of (2.6) written for Ω as well as for its complement.

More specifically, let us denote by $\mathcal{I}_+$ and $\mathcal{I}_-$ the identity (2.6) written for Ω_+ and Ω_-, respectively, and fix $\lambda \in \mathbf{R}$, $\lambda > \frac{1}{2}$. The next step is to create a new identity, $\mathcal{I}$, formally defined as $\mathcal{I} := (\lambda - \frac{1}{2})\mathcal{I}_+ + (\lambda + \frac{1}{2})\mathcal{I}_-$. In other words, we multiply both sides of (2.6) by $\lambda - \frac{1}{2}$, then multiply both sides of the version of (2.6), written for Ω_- in place of Ω, by $\lambda + \frac{1}{2}$ and, finally, add the resulting identities, side by side.

Turning to the actual details, it is helpful to remember in this process that the *outward* unit normal to Ω_- is $-\nu$, and that

$$(D_\nu u)\Big|_{\partial\Omega_\pm \times \mathbf{R}} = (\mp\tfrac{1}{2} I + K')f, \quad (\nabla_{\tan} u)\Big|_{\partial\Omega_+ \times \mathbf{R}} = (\nabla_{\tan} u)\Big|_{\partial\Omega_- \times \mathbf{R}}. \tag{2.8}$$

Recall next that the (parabolic) nontangential maximal function $(\cdot)^*$ is defined for $u:\Omega_\pm\times\mathbf{R}\to\mathbf{R}$ by

$$u^*(x,t):=\sup\{|u(y,t)|;\ |x-y|\le 2\,\mathrm{dist}\,(x,\partial\Omega)\},\quad x\in\partial\Omega,\ t\in\mathbf{R}. \tag{2.9}$$

It is convenient to keep in mind that, as is well known,

$$\|(\nabla u)^*\|_{L^2(\partial\Omega\times\mathbf{R})}\le C\|f\|_{L^2(\partial\Omega\times\mathbf{R})}. \tag{2.10}$$

At this stage, the left side of $\mathcal{I}$ reads

$$\begin{aligned} &-(\lambda-\tfrac12)\tfrac12\int_{\mathbf{R}}\int_{\partial\Omega}|(-\tfrac12 I+K')f|^2\nu_o\,d\sigma dt\\ &\qquad+(\lambda+\tfrac12)\tfrac12\int_{\mathbf{R}}\int_{\partial\Omega}|(\tfrac12 I+K')f|^2\nu_o\,d\sigma dt\\ &\qquad-\tfrac12\int_{\mathbf{R}}\int_{\partial\Omega}|\nabla_{\tan}u|^2\nu_o\,d\sigma dt. \end{aligned} \tag{2.11}$$

Call a term *good* if it is $\mathcal{O}\big(\|f\|_{L^2(\partial\Omega\times\mathbf{R})}\|(\lambda I+K')f\|_{L^2(\partial\Omega\times\mathbf{R})}\big)$; in particular, by the results in [1], [7], any term which is $\mathcal{O}\big(\|(\lambda I+K')f\|^2_{L^2(\partial\Omega\times\mathbf{R})}\big)$ is good.

Now, decomposing $(\mp\frac12 I+K')f=(\mp\frac12-\lambda)f+(\lambda I+K')f$ in the first two integrands in (2.11) we may conclude, after some algebra, that

$$\begin{aligned} \text{left side of }\mathcal{I}=&-(\lambda^2-\tfrac14)\tfrac12\int_{\mathbf{R}}\int_{\partial\Omega}|f|^2\nu_o\,d\sigma dt\\ &-\tfrac12\int_{\mathbf{R}}\int_{\partial\Omega}|\nabla_{\tan}u|^2\nu_o\,d\sigma dt+\text{good terms.} \end{aligned} \tag{2.12}$$

In fact, the same strategy yields

$$\begin{aligned} \text{right side of }\mathcal{I}=&-(\lambda-\tfrac12)\int_{\mathbf{R}}\iint_{\Omega_+}(\partial_t u)(D_{e_o}u)\,dxdt\\ &-(\lambda+\tfrac12)\int_{\mathbf{R}}\iint_{\Omega_-}(\partial_t u)(D_{e_o}u)\,dxdt+\text{good terms,} \end{aligned} \tag{2.13}$$

so that, all in all, $\mathcal{I}$ reads

$$\begin{aligned} &\tfrac12(\lambda^2-\tfrac14)\int_{\mathbf{R}}\int_{\partial\Omega}|f|^2(-\nu_o)\,d\sigma dt+\tfrac12\int_{\mathbf{R}}\int_{\partial\Omega}|\nabla_{\tan}u|^2(-\nu_o)\,d\sigma dt\\ &\quad=-(\lambda-\tfrac12)\int_{\mathbf{R}}\iint_{\Omega_+}(\partial_t u)(D_{e_o}u)\,dxdt\\ &\qquad-(\lambda+\tfrac12)\int_{\mathbf{R}}\iint_{\Omega_-}(\partial_t u)(D_{e_o}u)\,dxdt+\text{good terms.} \end{aligned} \tag{2.14}$$

