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ON THE KREIN-ŠMULIAN THEOREM FOR WEAKER TOPOLOGIES

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Abstract. We investigate possible extensions of the classical Krein-Šmulian theorem to various weak topologies. In particular, we show that if X is a WCG Banach space and τ is any locally convex topology weaker than the norm-topology, then for every τ-compact norm-bounded set H, $\overline{\mathrm{conv}}^{\tau} H$ is τ-compact. In arbitrary Banach spaces, the norm-fragmentability assumption on H is shown to be sufficient for the last property to hold.

A new proof to the following result is given: If a Banach space does not contain a copy of $\ell_1[0,1]$, then the Krein-Šmulian theorem holds for every topology τ induced by a norming set of functionals. We conclude that in such spaces a norm-bounded set is weakly compact if it is merely compact in the topology induced by a boundary. On the other hand, the same statement is obtained for all $C(K)$ and $\ell_1(\Gamma)$ spaces.

1. Introduction

A well-known result that goes back to M. Krein and V. Šmulian [23] says the following: the closed convex hull of a weakly compact subset of a Banach space X is weakly compact. It is known that this result also holds when the weak topology in X is replaced by any locally convex topology compatible with the dual pair $\langle X, X^* \rangle$; see [17, Corollary 9.9.6]. For which other topologies does this statement remain true?

Recent attention to this question is motivated by its connection with the Boundary Problem posed by G. Godefroy [14].

> Let X be a Banach space and B a *boundary* in the unit ball of X^*, i.e., such that $\|x\| = \max_{b \in B} b(x)$ holds for all $x \in X$. Denote by $\sigma(X, B)$ the topology in X of pointwise convergence

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> on B. Is a norm-bounded subset H of X weakly compact if it is merely compact with respect to $\sigma(X, B)$?

In [30] S. Simons gave a partial positive answer to this question in the case when H is a convex set. This establishes the equivalence between the Krein-Šmulian-type theorem for topologies $\sigma(X, B)$ generated by boundaries and the Boundary Problem itself. In other words, if one can prove that in a certain Banach space the $\sigma(X, B)$-closed convex hull of every norm-bounded $\sigma(X, B)$-compact subset is again $\sigma(X, B)$-compact, i.e., the analogue of the classical Krein-Šmulian Theorem holds, then by Simons' result for $\sigma(X, B)$, the Boundary Problem is solved positively in the given Banach space.

Even though the problem remains still open, to the best of our knowledge, considerable progress has been made by B. Cascales, G. Godefroy, G. Vera and others in a series of papers ([3], [4], [5], [6], [8], [9]). In particular, the Boundary Problem has been positively solved for all boundaries in spaces of continuous functions defined on a compact space [4], and for the particular boundary given by the set of extreme points (in the dual unit ball) for general Banach spaces [3]. In [5] the positive solution was shown to hold also for all Banach spaces not containing a copy of $\ell_1[0, 1]$. In fact, the following more general statement was proved.

Theorem 1.1 ([5]). *Suppose X does not contain a copy of $\ell_1[0, 1]$ and B is a norming subset of the unit ball of X^*. Then the $\sigma(X, B)$-closed convex hull of every $\sigma(X, B)$-compact norm-bounded set in X is $\sigma(X, B)$-compact.*

Here and in the sequel, by a norming set (also called a 1-norming set) for the Banach space $(X, \|\cdot\|)$ we mean a set $B \subset B_{X^*}$ such that $\|x\| = \sup_{b \in B} b(x)$ for all $x \in X$. For example, any boundary is a norming set.

So, Theorem 1.1 combined with the aforementioned result of Simons solves the Boundary Problem, in particular, for all separable, reflexive and, more generally, all weakly compactly generated (WCG for short) or weakly Lindelöf Banach spaces [16].

In the first part of this paper we recall that in order for a compact set H to have compact closed convex hull, every Radon measure on H must possess a barycenter, and vise versa. This last condition is shown to follow from the so-called Riemann-Lebesgue integrability of the identity mapping on H. Using recent results by V. Kadets et. al. (see [19], [20], [29]) we immediately obtain the Krein-Šmulian theorem for all topologies weaker than the norm topology of a given Banach space X, provided X is either WCG or X has an unconditional basis (possibly not countable) and fails to contain a copy of $\ell_1(\Gamma)$ over any uncountable set Γ. Furthermore, in Theorem 2.4 we obtain the same result for all compact sets fragmentable by the norm. This, in particular, generalizes an earlier result in [8].

In Section 3 we continue the discussion of the Krein-Šmulian theorem and give an alternative geometrical proof of Theorem 1.1. Our approach is based upon a straightforward construction of a sequence of independent functions (much in the spirit of [28]) whenever the conclusion of the theorem is violated. This subsequently allows us to embed a copy of $\ell_1[0,1]$ into the space. Our argument is self-contained and does not employ the non-trivial results used in the original proof in [5].

We further observe that in spite of Theorem 1.1, the Boundary Problem itself has positive solution in any space $\ell_1(\Gamma)$. This phenomenon is treated in Section 4. We will find that all spaces $\ell_1(\Gamma)$ and all $C(K)$-spaces are *angelic* in any topology generated by a boundary. In Proposition 4.3 this condition is shown to imply a positive solution the Boundary Problem.

Our notation and terminology are standard. We borrow some standard topological results from the books [13], [17], [21], [22], [27]. Our vector spaces are all real. If X is a Banach space, $B(X)$ denotes its closed unit ball, and X^* its topological dual space. For a locally convex space (X,τ) endowed with the topology τ its dual is denoted, as usual, by $(X,\tau)^*$. Whenever B is a subset of $(X,\tau)^*$, we write $\sigma(X,B)$ to denote the locally convex topology of convergence on functionals from B. Also we adopt the following short notation for weak topologies: $\sigma(X,X^*)$ in the usual Banach space sense is denoted by 'w' or 'w(τ)' for a general locally convex space with topology τ. Analogously, $\sigma(X^*,X)$ is denoted by w* or w$^*(\tau)$.

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2. The Krein-Šmulian theorem and barycenters

The study of compact convex sets is closely related to the existence of barycenters; see, for example, [10], [11], [26]. If H is a compact subset of the locally convex space (X,τ) we denote by $\mathcal{P}(H)$ the set of all Radon probabilities μ defined on the σ-algebra $\mathcal{B}(H)$ of τ-Borel subsets of H. A barycenter of μ is said to be a vector $x \in X$ such that the equality

$$x^*(x) = \int_H x^*(h)d\mu(h) \tag{1}$$

holds for every $x^* \in (X,\tau)^*$. Observe that the right hand side of equation (1) is well-defined, because $x^*|_H$ is τ-continuous and bounded, hence μ-integrable.

In general, a barycenter may not exist. However, its uniqueness follows immediately from the fact that $(X,\tau)^*$ separates the points of X. Let us denote by x_μ the barycenter of $\mu \in \mathcal{P}(H)$ whenever it exists. It is well known that

$$\overline{\text{conv}}^\tau H = \{x_\mu : \mu \in \mathcal{P}(H),\ \mu \text{ has a barycenter}\}; \tag{2}$$

see [26, Proposition 1.2] or [11, Theorem 2, p. 149].

The following lemma exhibits the classical link between barycenters and the Krein-Šmulian theorem.

Lemma 2.1. *Let H be a τ-compact set in a locally convex space (X,τ). Then $\overline{\text{conv}}^\tau H$ is τ-compact if and only if every measure $\mu \in \mathcal{P}(H)$ has a barycenter in X.*

Proof. If $\overline{\text{conv}}^\tau H$ is τ-compact, then every $\mu \in \mathcal{P}(H)$ has a barycenter by [11, Theorem 1, p. 148].

Conversely, let us suppose that every measure $\mu \in \mathcal{P}(H)$ has a barycenter and let us see that $\overline{\text{conv}}^\tau H$ is τ-compact. Since the mapping $\varphi : \mathcal{P}(H) \to X$ defined by $\varphi(\mu) = x_\mu$ is weak*-w(τ)-continuous, we obtain that $\varphi(\mathcal{P}(H))$ is w(τ)-compact. According to (2), $\overline{\text{conv}}^\tau H$ is also w(τ)-compact. The τ-compactness of $\overline{\text{conv}}^\tau H$ (that clearly follows from the classical Krein-Smulian's theorem, [17, Corollary 9.9.6]) is recalled below for the sake of completeness; since H is τ-compact, the closed convex hull $\overline{\text{conv}}^\tau H$ is precompact (τ-totally bounded). To prove the τ-compactness of $\overline{\text{conv}}^\tau H$ we show that every net in this set has a converging subnet. So, let us fix a net $\{y_\alpha\}$ in $\overline{\text{conv}}^\tau H$. The w($\tau$)-compactness implies the existence of a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ converging to some $y \in \overline{\text{conv}}^\tau H$ in the topology w(τ). In addition, the τ-total boundedness of $\overline{\text{conv}}^\tau H$ implies that there exists a further subnet $\{y_\gamma\}$ of $\{y_\beta\}$ which is τ-Cauchy. Since τ has a basis of neighborhoods of the origin consisting of w(τ)-closed sets, we conclude that actually $y = \tau - \lim_\gamma y_\gamma$ (see [17, Theorem 3.2.4]), and the proof is complete. □

As we will see in a moment, barycenters are related to the concept of the so-called Riemann-Lebesgue integral introduced in [20]. Let us briefly outline the definition.

Suppose that X is a Banach space, (Ω, Σ, μ) is a probability space and $f\colon \Omega \to X$ is a norm-bounded function, not necessarily measurable in any sense. Given a partition $\Pi = \{A_i\}_{i=1}^n$ of Ω into measurable sets and a collection $T = \{t_i\}_{i=1}^n$ of *sampling points*, i.e., $t_i \in A_i$, $i = \overline{1,n}$, we define the associated *Riemann-Lebesgue integral sum* as follows:

$$S(f, \Pi, T) = \sum_{i=1}^{n} f(t_i)\mu(A_i).$$

We endow $\{S(f,\Pi,T)\}_{\Pi,T}$ with a net structure by defining a partial order by the rule: $\Pi_1 \succ \Pi_2$ if and only if every element of Π_1 is contained in some element of Π_2. If this net converges to some element x in the norm topology, then f is called Riemann-Lebesgue integrable, and x is then defined as its Riemann-Lebesgue integral. We refer the reader to [2], [7], [19], [20], [29] for detailed treatments of this and related notions.

Notice that if f is strongly measurable then its Bochner integrability is equivalent to the convergence of the entire net of its Riemann-Lebesgue integral sums (see [20]).

Assume now that the Banach space X is also endowed with another locally convex topology τ that is weaker than the norm topology. If H is a τ-compact set in X and the identity map $\mathrm{id}\colon H \to X$ is Riemann-Lebesgue integrable with respect to a measure $\mu \in \mathcal{P}(H)$ then its integral is the barycenter of μ. More generally, if the net of the Riemann-Lebesgue integral sums of $\mathrm{id}\colon H \to X$ has a cluster point, then this point is the barycenter of μ. Indeed, if $x = \lim_\alpha S(\mathrm{id}, \Pi_\alpha, T_\alpha)$ for some subnet, then for every $x^* \in (X,\tau)^*$ we have

$$x^*(x) = \lim_\alpha x^*(S(\mathrm{id}, \Pi_\alpha, T_\alpha)) = \lim_\alpha S(x^*|_H, \Pi_\alpha, T_\alpha) = \int_H x^*(h)d\mu(h),$$

since the last integral converges in the conventional Lebesgue sense.

Certain geometric conditions on the Banach space are shown to guarantee the existence of a cluster point for any measure μ. From [19, Theorem 4.1] and [29, Theorem 2.1.2], where such conditions are formulated, we immediately obtain the following result.

THEOREM 2.2. *Let X be a Banach space satisfying either of the two conditions below:*

(i) *X is a WCG-space.*
(ii) *X has an unconditional basis (possibly not countable) and fails to contain a copy of $\ell_1(\Gamma)$ over uncountable Γ.*

Let also τ be a locally convex topology on X weaker than the norm-topology. Then the τ-closed convex hull of any τ-compact norm-bounded subset H of X is τ-compact.

Although the geometric assumptions on the space X in this theorem are obviously more restrictive than in Theorem 1.1, the conclusion holds for more general topologies.

Next, using Lemma 2.1 and the above ideas we isolate a class of compact sets (for topologies weaker than the weak topology) in a Banach space for which the Krein-Šmulian theorem holds. We will use the notion of fragmentability, originally introduced by Jayne and Rogers [18] and defined as follows.

DEFINITION 2.3. Let (Z,τ) be a topological space and ρ a metric on Z. We say that (Z,τ) is *fragmented by ρ* (or *ρ-fragmented*) if for each non-empty subset C of Z and for each $\varepsilon > 0$ there exists a τ-open subset U of Z such that $U \cap C \neq \emptyset$ and $\rho - \mathrm{diam}(U \cap C) \leq \varepsilon$.