A word of explanation with regard to the last two solid integrals above—which do not appear in the elliptic case—is in order here. An attempt to handle them based purely on size estimates of square-function type (note that, informally speaking, $(\partial_t u)(D_{e_o}u)$ is the equivalent of 3/2 spatial derivatives on each u), runs into the problem that each is $\mathcal{O}(\|f\|^2_{L^2(\partial\Omega\times\mathbf{R})})$, which, of course, does not suit our purposes. In fact, when taken separately, none of the two integrals in question is a 'good' term. Instead, our strategy is to show that, when *simultaneously* considered, and after further algebraic manipulations which allow certain suitable *cancellations* to occur, the first two solid integrals in the right hand side of (2.14) eventually amount to a combination of 'good terms.' Thus, the key element in our analysis is the algebraic interplay between the two versions of the identity (2.6) written for Ω_+ and Ω_-, respectively. This is achieved by integrating by parts, as dictated by the homogeneity of the various terms considered. In particular, as far as the time variable is concerned, we shall need to factor out ∂_t into a product of fractional derivatives.

To this end, for $0<\alpha<1$ introduce D_t^α, the fractional derivative operator of order α in the time variable as the Fourier multiplier operator corresponding to the symbol $|\tau|^\alpha$. Thus, among other things,

$$D_t^\alpha \text{ is symmetric, and } \partial_t = -H\, D_t^\alpha\, D_t^{1-\alpha} \text{ for any } 0<\alpha<1, \tag{2.15}$$

where H stands for the usual Hilbert transform on $\mathbf{R}$ (cf. [23]). As is well-known,

$$\begin{aligned} &(D_t^{1/2}u)\Big|_{\partial\Omega_+\times\mathbf{R}} = (D_t^{1/2}u)\Big|_{\partial\Omega_-\times\mathbf{R}} \quad \text{and} \\ &\|(D_t^{1/2}u)^*\|_{L^2(\partial\Omega\times\mathbf{R})} \le C\|f\|_{L^2(\partial\Omega\times\mathbf{R})}. \end{aligned} \tag{2.16}$$

For further use, we also note here the following form of Green's first identity,

$$\begin{aligned} \pm\int_{\mathbf{R}}\int_{\partial\Omega_\pm} v\,(D_\nu w)\,d\sigma dt = \int_{\mathbf{R}}\iint_{\Omega_\pm}\langle\nabla v,\nabla w\rangle\,dxdt \\ + \int_{\mathbf{R}}\iint_{\Omega_\pm} v\,(\partial_t w)\,dxdt, \end{aligned} \tag{2.17}$$

valid for any two C^2 functions v,w in $\Omega_\pm\times\mathbf{R}$, which are well-behaved at infinity and near the boundary, and such that w is caloric. We write

$$\begin{aligned} -\int_{\mathbf{R}}\iint_{\Omega_\pm}&(\partial_t u)(D_{e_o}u)\,dxdt \\ &= -\int_{\mathbf{R}}\iint_{\Omega_\pm}(HD_t^{3/4}u)(D_t^{1/4}D_{e_o}u)\,dxdt \\ &\le \tfrac12\int_{\mathbf{R}}\iint_{\Omega_\pm}|HD_t^{3/4}u|^2\,dxdt \\ &\qquad + \tfrac12\int_{\mathbf{R}}\iint_{\Omega_\pm}|D_t^{1/4}D_{e_o}u|^2\,dxdt \\ &=: A_\pm + B_\pm. \end{aligned} \tag{2.18}$$

Now, the identity (2.17) with $v := HD_t^{1/2}u$, $w := u$, gives

$$\begin{aligned}(2.19)\qquad 2\,A_\pm &= \int_{\mathbf{R}}\iint_{\Omega_\pm} |HD_t^{3/4}u|^2\,dxdt = \int_{\mathbf{R}}\iint_{\Omega_\pm} (HD_t^{1/2}u)\partial_t u\,dxdt \\ &= \pm\int_{\mathbf{R}}\int_{\partial\Omega_\pm} (HD_t^{1/2}u)(D_\nu u)\,d\sigma dt \\ &\quad - \int_{\mathbf{R}}\iint_{\Omega_\pm} \langle\nabla(HD_t^{1/2}u),\nabla u\rangle\,dxdt \\ &= \pm\int_{\mathbf{R}}\int_{\partial\Omega_\pm} (HD_t^{1/2}u)(D_\nu u)\,d\sigma dt.\end{aligned}$$