A great variety of sufficient conditions for norm-fragmentability of a subset in a Banach space can be found in the literature: weakly compact sets of

Banach spaces are norm-fragmented; see [24]. More generally, sets which are Lindelöf for the weak topology and compact with respect to the topology generated by a norming set of functionals are fragmented, too; see [6], [8].

Theorem 2.4. *Let X be a Banach space and τ any locally convex topology in X weaker than the norm-topology. If $H \subset X$ is a τ-compact norm-bounded set fragmented by the norm, then $\overline{\text{conv}}^\tau H$ is τ-compact. Furthermore,*

$$\overline{\text{conv}}^\tau H = \overline{\text{conv}}^{\|\cdot\|} H.$$

First we show that a fragmentable set can be essentially split into subsets of small diameter.

Lemma 2.5. *Let (H,τ) be a compact space fragmented by a metric ρ and $\mu \in \mathcal{P}(H)$. Then for every $\varepsilon > 0$ there is a finite partition $A_1, A_2, \ldots, A_m$ of H in $\mathcal{B}(H)$ such that:*

(i) $\rho - \text{diam}(A_i) < \varepsilon,\ i = 1, 2, \ldots, m-1.$
(ii) $\mu(A_m) < \varepsilon$.

Proof. Let $\mathcal{A} = \{A \in \mathcal{B}(H) : \rho - \text{diam}(A) < \varepsilon\}$ and let $\mathcal{F}$ be the family consisting of finite unions of elements in $\mathcal{A}$. The ρ-fragmentability of (H,τ) implies that $\mathcal{A}$ is not empty. Thus $\mathcal{F}$ is not empty either. Let us define $\alpha = \sup\{\mu(B) : B \in \mathcal{F}\}$ and pick a sequence (B_n) in $\mathcal{F}$ such that $\alpha = \lim_n \mu(B_n)$. If $E_n = \bigcup_{k=1}^n B_k$, we still have $\alpha = \lim_n \mu(E_n) = \mu(\bigcup_{n=1}^\infty E_n)$. We claim that

$$\mu\left(H \setminus \bigcup_{n=1}^{\infty} E_n\right) = 0. \tag{3}$$

If this is not the case, then there is a compact set $K \subset H \setminus (\bigcup_{n=1}^\infty E_n)$ such that $\mu(K) > 0$. The restriction $\mu|_K$ of μ to the Borel sets of K is a Radon measure that has a non empty support $F \subset K$. The ρ-fragmentability of (H,τ) applied to F implies that there is an open set $O \subset H$ such that $O \cap F \neq \emptyset$ and $\rho - \text{diam}(O \cap F) < \varepsilon$. We also have $\mu(O \cap F) > 0$ because F is the support of $\mu|_K$. Consequently,

$$\alpha \geq \lim_n \mu(E_n \cup (O \cap F)) = \lim_n \mu(E_n) + \mu(O \cap F) = \alpha + \mu(O \cap F) > \alpha$$

and we reach the contradiction that establishes the validity of (3). Since

$$\lim_n \mu(H \setminus E_n) = \mu\left(H \setminus \bigcup_{n=1}^{\infty} E_n\right) = 0,$$

we can find an $m \in \mathbf{N}$ such that $\mu(H \setminus E_m) < \varepsilon$. Put $A_m = H \setminus E_m$. Then A_m satisfies (ii), and clearly E_m can be split as required in (i). □

Let us point out that the above lemma is very much like an argument used in the proof of Theorem 2.3 in [25]: If we assume that ρ is lower semi-continuous with respect to τ in our lemma, then we can take $A_1, A_2, \ldots, A_{m-1}$

to be compact (by adapting the first part of the proof of Theorem 2.3 in [25] to this situation).

Proof of Theorem 2.4. We show that the identity mapping $\mathrm{id} : H \to X$ is Riemann-Lebesgue integrable with respect to any measure $\mu \in \mathcal{P}(H)$. According to Lemma 2.1 and the preceding discussion, this implies the first part of the theorem. Moreover, from (2) we conclude that $\overline{\mathrm{conv}}^{\tau} H$ lies in the closure of all possible Riemann-Lebesgue integral sums of id, which is obviously a subset of $\overline{\mathrm{conv}}^{\|\cdot\|} H$. This implies the second part.

So, let us fix $\mu \in \mathcal{P}(H)$. Without loss of generality we can assume that H lies in the unit ball of X. For any given $k \in \mathbf{N}$, using Lemma 2.5 we can find a finite partition $V_1^k, V_2^k, \dots, V_{n_k}^k$ of H in $\mathcal{B}(H)$ such that

$$\|\cdot\| - \mathrm{diam}\, V_i^k < \frac{1}{2^{k+1}}, \; i = \overline{1, n_k - 1}, \tag{4}$$

and

$$\mu(V_{n_k}^k) < \frac{1}{2^{k+1}}. \tag{5}$$

Let us now denote $A_{i_1 i_2 \dots i_k} = \bigcap_{j=1}^k V_{i_j}^j$, where $1 \le i_j \le n_j$, $1 \le j \le k$, and define a sequence of partitions of H as follows:

$$\Pi_k = \{A_{i_1 \dots i_k} : 1 \le i_j \le n_j, 1 \le j \le k\}.$$

For each $k \in \mathbf{N}$ we also fix an arbitrary set of sampling points $T_k = \{t_{i_1 \dots i_k} : t_{i_1 \dots i_k} \in A_{i_1 \dots i_k}\}$. We claim that the limit $\lim_{k\to\infty} S(\mathrm{id}, \Pi_k, T_k)$ exists in the norm-topology, and is a limit point of the integral sums, even though the sequence $\{\Pi_k, T_k\}$ is not a proper subnet.

Indeed, in view of (4) and (5), we have

$$\begin{aligned}
&\left\| S(\mathrm{id}, \Pi_k, T_k) - S(\mathrm{id}, \Pi_{k+1}, T_{k+1}) \right\| \\
&\quad = \left\| \sum_{\substack{1 \le j \le k \\ 1 \le i_j \le n_j}} t_{i_1 \dots i_k} \mu(A_{i_1 \dots i_k}) - \sum_{\substack{1 \le j \le k+1 \\ 1 \le i_j \le n_j}} t_{i_1 \dots i_k i_{k+1}} \mu(A_{i_1 \dots i_k i_{k+1}}) \right\| \\
&\quad = \left\| \sum_{\substack{1 \le j \le k \\ 1 \le i_j \le n_j}} t_{i_1 \dots i_k} \sum_{i_{k+1}=1}^{n_{k+1}} \mu(A_{i_1 \dots i_k i_{k+1}}) - \sum_{\substack{1 \le j \le k+1 \\ 1 \le i_j \le n_j}} t_{i_1 \dots i_k i_{k+1}} \mu(A_{i_1 \dots i_k i_{k+1}}) \right\| \\
&\quad \le \sum_{\substack{1 \le j \le k+1 \\ 1 \le i_j \le n_j}} \left\| t_{i_1 \dots i_k} - t_{i_1 \dots i_k i_{k+1}} \right\| \mu(A_{i_1 \dots i_k i_{k+1}}) \le \frac{3}{2^{k+1}}.
\end{aligned}$$

So, the sequence $\{S(\mathrm{id}, \Pi_k, T_k)\}$ converges to some vector $x \in X$. An easy computation also gives the estimate

$$\|S(\mathrm{id}, \Pi_k, T_k) - x\| \leq \frac{3}{2^k}, \quad k = 1, 2, \ldots$$

Given $\varepsilon > 0$ take $k \in \mathbf{N}$ so that $9/2^{k+1} < \varepsilon$. If $\Pi \succ \Pi_k$ and T is any collection of sampling points in Π, the same calculations as above show that

$$\|S(\mathrm{id}, \Pi, T) - S(\mathrm{id}, \Pi_k, T_k)\| \leq \frac{3}{2^{k+1}},$$

and hence

$$\|S(\mathrm{id}, \Pi, T) - x\| \leq \frac{3}{2^{k+1}} + \frac{3}{2^k} = \frac{9}{2^{k+1}} < \varepsilon.$$

This proves the desired result and finishes the argument. □

Let us note that Theorem 2.4 applied to the spaces of Bochner integrable functions considered in [8, Example E] yields the main results of [1] as a consequence. Furthermore, if X is an Asplund space (i.e., X^* has the Radon-Nikodým property, or, equivalently, the w*-compact subsets of X^* are norm-fragmented), then according to our theorem, for every w*-compact subset H of X^* we have the equality $\overline{\mathrm{conv}}^{\mathrm{w}^*} H = \overline{\mathrm{conv}}^{\|\cdot\|} H$. This gives an alternative proof of [24, Theorem 2.3].

We conclude this section with several remarks.

First we comment on the fact that Lemma 2.5 implies strong μ-measurability of id. Hence, id is Bochner integrable and its Riemann-Lebesgue integral x that we found at the end of the proof of Theorem 2.4 is in fact also its Bochner integral.

We also remark that τ-compact sets as in Theorem 2.4 are not automatically norm-bounded even if τ is generated by a norming set of functionals. Indeed, consider $X = \ell_1$ and τ induced by the coordinate-axis vectors $\{e_n\}_{n \in N} \subset \ell_\infty$. Set $H = \{ne_n\}_{n \in N} \cup \{0\} \subset \ell_1$. Then H is unbounded, yet τ-compact.

3. A new proof of Theorem 1.1

In this section we give an alternative proof of Theorem 1.1 stated in the introduction. Our approach is based on a geometric construction of an independent sequence of functions on a τ-compact set ($\tau = \sigma(X, B)$) with non-compact convex hull. After a short argument, presented in the original proof in [5], this implies existence of a copy of $\ell_1[0, 1]$ in X.

So, for the rest of this section we assume that there exists a norm-bounded τ-compact set H in X such that $\overline{\mathrm{conv}}^\tau H$ is not τ-compact, and we show that X then contains a copy of $\ell_1[0, 1]$. For purely technical reasons we also assume without loss of generality that H is contained in the unit ball of X and that the norming set B inducing τ is absolutely convex.

In view of Lemma 2.1 there is a measure $\mu \in \mathcal{P}(H)$ without a barycenter. We can decompose μ into the sum of its purely atomic part μ_a and its atomless part. The purely atomic part always has a barycenter. To see this, we recall that the Radon probability μ has at most countably many disjoint atoms that are singletons $(h_i)_i$. Hence, $\mu_a = \sum_i \lambda_i \delta_{h_i}$, with $\lambda_i \geq 0$ and $\sum_i \lambda_i \leq 1$, and thus $x_{\mu_a} = \sum_i \lambda_i h_i$ is the barycenter for μ_a. This observation implies that only the atomless part of μ does not have a barycenter.

So, from now on we assume that μ has no atoms. We also identify H with the support of μ, so that every open set in H has positive measure with respect to μ.

Our plan is to pick a sequence of functionals $(f_n)_{n\in\mathbf{N}}$ in B so that

$$H \bigcap \left(\bigcap_{m\in M} \{f_m > r+\delta\} \right) \bigcap \left(\bigcap_{n\in N} \{f_n < r\} \right) \neq \emptyset \tag{6}$$

holds for any two disjoint sets of natural numbers M and N, and two fixed real numbers r and δ, $\delta > 0$. Such a sequence is called independent over H (see [28]). Every Banach space, which contains an independent sequence over a compact set, also contains a copy of $\ell_1[0,1]$ (see Lemma B in [5]).

Our construction is based on the following lemmas.

Lemma 3.1. *There exists an $\varepsilon > 0$ and a Borel set A in H with $\mu(A) > 0$, such that for every Borel subset B in A with $\mu(B) > 0$ and every $h \in \overline{\text{conv}}^\tau H$ there is an $f \in B$ satisfying the following inequality:*

$$f(h) > \varepsilon + \frac{1}{\mu(B)} \int_B f(s)d\mu(s). \tag{7}$$

Proof. Suppose, on the contrary, that for any $\varepsilon > 0$ and any measurable $A \subset H$ there is a $B \subset A$ and $h \in \overline{\text{conv}}^\tau H$ such that

$$f(h) \leq \varepsilon + \frac{1}{\mu(B)} \int_B f(s)d\mu(s),$$

whenever $f \in B$.

Let $\varepsilon_n = 1/2^n$, $n \in \mathbf{N}$. By an exhaustion argument, using the previous inequality for $\varepsilon_1 = 1/2$, we can find a sequence $(h_n^1)_{n\in\mathbf{N}} \subset \overline{\text{conv}}^\tau H$ and a pairwise disjoint sequence $(A_n^1)_{n\in\mathbf{N}}$ in $\mathcal{B}(H)$ such that $\mu(H \setminus \bigcup_{n=1}^\infty A_n^1) = 0$ and

$$f(h_n^1) \leq \varepsilon_1 + \frac{1}{\mu(A_n^1)} \int_{A_n^1} f(s)d\mu(s),$$

for all $f \in B$ and $n \in \mathbf{N}$. Hence, as B is absolutely convex, we have

$$\left| f(h_n^1) - \frac{1}{\mu(A_n^1)} \int_{A_n^1} f(s)d\mu(s) \right| \leq \varepsilon_1,$$

for all $f \in B$ and $n \in \mathbf{N}$. Letting $h^1 = \sum_{n=1}^{\infty} \mu(A_n^1) h_n^1$ and adding up the previous inequalities we get

$$\left| f(h^1) - \int_H f(s) d\mu(s) \right| \leq \varepsilon_1.$$

In the same manner, for every $n \in \mathbf{N}$ we can construct an $h^n \in \overline{\text{conv}}^\tau H$ so that

$$\left| f(h^n) - \int_H f(s) d\mu(s) \right| \leq \varepsilon_n,$$

for all $f \in B$. Since B is norming, it follows that $\left\| h^n - h^{n+1} \right\| \leq \varepsilon_n + \varepsilon_{n+1}$, and hence the limit $h = \| \cdot \| - \lim_{n \to \infty} h_n$ exists. Passing to limits in the previous inequality we see that h is the barycenter of μ, which contradicts our assumption. □

Note that since μ is a regular measure, A can be chosen to be closed. Furthermore, restricting μ to the set A we can and do assume that A is in fact the entire space H.