In the last equality above we have used

$$\begin{aligned}(2.20)\qquad &\int_{\mathbf{R}}\iint_{\Omega_\pm} \langle\nabla(HD_t^{1/2}u),\nabla u\rangle\,dxdt \\ &\qquad = \int_{\mathbf{R}}\iint_{\Omega_\pm} \langle H(\nabla D_t^{1/4}u),\nabla D_t^{1/4}u\rangle\,dxdt = 0,\end{aligned}$$

by the antisymmetry of the Hilbert transform. If we now invoke the absence of a jump for $HD_t^{1/2}u$ across $\partial\Omega\times\mathbf{R}$, i.e.,

$$(2.21)\qquad (HD_t^{1/2}u)\Big|_{\partial\Omega_+\times\mathbf{R}} = (HD_t^{1/2}u)\Big|_{\partial\Omega_-\times\mathbf{R}},$$

in concert with the identity

$$(2.22)\qquad (\lambda-\tfrac12)(-\tfrac12 I+K')f-(\lambda+\tfrac12)(\tfrac12 I+K')f = -(\lambda I+K')f,$$

we may eventually write

$$\begin{aligned}(2.23)\quad (\lambda-\tfrac12)A_+ + (\lambda+\tfrac12)A_- &= \mathcal{O}\Big(\|f\|_{L^2(\partial\Omega\times\mathbf{R})}\|(\lambda I+K')f\|_{L^2(\partial\Omega\times\mathbf{R})}\Big) \\ &= \text{good term}.\end{aligned}$$

On the other hand, the identity (2.17) written for $v := w = D_t^{1/4}u$ allows us to estimate

$$\begin{aligned}(2.24)\quad \int_{\mathbf{R}}\iint_{\Omega_\pm} |D_t^{1/4}D_{e_o}u|^2\,dxdt &\le \int_{\mathbf{R}}\iint_{\Omega_\pm} |D_t^{1/4}\nabla u|^2\,dxdt \\ &= \int_{\mathbf{R}}\iint_{\Omega_\pm} \langle\nabla D_t^{1/4}u,\nabla D_t^{1/4}u\rangle\,dxdt \\ &= \pm\int_{\mathbf{R}}\int_{\partial\Omega_\pm} (D_t^{1/2}u)(D_\nu u)\,d\sigma dt \\ &\quad - \int_{\mathbf{R}}\iint_{\Omega_\pm} (D_t^{1/4}u)\partial_t(D_t^{1/4}u)\,dxdt \\ &= \pm\int_{\mathbf{R}}\int_{\partial\Omega_\pm} (D_t^{1/2}u)(D_\nu u)\,d\sigma dt,\end{aligned}$$

where the last step utilizes the antisymmetry of ∂_t. Thus, relying on (2.8), (2.16) much as before, we get

$$(2.25)\quad (\lambda-\tfrac{1}{2})B_+ + (\lambda+\tfrac{1}{2})B_- = \mathcal{O}\Big(\|f\|_{L^2(\partial\Omega\times\mathbf{R})}\|(\lambda I+K')f\|_{L^2(\partial\Omega\times\mathbf{R})}\Big) = \text{good term.}$$

All in all, from (2.25), (2.23), (2.18), (2.14) and (2.2), we arrive at the conclusion that for each $\lambda > \frac{1}{2}$ there exists a finite constant $C_\lambda = C(\partial\Omega, n, \lambda) > 0$ so that

$$(2.26)\qquad \|f\|^2_{L^2(\partial\Omega\times\mathbf{R})} \le C_\lambda \|f\|_{L^2(\partial\Omega\times\mathbf{R})} \cdot \|(\lambda I+K')f\|_{L^2(\partial\Omega\times\mathbf{R})}.$$

In fact, we claim that matters can be arranged so that the map $(\frac{1}{2},\infty) \ni \lambda \mapsto C_\lambda \in (0,\infty)$ is bounded. An inspection of our proof shows that $C_\lambda \le C(\partial\Omega)(\lambda^2-\frac{1}{4})^{-1}$, which serves our purpose only as long as λ stays away from $\frac{1}{2}$. (The special role of $\lambda=\frac{1}{2}$ is highlighted by the first identity in (2.8).) Nonetheless, the estimate (2.1) with $\lambda=\frac{1}{2}$ has been established in [2]. This, in turn, remains true for λ near $\frac{1}{2}$ as an elementary perturbation argument shows.

At this point, (2.1) is justified and, hence, the proof of Theorem 1.1 is finished. □

Proof of Corollary 1.2. If $Rf(x,t) := f(x,-t)$ is the reflection in time, it is clear that

$$(2.27)\qquad K = R\,(K')^t\,R,$$

where the superscript 't' indicates transposition. It follows that $\lambda I+K = R\,(\lambda I+K')^t\,R$ which, together with Theorem 1.1 and duality, takes care of the claim about the operator (1.6).