We say that a set $K \subset X$ has a finite ε-net if there is a finite subset F of K such that $K \subset \bigcup_{x \in F} \{y \in X : \|y - x\| \leq \varepsilon\}$. It is a basic fact that every norm-compact set has a finite ε-net for all $\varepsilon > 0$.

From now on, we fix $\varepsilon > 0$ as in Lemma 3.1.

LEMMA 3.2. *For any norm-compact set $K \subset \overline{\text{conv}}^\tau H$, any collection of open sets $(U_i)_{i=1}^n$ in H and any positive numbers $(\lambda_i)_{i=1}^n$ with $\sum_{i=1}^n \lambda_i = 1$ there are open sets $(V_i)_{i=1}^n$ satisfying the following conditions:*

(i) $V_i \subset U_i$, $i = \overline{1, n}$.
(ii) $\text{dist}(K, \sum_{i=1}^n \lambda_i v_i) > \varepsilon/2$, *whenever* $v_i \in V_i$, $i = \overline{1, n}$.

Proof. First we find in each U_i a Borel subset W_i so that $\mu(W_i) = \lambda_i \mu(W) > 0$, where $W = \bigcup_{i=1}^n W_i$ and $W_i \cap W_j = \emptyset$, $i \neq j$. Indeed, since μ is atomless, we can pick disjoint Borel sets $A_i \subset U_i$, $i = \overline{1, n}$, such that $\mu(A_i) = \mu(A_j) > 0$ whenever $i \neq j$. By the same token, there are sets $W_i \subset A_i$ such that $\mu(W_i) = \lambda_i \mu(A_i)$, $i = \overline{1, n}$. Clearly, these sets fulfill our requirement.

Let us fix any finite $(\varepsilon/2)$-net $(h_k)_{k=1}^N$ in K. In view of Lemma 3.1 there is an $f \in B$ verifying

$$\begin{aligned} f(h_1) &> \varepsilon + \frac{1}{\mu(W)} \int_W f(s) d\mu(s) \\ &= \varepsilon + \sum_{i=1}^n \frac{\lambda_i}{\mu(W_i)} \int_{W_i} f(s) d\mu(s). \end{aligned}$$

Then for every $i = \overline{1,n}$ one can find $(w_{ij})_{j=1}^M \subset W_i$ such that

$$f(h_1) > \varepsilon + \sum_{i=1}^n \lambda_i \sum_{j=1}^M \frac{1}{M} f(w_{ij})$$
$$= \varepsilon + \sum_{j=1}^M \frac{1}{M} \sum_{i=1}^n \lambda_i f(w_{ij}).$$

Thus,

$$\sum_{j=1}^M \frac{1}{M} \left| f(h_1) - \sum_{i=1}^n \lambda_i f(w_{ij}) \right| > \varepsilon.$$

So, for some j_0 we have

$$\left| f(h_1) - \sum_{i=1}^n \lambda_i f(w_{ij_0}) \right| > \varepsilon.$$

Since $w_{ij_0} \in W_i \subset U_i$, there are open subsets $W_i^1 \subset U_i$ such that the inequality

$$\left| f(h_1) - \sum_{i=1}^n \lambda_i f(w_i) \right| > \varepsilon$$

holds for all w_i in W_i^1, $i = \overline{1,n}$. As a consequence, we have

$$\left\| h_1 - \sum_{i=1}^n \lambda_i w_i \right\| > \varepsilon,$$

whenever $w_i \in W_i^1$, $i = \overline{1,n}$.

Doing the same for $(W_i^1)_{i=1}^n$ instead of $(U_i)_{i=1}^n$, and h_2 instead of h_1, we obtain open sets $W_i^2 \subset W_i^1$ with

$$\left\| h_2 - \sum_{i=1}^n \lambda_i w_i \right\| > \varepsilon,$$

whenever $w_i \in W_i^2$, $i = \overline{1,n}$.

Continuing the process we end up with open sets $V_i = W_i^N$. It is clear from our construction that

$$\left\| h - \sum_{i=1}^n \lambda_i v_i \right\| > \frac{\varepsilon}{2},$$

for all $h \in K$ and $v_i \in V_i$. So, conditions (i) and (ii) are satisfied. □

LEMMA 3.3. *For any norm-compact set $K \subset \overline{\text{conv}}^\tau H$ and any collection of open sets $(U_i)_{i=1}^n$ in H there are open sets $(V_i)_{i=1}^n$ satisfying the following conditions:*

(i) $V_i \subset U_i$, $i = \overline{1,n}$.

(ii) $\operatorname{dist}(K, \sum_{i=1}^n \lambda_i v_i) > \varepsilon/4$, *whenever* $v_i \in V_i$, $i = \overline{1,n}$, *and* $\lambda_i \geq 0$ *with* $\sum_{i=1}^n \lambda_i = 1$.

Proof. To prove this lemma we fix a finite $\varepsilon/4$-net in the set

$$\left\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \sum_{i=1}^n \lambda_i = 1,\ \lambda_i \geq 0\right\}$$

equipped with the metric $\rho((\lambda_i), (\nu_i)) = \sum_{i=1}^n |\lambda_i - \nu_i|$. Then we apply Lemma 3.2 successively to all elements of the net. □

LEMMA 3.4. *For any collection of open sets* $(U_i)_{i=1}^n$ *in* H *there exist* $f \in B$ *and two constants* a *and* b *with* $b - a \geq \varepsilon/8$ *such that*

$$\{f > b\} \cap U_i \neq \emptyset,$$
$$\{f < a\} \cap U_i \neq \emptyset,$$

for all $i = \overline{1,n}$.

Proof. Let us fix arbitrary vectors $u_i \in U_i$, $i = \overline{1,n}$, and set $K = \operatorname{conv}(u_i)_{i=1}^n$. By Lemma 3.3, there are vectors $v_i \in U_i$ such that if $L = \operatorname{conv}(v_i)_{i=1}^n$, then $\operatorname{dist}(K, L) > \varepsilon/4$.

By the geometric version of the Hahn-Banach Theorem, there exists a $g \in B(X^*)$, $\|g\| = 1$, separating $K - L$ from the ball $(\varepsilon/4)B(X)$, i.e.,

$$g(k - l) > \frac{\varepsilon}{4},$$

for all $k \in K$, $l \in L$. Since the w*-closure of B coincides with the entire space $B(X^*)$, we can find an $f \in B$ for which the inequality

$$f(k - l) > \frac{\varepsilon}{4}$$

holds, whenever $k \in K$ and $l \in L$.

Now it is easy to see that the constants $a = \sup_{l \in L} f(l) + \varepsilon/16$ and $b = \inf_{k \in K} f(k) - \varepsilon/16$ meet the desired conditions. □

Construction of the independent sequence. First, applying Lemma 3.4 to $U_1 = U_2 = \cdots = U_n = H$ we find $f_1 \in B$ and constants a_1, b_1 with $b_1 - a_1 \geq \varepsilon/8$ such that

$$U_1 = \{f_1 > b_1\} \cap H \neq \emptyset,$$
$$U_2 = \{f_1 < a_1\} \cap H \neq \emptyset.$$

Then we apply Lemma 3.4 to U_1, U_2 and get $f_2 \in B$, a_2, b_2 with $b_2 - a_2 \geq \varepsilon/8$ such that

$$\{f_2 > b_2\} \cap U_i \neq \emptyset,$$
$$\{f_2 < a_2\} \cap U_i \neq \emptyset,$$

for $i = 1, 2$. It is clear how to continue the process to obtain sequences $(f_n)_{n\in\mathbf{N}} \subset B$ and $(b_n, a_n)_{n\in\mathbf{N}}$, $b_n - a_n \geq \varepsilon/8$, such that for all finite disjoint sets M and N in $\mathbf{N}$ we have

$$H \cap \left(\bigcap_{m\in M} \{f_m > b_m\} \right) \bigcap \left(\bigcap_{n\in N} \{f_n < a_n\} \right) \neq \emptyset.$$

Of course we can assume that $|a_n - a| < \varepsilon/32$, for some constant a and every $n \in \mathbf{N}$. Then, letting $\delta = \varepsilon/32$ and $r = a + \varepsilon/32$, we finally get

$$H \cap \left(\bigcap_{m\in M} \{f_m > r + \delta\} \right) \bigcap \left(\bigcap_{n\in N} \{f_n < r\} \right) \neq \emptyset,$$

whenever M and N are finite disjoint subsets of $\mathbf{N}$. The proof is complete. □

As explained in the introduction, as a consequence of Theorem 1.1 and Simons' result [30], we obtain the positive solution to the Boundary Problem in spaces not containing $\ell_1[0, 1]$. Surprisingly, this is also true for any space $\ell_1(\Gamma)$ in the canonical norm. In the next section we discuss the Boundary Problem in the classical $\ell_1(\Gamma)$ and $C(K)$-spaces in more detail and prove even stronger results for these spaces.

4. Angelic spaces and the Boundary Problem

To motivate the results in this section we start with the following easy fact, which, in particular, yields the positive solution to the Boundary Problem under certain restrictions on the boundary.

Proposition 4.1. *Let X be a Banach space, D a norming subset of $B(X^*)$ and H a norm bounded $\sigma(X, D)$-compact subset of X. If D is dense in $B(X^*)$ in the topology of uniform convergence on countable subsets of H, then H is weakly compact.*

Proof. It suffices to prove that H is weakly countably compact, which implies that H is weakly compact by the Eberlein-Šmulyan Theorem. Take any sequence (x_n) in H and let $x_0 \in H$ be a $\sigma(X, D)$-cluster point of (x_n). For any $x^* \in B_{X^*}$ and $\varepsilon > 0$ (iii) implies that there is a point $d^* \in D$ such that

$$|x^*(x_n) - d^*(x_n)| < \varepsilon, \text{ for } n = 0, 1, 2 \ldots$$

From this we deduce that x_0 is also a $\sigma(X, X^*)$-cluster point of X^*, and the proof is complete. □

We stress that if, in the above proposition, D is moreover absolutely convex, then the fact that H is weakly compact implies that D is dense in $B(X^*)$ in the topology of uniform convergence on countable subsets of H (in fact, in the topology of uniform convergence on H), bearing in mind that the closures of D in the Mackey topology $\mu(X^*, X)$ and the weak* topology $\sigma(X^*, X)$ coincide and that $\overline{D}^{\sigma(X^*,X)} = B(X^*)$; see [27].

It is interesting to note that the assertion of Proposition 4.1 also holds if we assume that there is a boundary $B' \subset B_{X^*}$ such that:

(α) Each $x^* \in B'$ is in the closure of D for the topology of uniform convergence on countable subsets of H.

(β) Norm bounded and $\sigma(X, B')$- relatively countably compact subsets of X are weakly relatively compact.

This idea was used in [4] for $X = C(K)$ and $B' = K \cup \{-K\} \subset B(C(K)^*)$ to solve the Boundary Problem for $C(K)$-spaces. We now establish a purely topological statement giving a new proof of this result, not only for $C(K)$, but also for all spaces $\ell^1(\Gamma)$ in their canonical norms. In fact, we prove that those spaces are angelic (see the definition below) in the topology induced by a boundary.

Definition 4.2 (Fremlin). A regular topological space E is angelic if every relatively countably compact subset A of E is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from A.

In angelic spaces the different concepts of compactness and relative compactness coincide: The (relatively) countably compact, (relatively) compact and (relatively) sequentially compact subsets are the same (see [13]). Examples of angelic spaces include $C(K)$ endowed with the topology $t_p(K)$ of pointwise convergence on a countably compact space K (see [15], [22]), and all Banach spaces with their weak topologies.

The relation between angelicity and the Boundary Problem is seen from the following proposition.