Next, recall that if

$$(2.28)\quad \mathcal{D}f(x,t) := \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial\nu_y}\Gamma(x-y,t-s)f(y,s)\,d\sigma_y ds, \quad x\notin\partial\Omega,\ t\in\mathbf{R},$$

stands for the caloric double layer potential, then $\mathcal{D}f|_{\partial\Omega\times\mathbf{R}} = (\frac{1}{2}I+K)f$ and

$$(2.29)\qquad u = \mathcal{D}(u|_{\partial\Omega\times\mathbf{R}}) - \mathcal{S}(D_\nu u)$$

for any (reasonably well-behaved) caloric function u in $\Omega\times\mathbf{R}$. In particular, substituting $u := \mathcal{S}f$, with $f\in L^p(\partial\Omega\times\mathbf{R})$, in (2.29) yields—after some algebra,

$$(2.30)\qquad KS = SK',$$

where $Sf := \mathcal{S}f|_{\partial\Omega\times\mathbf{R}}$ stands for the boundary trace of the caloric single layer potential. Thus, S intertwines $\lambda I+K$ and $\lambda I+K'$, so that the desired claim (about the operator (1.7)) follows from Theorem 1.1, if we recall from [2], [3], that

$$S : L^p(\partial\Omega \times \mathbf{R}) \longrightarrow L^p_{1,1/2}(\partial\Omega \times \mathbf{R}) \tag{2.31}$$

is an isomorphism for p near 2. □

Proof of Theorem 1.3. Let us recall from [2], [3] that for each p near 2,

$$\begin{aligned}&(\partial_t - \Delta)U = 0 \text{ in } \Omega \times \mathbf{R} \text{ and } (\nabla U)^*,\ (D_t^{1/2}U)^* \in L^p(\partial\Omega \times \mathbf{R})\\ &\text{if and only if } U = \mathcal{S}f \text{ in } \Omega \times \mathbf{R} \text{ for some } f \in L^p(\partial\Omega \times \mathbf{R}).\end{aligned} \tag{2.32}$$

Given that

$$(D_\nu \mathcal{S}f)|_{\partial\Omega\pm\times\mathbf{R}} = \Big(\mp\tfrac{1}{2}I + K'\Big)f \quad \text{and} \quad \mathcal{S}f|_{\partial\Omega\pm\times\mathbf{R}} = Sf, \tag{2.33}$$

the transmission boundary problem (1.8) becomes equivalent to finding two (unique) functions $f, g \in L^p(\partial\Omega \times \mathbf{R})$ so that

$$\begin{cases} Sf - Sg = F, \\ \Big(-\tfrac{1}{2}I + K'\Big)f - \mu\Big(\tfrac{1}{2}I + K'\Big)g = G. \end{cases} \tag{2.34}$$

The first line in (2.34) entails $g = f - S^{-1}F$. In turn, when further substituted in the second line in (2.34) this leads to an equation of the form

$$\Big(\lambda I + K'\Big)f = \tilde{G}, \tag{2.35}$$

where

$$\lambda := \frac{1}{2}\cdot\frac{\mu+1}{\mu-1} \quad \text{and} \quad \tilde{G} := \frac{1}{1-\mu}\Big[G - \mu(\tfrac{1}{2}I + K')S^{-1}F\Big] \in L^p(\partial\Omega\times\mathbf{R}). \tag{2.36}$$

Note that $\lambda \in \mathbf{R}$ satisfies $|\lambda| > \frac{1}{2}$, so Theorem 1.1 applies and allows us to write the (unique) solution of (2.34) in the form

$$f = \Big(\lambda I + K'\Big)^{-1}\tilde{G}, \quad g = \Big(\lambda I + K'\Big)^{-1}\tilde{G} - S^{-1}F. \tag{2.37}$$

From these and the properties of the layer potentials involved (cf. (2.10, (2.16)), the estimate (1.9) follows. □

In closing, we would like to point out that—as an inspection of the above proof reveals—the well-posedness of the problem (1.8) for *all* values of the transmission parameter μ (i.e., $\mu > 0$, $\mu \neq 1$) is in fact *equivalent* to the validity of Theorem 1.1 (assuming that p is near 2).

3. The adaptation to finite cylinders

For Ω a *bounded* Lipschitz domain in $\mathbf{R}^n$ and $T > 0$ a fixed, finite number, introduce the space $L^p_{1,1/2}(\partial\Omega \times (0,T))$ as the collection of restrictions $F|_{\partial\Omega\times(0,T)}$ of functions $F \in L^p_{1,1/2}(\partial\Omega \times \mathbf{R})$ with the extra property that $F \equiv 0$ for $t < 0$. Next, consider the initial transmission boundary value problem

$$
\left\{
\begin{array}{l}
(\partial_t - \Delta)u_\pm = 0 \text{ in } \Omega_\pm \times (0,T),\\
(\nabla u_\pm)^*,\ (D_t^{1/2} u_\pm)^* \in L^p(\partial\Omega \times (0,T)),\\
u_\pm(\cdot,0) \equiv 0 \text{ in } \Omega_\pm,\\
(u_+)\big|_{\partial\Omega\times(0,T)} - (u_-)\big|_{\partial\Omega\times(0,T)} = F \in L^p_{1,1/2}(\partial\Omega \times (0,T)),\\
(D_\nu u_+)\big|_{\partial\Omega\times(0,T)} - \mu(D_\nu u_-)\big|_{\partial\Omega\times(0,T)} = G \in L^p(\partial\Omega \times (0,T)).
\end{array}
\right.
\tag{3.1}
$$

Here, once again, the transmission parameter $\mu \in \mathbf{R}$ is assumed to satisfy $\mu > 0$, $\mu \neq 1$.