Proposition 4.3. *Let X be a Banach space and let $B \subset B(X^*)$ be a boundary for X such that $(X, \sigma(X, B))$ is angelic. Then a subset H of X is weakly compact if (and only if) H is norm bounded and $\sigma(X, B)$-countably compact.*

Proof. In view of the Eberlein-Šmulyan Theorem, we only have to prove that if H is norm bounded and $\sigma(X, B)$-compact, then H is $\sigma(X, X^*)$-sequentially compact. Since the space $(X, \sigma(X, B))$ is angelic, for each sequence (x_n) in H there is a subsequence (x_{n_k}) and a point $x_0 \in H$ such that $x_0 = \sigma(X, B)-\lim_k x_{n_k}$. Now, Corollary 11 in [30] (alternatively, [31, Theorem on p. 70]) applies in a straightforward manner to ensure that $x_0 = \sigma(X, X^*) - \lim_k x_{n_k}$. The proof is complete. □

It is not difficult to prove that if X is a separable Banach space then for any boundary $B \subset B(X^*)$ the space $(X, \sigma(X, B))$ is angelic, although there are boundaries in the nonseparable case that also provide angelic topologies, such as the one with $C(K)$ we mentioned above.

Another example of this phenomenon is given in the following proposition.

Proposition 4.4. *Let Γ be any set and $D = \{-1,1\}^\Gamma$ the set of the extreme points of $B(\ell_\infty(\Gamma))$. Then:*

(i) *$(\ell_1(\Gamma), \sigma(\ell_1(\Gamma), D))$ is angelic.*

(ii) *If $H \subset \ell_1(\Gamma)$ is $\|\cdot\|_1$-bounded and $\sigma(\ell_1(\Gamma), D)$-compact then H is weakly compact.*

Proof. To prove (i) observe first that $D \subset B(\ell_\infty(\Gamma))$, $(D, \sigma(\ell_\infty(\Gamma), \ell_1(\Gamma)))$ is compact and $(C(D), t_p(D))$ is angelic; see [13]. The natural embedding

$$(\ell_1(\Gamma), \sigma(\ell_1(\Gamma), D)) \to (C(D), t_p(D))$$

is a homeomorphism onto its image. Then the angelicity of the space $\ell_1(\Gamma)$ in the topology $\sigma(\ell_1(\Gamma), D)$ follows from the angelicity of $(C(D), t_p(D))$. Statement (ii) is a straightforward consequence of Proposition 4.3. □

In Theorem 4.9 we will prove that statements such as those in Proposition 4.4 hold for all boundaries of $B(\ell_\infty(\Gamma))$. Still let us remark that (ii) was previously obtained in [17, Theorem 10.5.2] using Schur's Lemma for $\ell_1(\Gamma)$.

The next lemma will allow us to transfer the angelic property from one topology to another. We will use it later in the proofs of Theorems 4.8 and 4.9.

Lemma 4.5. *Let X be a non-empty set and τ, $\mathfrak{T}$ two Hausdorff topologies on X such that (X, τ) is regular and $(X, \mathfrak{T})$ is angelic. Assume that for every sequence (x_n) in X with a τ-cluster point $x \in X$, x is a $\mathfrak{T}$-cluster point of (x_n). Then the following assertions hold:*

(i) *If $L \subset X$ is τ-relatively countably compact, then L is $\mathfrak{T}$-relatively compact.*

(ii) *If $L \subset X$ is τ-compact, then L is $\mathfrak{T}$-compact.*

(iii) *(X, τ) is an angelic space.*

Proof. Statement (i) is a straightforward consequence of the assumptions on τ-cluster points of sequences in X and the fact that $(X, \mathfrak{T})$ is angelic.

Let us prove (ii). If $L \subset X$ is τ-compact, then L is $\mathfrak{T}$-relatively compact by (i). To finish the proof of (ii) it will be enough to show that L is $\mathfrak{T}$-closed. Pick $x \in \overline{L}^{\mathfrak{T}}$. Using the fact that $(X, \mathfrak{T})$ is angelic, there is a sequence (x_n) in L with

$$x = \mathfrak{T} - \lim_{n\to\infty} x_n. \tag{8}$$

By τ-compactness, there is $y \in L$ which is a τ-cluster point of (x_n). Our assumption implies that y is a $\mathfrak{T}$-cluster point of (x_n). Hence, by (8), $y = x$ and thus $x \in L$. This concludes the proof of (ii).

The proof of (iii) relies upon the following claim.

CLAIM 4.6. If L is a τ-relatively countably compact and countable subset of X, then

$$\overline{L}^{\tau} = \overline{L}^{\mathfrak{T}} \tag{9}$$

and the topologies τ and $\mathfrak{T}$ coincide on $\overline{L}^{\mathfrak{T}}$.

Suppose for the moment that the claim is true and let us prove that (X, τ) is angelic. To this end we will show that if $A \subset X$ is τ-relatively countably compact then $\overline{A}^{\tau} = \overline{A}^{\mathfrak{T}}$ is τ-compact and τ and $\mathfrak{T}$ coincide on $\overline{A}^{\tau}$. By (i) we already know that $\overline{A}^{\mathfrak{T}}$ is $\mathfrak{T}$-compact. Now we will prove that the identity map

$$\mathrm{id} : (\overline{A}^{\mathfrak{T}}, \mathfrak{T}) \longrightarrow (\overline{A}^{\mathfrak{T}}, \tau)$$

is continuous, that is, we will show that any τ-closed subset of $\overline{A}^{\mathfrak{T}}$ is $\mathfrak{T}$-closed. Indeed, take a τ-closed subset F of $\overline{A}^{\mathfrak{T}}$. Pick any $x \in \overline{F}^{\mathfrak{T}}$. The angelicity of $(X, \mathfrak{T})$ provides us with a sequence (x_n) in F such that

$$x = \mathfrak{T} - \lim_{n\to\infty} x_n.$$

Now for every $n \in \mathbf{N}$ we can also take (x_{mn}) in A such that

$$x_n = \mathfrak{T} - \lim_{m\to\infty} x_{mn}.$$

If we define $L = \{x_{mn} : m, n \in \mathbf{N}\}$ then the claim tells us that τ and $\mathfrak{T}$ coincide on $\overline{L}^{\mathfrak{T}}$ and so

$$x = \tau - \lim_{n\to\infty} x_n,$$

which implies that $x \in F$. So $\overline{A}^{\mathfrak{T}}$ is τ-compact and τ and $\mathfrak{T}$ coincide on $\overline{A}^{\mathfrak{T}}$. Hence, since $\overline{A}^{\mathfrak{T}}$ is τ-closed, we obtain $\overline{A}^{\tau} \subset \overline{A}^{\mathfrak{T}}$. On the other hand, as $A \subset \overline{A}^{\mathfrak{T}}$, A is τ-relatively compact and so $\overline{A}^{\tau}$ is $\mathfrak{T}$-compact by (ii). Therefore $\overline{A}^{\mathfrak{T}} \subset \overline{A}^{\tau}$, and thus $\overline{A}^{\mathfrak{T}} = \overline{A}^{\tau}$, and the proof is complete.