Much as in the case of infinite cylinders, the well-posedness of the above transmission problem (at least for $1 < p \leq 2$) for all μ's is equivalent to the invertibility of $\lambda I + K'$ on $L^p(\partial\Omega \times (0,T))$ for all $\lambda \in \mathbf{R}$, $|\lambda| > \frac{1}{2}$.

THEOREM 3.1. *For any bounded Lipschitz domain Ω there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ so that for any $2 - \varepsilon < p < 2 + \varepsilon$ the problem* (3.1) *has a unique solution.*

Proof. Once again, it suffices to treat the case $p = 2$; the extension to $p \in (2-\varepsilon, 2+\varepsilon)$ for some small $\varepsilon > 0$ follows from the general stability results alluded to before. For the time being, let us continue to assume—as we have done in the preceding sections—that Ω is the *unbounded* domain above the graph of a Lipschitz function $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$. Our goal is to prove a version of our previous results in which $\Omega \times \mathbf{R}$ is replaced by $\Omega \times (0,T)$.

The key idea is to show that, besides being well posed, the transmission boundary problem (1.8) with $p = 2$ has the property that

$$
F \equiv 0 \text{ for } t < 0 \text{ and } G \equiv 0 \text{ for } t < 0 \implies u_\pm \equiv 0 \text{ for } t < 0. \tag{3.2}
$$

With this goal in mind, Green's formula (cf. (2.17)) plus the fact that $u_\pm$ are caloric in $\Omega_\pm \times \mathbf{R}$ gives

$$
\begin{aligned}
&\int_{-\infty}^0 \iint_{\Omega_\pm} |\nabla u_\pm|^2 \, dxdt \\
&\qquad = -\iint_{\Omega_\pm} |u_\pm(x,0)|^2 \, dx \pm \int_{-\infty}^0 \int_{\partial\Omega} (D_\nu u_\pm)\, u_\pm \, d\sigma dt.
\end{aligned}
\tag{3.3}
$$

The assumption that $F \equiv 0$ and $G \equiv 0$ for $t < 0$ together with the transmission boundary conditions in (1.8) ensure that the boundary terms in (3.3) (in the two versions, corresponding to the choices $\pm$) are multiples of each other, by a factor of μ. Thus, combining the identities (3.3) in such a way that these boundary integrals cancel leads to the conclusion that

$$\begin{aligned}(3.4)\qquad &\int_{-\infty}^{0}\iint_{\Omega_+}|\nabla u_+|^2\,dxdt+\iint_{\Omega_+}|u_+(x,0)|^2\,dx\\ &\qquad+\mu\int_{-\infty}^{0}\iint_{\Omega_-}|\nabla u_-|^2\,dxdt+\mu\iint_{\Omega_-}|u_-(x,0)|^2\,dx=0.\end{aligned}$$

This clearly forces $u_\pm \equiv 0$ for $t < 0$ as desired.

Going further, (3.2) and the connection between the transmission boundary problems we are considering, on the one hand, and the classical caloric layer potential operators, on the other hand (cf. the remark made at the end of Section 2), allow us to conclude that

$$(3.5)\qquad \lambda I + K' : L^2(\partial\Omega\times(0,T)) \longrightarrow L^2(\partial\Omega\times(0,T))$$

is invertible for each λ real with $|\lambda| > \frac{1}{2}$.

The version of this result corresponding to the case when Ω is a *bounded* Lipschitz domain can then be proved by semi-standard arguments. More specifically, the property of the singular integral operators (we are dealing with) of being bounded from below, modulo compact operators, on cylinders of finite height can be localized, and we may write

$$(3.6)\qquad \|f\|_{L^2(\partial\Omega\times(0,T))} \leq C\|(\lambda I + K')f\|_{L^2(\partial\Omega\times(0,T))} + \|\,\mathrm{Comp}\,f\|,$$

whenever $\lambda \in \mathbf{R}$, $|\lambda| > \frac{1}{2}$. Here, Ω is an arbitrary, *bounded* Lipschitz domain, and Comp denotes generic compact operators on $L^2(\partial\Omega\times(0,T))$. For more details in similar circumstances see §10 in [19].

In particular, for each bounded Lipschitz domain Ω, the operator $\lambda I + K'$ is Fredholm with index zero on $L^2(\partial\Omega\times(0,T))$, provided $\lambda \in \mathbf{R}$ satisfies $|\lambda| > \frac{1}{2}$. Finally, the fact that K' does not have any real eigenvalues outside the interval $[-\frac{1}{2}, \frac{1}{2}]$ can be proved by adapting a classical argument of Kellogg (cf. [15]), as in [5]; we omit the details. Instead, we remark that, for this segment of the proof, one can alternatively proceed as follows.

The idea is that the residual terms in (3.6) are $\leq C_T\|f\|_{L^p(\partial\Omega\times(0,T))}$, where the intervening constant satisfies $C_T \to 0$ as $T \to 0^+$. In particular, these residual terms can be hidden in the left side for T small, yielding a genuine bound from below for the operator $\lambda I + K'$. As before, this implies that the operator in question is invertible. Finally, the smallness assumption on T can be lifted by means of a well-known bootstrap argument (as in [7]).

At this stage, we may therefore conclude that $\lambda I + K'$ is an invertible operator on the space $L^2(\partial\Omega\times(0,T))$ for each λ real with $|\lambda| \geq \frac{1}{2}$. With this in hand, the well-posedness of the initial transmission boundary problem

(3.1), for an arbitrary bounded Lipschitz domain Ω, can be handled much as we have done for Theorem 1.3. $\square$

It is worth singling out a useful consequence of the above proof (cf. also the remark at the end of Section 2).

COROLLARY 3.2. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, and fix $T > 0$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ so that for any $2 - \varepsilon < p < 2 + \varepsilon$ the operators*

$$\lambda I + K : L^p(\partial\Omega \times (0,T)) \longrightarrow L^p(\partial\Omega \times (0,T)), \tag{3.7}$$

$$\lambda I + K : L^p_{1,1/2}(\partial\Omega \times (0,T)) \longrightarrow L^p_{1,1/2}(\partial\Omega \times (0,T)) \tag{3.8}$$

are invertible whenever $\lambda \in \mathbf{R}$ satisfies $|\lambda| \geq \frac{1}{2}$.

4. The case of time-varying domains

In this section we shall study the problem (1.8) in the setting of time-varying domains, considered in [18], [11], [12]; consequently, we shall follow the notation introduced and employed there, with only minor variations. More specifically, for $n \geq 2$ let

$$\Omega := \Big\{(x_0, x, t) \in \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R};\, x_0 > A(x,t)\Big\}, \tag{4.1}$$

where, among other things, the function $A : \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}$ satisfies

$$|A(x,t) - A(y,t)| \leq \beta |x - y|, \quad x, y \in \mathbf{R}^{n-1},\, t \in \mathbf{R}, \tag{4.2}$$

i.e., is Lipschitz in the space variable, uniformly in time. In addition, we shall impose on A a half-order smoothness condition in the time variable to the effect that $\mathbb{D}_n A$ (cf. (4.4) below) belongs to the parabolic version of BMO in $\mathbf{R}^{n-1} \times \mathbf{R}$.

In order to make this latter condition a little more transparent, we need more notation. Recall that if the parabolic norm $\|(x,t)\|$ is defined as the unique positive solution τ of the equation

$$\sum_{j=1}^{n-1} \frac{x_j^2}{\tau^2} + \frac{t^2}{\tau^4} = 1, \tag{4.3}$$

then

$$\mathbb{D}_n A(x,t) := \Big(\frac{\tau}{\|(\xi,\tau)\|} \hat{A}(\xi,\tau) \Big)^{\vee}(x,t), \tag{4.4}$$

where $\wedge$, $\vee$ denote the Fourier transform and its inverse, respectively, and where ξ, τ denote, respectively, the space and time variables on the Fourier transform side.

Going further, recall that $\mathrm{BMO}_{\mathrm{par}}(\mathbf{R}^{n-1}\times\mathbf{R})$, the parabolic BMO in $\mathbf{R}^{n-1}\times\mathbf{R}$, is the space of all locally integrable functions f, modulo constants, with the property that

$$\|f\|_{\mathrm{BMO}_{\mathrm{par}}(\mathbf{R}^{n-1}\times\mathbf{R})} := \sup_B \inf_{m\in\mathbf{R}} \frac{1}{|B|}\int_B |f(z)-m|\,dz < +\infty, \tag{4.5}$$

where the supremum is taken over all parabolic balls in $\mathbf{R}^{n-1}\times\mathbf{R}$, i.e., sets of the form $B=B_r(z_o)=\{z\in\mathbf{R}^{n-1}\times\mathbf{R};\ \|z-z_o\|<r\}$.

In this context, it is customary to let $d\sigma_t(x)dt$ play the role of the surface measure on $\partial\Omega$. Here, for each $t\in\mathbf{R}$, $d\sigma_t$ is the actual surface measure on the cross-section

$$\Omega_t := \Big\{(x_0,x,t)\in\mathbf{R}\times\mathbf{R}^{n-1}\times\{t\};\ x_0>A(x,t)\Big\}. \tag{4.6}$$

Later on, we shall also need ν_t, the outer unit normal to $\partial\Omega_t$, when the latter is regarded as a subset of $\mathbf{R}\times\mathbf{R}^{n-1}$.

With the above convention in mind (regarding the surface measure), we can define $L^p(\partial\Omega)$ in a natural fashion. Alternatively, one can define this space via pull-back to the Euclidean setting. For example, parabolic Sobolev spaces can be introduced as follows. If $1<p<\infty$, and $\pi:\partial\Omega\to\mathbf{R}^n$ is the projection $\pi(A(x,t),x,t):=(x,t)$, then $f\in L^p_{1,1/2}(\partial\Omega)$ if and only if $f\circ\pi^{-1}\in L^p_{1,1/2}(\mathbf{R}^{n-1}\times\mathbf{R})$, with equivalence of norms.

Next, we discuss the nontangential maximal operator in the current setting. Given $\alpha>0$ and $(P,t)=(p_0,p,t)\in\partial\Omega$, consider the (nontangential) parabolic cone

$$\gamma(P,t) := \Big\{(q_0,q,s)\in\Omega;\ \|(p-q,t-s)\|<\alpha|q_0-A(p,t)|\Big\}. \tag{4.7}$$

For an arbitrary $h:\Omega\to\mathbf{R}$ we then introduce

$$h^*(P,t) := \sup\Big\{|h(Q,s)|;\ (Q,s)\in\gamma(P,t)\Big\}. \tag{4.8}$$

Similar considerations apply for functions defined in $\Omega_- := (\mathbf{R}\times\mathbf{R}^{n-1}\times\mathbf{R})\setminus\bar{\Omega}_+$, where we set $\Omega_+:=\Omega$. The nontangential trace on the boundary is then defined by insisting that the boundary point is approached from within the nontangential approach region (4.7).

Finally, there remains to define $D_t^{1/2}u$, the 1/2 order time derivative of a function u in the non-cylindrical case considered here. This will be done "tangentially", after an appropriate pull-back. Somewhat more specifically, let ρ denote the Dahlberg-Kenig-Stein mapping of the half-space $\mathbf{R}^{n+1}_+=\{(\lambda,x,t);\ \lambda>0,\ (x,t)\in\mathbf{R}^{n-1}\times\mathbf{R}\}$ onto Ω. In particular, ρ preserves the time component; see §2 of [12] for a detailed exposition. If we now identify $\mathbf{R}^{n+1}_+$ with Ω under this mapping then we can define $D_t^{1/2}u$ as the half order time-derivative of $u\circ\rho$.

We are now ready to state our first result in this section.

THEOREM 4.1. *For each $\beta > 0$ finite, there exits a (typically small) $\beta_o > 0$ and $\varepsilon > 0$ with the following significance. Assume that Ω is as in* (4.1) *for a function $A(x,t)$ satisfying*

$$\|\nabla_x A\|_{L^\infty(\mathbf{R}^{n-1}\times\mathbf{R})} \le \beta, \qquad \|\mathbb{D}_n A\|_{BMO_{\mathrm{par}}(\mathbf{R}^{n-1}\times\mathbf{R})} \le \beta_o, \tag{4.9}$$

and, for $1 < p < \infty$, $\mu \in \mathbf{R}$, $\mu > 0$, $\mu \neq 1$, consider the transmission boundary value problem

$$\begin{cases} (\partial_t - \Delta)u_\pm = 0 \ \text{ in } \Omega_\pm, \\ (\nabla_{x_0,x} u_\pm)^*,\ (D_t^{1/2} u_\pm)^* \in L^p(\partial\Omega), \\ (u_+)\big|_{\partial\Omega} - (u_-)\big|_{\partial\Omega} = F \ \text{ a.e. on } \partial\Omega, \\ (D_{\nu_t} u_+)\big|_{\partial\Omega_t} - \mu(D_{\nu_t} u_-)\big|_{\partial\Omega_t} = G \ \text{ a.e. on } \partial\Omega_t, \ \text{ for each } t \in \mathbf{R}. \end{cases} \tag{4.10}$$

Then, for each $2 - \varepsilon < p < 2 + \varepsilon$ and each pair of functions $F \in L^p_{1,1/2}(\partial\Omega)$, $G \in L^p(\partial\Omega)$, the problem (4.10) *has a unique solution.*

Proof. We follow the same strategy as before, and work with layer potentials. The case of the single layer has already been handled in [11], so it remains for us to establish the analogue of (2.1) in this setting. Recall that

$$K' f(P,t) := \lim_{\epsilon\to 0+} \int_{-\infty}^{t-\epsilon} \int_{\partial\Omega_t} \frac{\partial}{\partial\nu_t(Q)} \Gamma(P-Q, t-s) f(Q,s)\, d\sigma_t(Q) ds, \tag{4.11}$$
$$P \in \partial\Omega_t,\ t \in \mathbf{R}.$$

The general idea is to repeat the same steps as in the proof for the cylindrical case. Integrations by parts in the space variables create no additional problems since they are performed in the cross-section Ω_t while keeping t fixed. The main difference is the failure of the identity $\int_{\mathbf{R}} \int_{\Omega_t} (D_t^\alpha u) v \, d\sigma_t dt = \int_{\mathbf{R}} \int_{\Omega_t} u (D_t^\alpha v)\, d\sigma_t dt$, due to the dependence of the cross-section Ω_t on t. In this context, the above formula holds only modulo error terms. Now, the crucial observation—itself a consequence of the estimates in [12]—is that a generic error term can be bounded by $\eta(\beta_o)\|f\|^2_{L^2(\partial\Omega)}$, where $\eta(\beta_o) \to 0$ as $\beta_o \to 0$. Consequently, $\eta(\beta_o)\|f\|_{L^2(\partial\Omega)}$ can be absorbed in the left hand side of (2.1) and, hence, has no significant overall effect, as far as our main goal is concerned. We remark that the uniformity of our estimates with respect to the parameter λ ultimately allows us to take β_o independent of λ, which is an essential point in our analysis.

Let us illustrate this general scheme by considering one paradigmatic example, i.e., the integration by parts formula (2.20). In the time-varying case, an error term appears which we claim is small with β_o. Indeed, this error is precisely the term III in (5.9) of [12]. As proved on pp. 388–392 of [12], this obeys $|III| \le C_{\beta_o}\|f\|^2_{L^2}$, where C_{β_o} can be chosen small with β_o. □

Implicit in the above proof is the following corollary.

Corollary 4.2. *Assume that Ω is as in* (4.1) *for a function $A(x,t)$ satisfying* (4.9)*, where $\beta > 0$ is finite and $\beta_o > 0$ is sufficiently small (relative to β). With K' as in* (4.11) *consider the operator*

$$\lambda I + K' : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega). \tag{4.12}$$

Then there exists $\varepsilon > 0$ so that (4.12) *is invertible whenever $2-\varepsilon < p < 2+\varepsilon$ and $\lambda \in \mathbf{R}$ satisfies $|\lambda| \geq \frac{1}{2}$.*

References

[1] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitziennes*, Ann. of Math. (2) **116** (1982), 361–387. MR **84m:**42027

[2] R. M. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math. **111** (1989), 339–379. MR **90d:**35118

[3] ______, *The initial-Neumann problem for the heat equation in Lipschitz cylinders*, Trans. Amer. Math. Soc. **320** (1990), 1–52. MR **90k:**35112

[4] M. Costabel, *Boundary integral operators for the heat equation*, Integral Equations Operator Theory **13** (1990), 498–552. MR **91j:**35119

[5] L. Escauriaza, E. B. Fabes, and G. Verchota, *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. Amer. Math. Soc. **115** (1992), 1069–1076. MR **92j:**35020

[6] L. Escauriaza and J. K. Seo, *Regularity properties of solutions to transmission problems*, Trans. Amer. Math. Soc. **338** (1993), 405–430. MR **93j:**35039

[7] E. B. Fabes and N. M. Rivière, *Dirichlet and Neumann problems for the heat equation in C^1-cylinders*, Harmonic analysis in Euclidean spaces (Williams Coll., Williamstown, Mass, 1978), Proc. Sympos. Pure Math., XXXV, Part 2, Amer. Math. Soc., Providence, R.I., 1979, pp. 179–196. MR **81b:**35044

[8] E. B. Fabes, M. Sand, and J. K. Seo, *The spectral radius of the classical layer potentials on convex domains*, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), IMA Vol. Math. Appl., 42, Springer, New York, 1992, pp. 129–137. MR **93d:**35026

[9] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964. MR 31#6062

[10] F. Hettlich and W. Rundell, *Identification of a discontinuous source in the heat equation*, Inverse Problems **17** (2001), 1465–1482. MR **2002i:**35202

[11] S. Hofmann, *Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials*, Duke Math. J. **90** (1997), 209–259. MR **99b:**42018

[12] S. Hofmann and J. L. Lewis, *L^2-Solvability and representation by caloric layer potentials in time-varying domains*, Ann. of Math. (2) **144** (1996), 349–420. MR **97h:**35072

[13] T. Jakab, Ph.D. Thesis, University of Missouri, 2003.

[14] N. Kalton and M. Mitrea, *Stability results on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc. **350** (1998), 3903–3922. MR **98m:**46094

[15] O. D. Kellogg, *Foundations of potential theory*, Grundlehren der Mathematischen Wissenschaften, Band 31, Springer-Verlag, Berlin-New York, 1967. MR 36#5369

[16] C. E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, vol. 83, American Mathematical Society, Providence, R.I., 1994. MR **96a:**35040

[17] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, R.I., 1967. MR 39#3159b

[18] J. L. Lewis and M. A. M. Murray, *The method of layer potentials for the heat equation in time-varying domains*, Mem. Amer. Math. Soc. **545** (1995). MR **96e:**35059

[19] D. Mitrea, M. Mitrea, and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Mem. Amer. Math. Soc. **150** (2001). MR **2002g:**58026

[20] M. Mitrea, *The initial Dirichlet boundary value problem for general second order parabolic systems in nonsmooth manifolds*, Comm. Partial Differential Equations **26** (2001), 1975–2036. MR **2002k:**35137

[21] J. Nečas, *Les Méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967. MR 37#3168

[22] Z. W. Shen, *Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders*, Amer. J. Math. **113** (1991), 293–373. MR **92a:**35133

[23] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, no. 30, Princeton University Press, Princeton, N.J., 1970. MR 44#7280

[24] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611. MR **86e:**35038

S. Hofmann, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: hofmann@math.missouri.edu

J. Lewis, Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

E-mail address: john@ms.uky.edu

M. Mitrea, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: marius@math.missouri.edu