Let us now prove Claim 4.6.

From the assumptions we have $\overline{L}^{\tau} \subset \overline{L}^{\mathfrak{T}}$. Conversely, if we pick x in the $\mathfrak{T}$-compact subset $\overline{L}^{\mathfrak{T}}$, then the angelicity of $(X, \mathfrak{T})$ ensures the existence of a sequence (x_n) in L such that

$$x = \mathfrak{T} - \lim_{n\to\infty} x_n. \tag{10}$$

By the τ-relatively countably compactness of L there is $y \in \overline{L}^{\tau}$ which is a τ-cluster point of (x_n). Therefore y is a $\mathfrak{T}$-cluster point of (x_n). Hence, by (10), $y = x$ and thus $x \in \overline{L}^{\tau}$, which implies the equality (9).

To prove that the topologies τ and $\mathfrak{T}$ coincide on $H := \overline{L}^{\mathfrak{T}}$ it suffices to show, by compactness, that the identity map

$$\mathrm{id} : (H, \mathfrak{T}) \longrightarrow (H, \tau)$$

is continuous. To this end we will establish that any τ-closed subset F of H is $\mathfrak{T}$-closed. Indeed, as (H, τ) is a regular topological space we have

$$F = \cap\{\overline{U}^{\tau} : F \subset U \subset H, U \text{ is } \tau\text{-open in } H\}.$$

On the other hand, for any such U we have that $\overline{U}^{\tau} = \overline{U \cap L}^{\tau}$ and we can apply the equality (9) to $U \cap L$ to conclude that

$$\overline{U \cap L}^{\tau} = \overline{U \cap L}^{\mathfrak{T}}.$$

This implies

$$F = \cap\{\overline{U \cap L}^{\mathfrak{T}} : F \subset U \subset H, U \text{ is } \tau\text{-open in } H\}$$

and so F is $\mathfrak{T}$-closed. □

Note that the hypothesis on L in Claim 4.6, namely that L is countable and relatively countably compact in X, does not imply (in general) that L is relatively compact in X. Indeed, take $\beta\mathbf{N}$ to be the Stone-Čech compactification of the natural numbers $\mathbf{N}$ and pick a point $p \in \beta\mathbf{N} \setminus \mathbf{N}$. Take $X := \beta\mathbf{N} \setminus \{p\}$ and $L = \mathbf{N}$. It is well known that an infinite set in L cannot have a unique cluster point in $\beta\mathbf{N} \setminus \mathbf{N}$. This proves that L is relatively countably compact in X, but its closure in X, $\overline{L}^{X} = X$, is not compact.

The lemma below can be found in [4]. Here we include a slightly different proof that does not use the Urysohn Lemma.

LEMMA 4.7. *Let K be a compact space and $B \subset B(C(K)^*)$ a boundary for $(C(K), \|\cdot\|_\infty)$. If (f_n) is an arbitrary sequence in $C(K)$ and $x \in K$, then there is $\mu \in B$ such that*

$$f_n(x) = \int_K f_n d\mu$$

for every $n \in \mathbf{N}$.

Proof. If we define the continuous function $g : K \to [0, 1]$ by

$$g(t) := 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(t) - f_n(x)|}{1 + |f_n(t) - f_n(x)|}, \quad t \in K,$$

then

$$F = \bigcap_{n=1}^{\infty} \{y \in K; f_n(y) = f_n(x)\} = \{y \in K : g(y) = 1 = \|g\|_\infty\}. \tag{11}$$

Since B is a boundary, there exists $\mu \in B$ such that $\int_K g d\mu = 1$. From this we obtain

$$1 = \|\mu\| = |\mu|(K) \geq \int_K g d|\mu| \geq \int_K g d\mu = 1. \tag{12}$$

In other words,

$$0 = |\mu|(K) - \int_K g d|\mu| = \int_K (1-g) d|\mu|.$$

Since $1 - g \geq 0$, we obtain $|\mu|(\{y \in K : 1 - g(y) > 0\}) = 0$, that is, $|\mu|(K \setminus F) = 0$. Therefore, for every $n \in \mathbf{N}$ we have

$$\int_K f_n d\mu = \int_F f_n d\mu = \int_F f_n(x) d\mu = f_n(x),$$

because $\mu(F) = \int_F g d\mu = \int_K g d\mu = 1$ by the equalities (11) and (12). □

We naturally arrive at the following result.

THEOREM 4.8 ([4]). *Let K be a compact space and $B \subset B(C(K)^*)$ a boundary for $C(K)$. Then $(C(K), \sigma(C(K), B))$ is an angelic space. Consequently, if $H \subset C(K)$ is norm bounded and $\sigma(C(K), B)$-countably compact, then H is weakly compact.*

Proof. The space $(C(K), t_p(K))$ is angelic ([15], [22]; see also [13]). Bearing this in mind, the first part of the theorem follows from Lemmas 4.7 and 4.5 applied to $X = C(K)$, $\tau = \sigma(C(K), B)$ and $\mathfrak{T} = t_p(K)$.

The second part of the theorem follows from Proposition 4.3. □

The game we played with the spaces $C(K)$ can also be played with $\ell_1(\Gamma)$.

THEOREM 4.9. *Let Γ be any set and $B \subset B(\ell_\infty(\Gamma))$ a boundary for $(\ell_1(\Gamma), \|\cdot\|_1)$. Then:*

(i) *$(\ell_1(\Gamma), \sigma(\ell_1(\Gamma), B))$ is angelic.*
(ii) *If $H \subset \ell_1(\Gamma)$ is $\|\cdot\|_1$-bounded and $\sigma(X, B)$-compact then H is weakly compact.*

Proof. The fact that B is a boundary implies that for any countable subset $A \subset \Gamma$ and any family of signs $(y_\gamma)_{\gamma \in A} \in \{-1, 1\}^A$, there is $(b_\gamma)_{\gamma \in \Gamma}$ in B such that $b_\gamma = y_\gamma$, for $\gamma \in A$. Hence, if $D = \{-1, 1\}^\Gamma$, $d^* \in D$, and we take a sequence $(z_n)_n \in \ell_1(\Gamma)$, there is $b^* \in B$ such that

$$d^*(z_n) = b^*(z_n)$$

for every $n \in \mathbb{N}$. By Proposition 4.4, the space $\ell^1(\Gamma)$ is angelic in the topology $\sigma(\ell_1(\Gamma), D)$. Therefore statement (i) follows from Lemma 4.5 applied to $\tau = \sigma(\ell_1(\Gamma), B)$ and $\mathfrak{T} = \sigma(\ell_1(\Gamma), D)$. Statement (ii) is now a consequence of Proposition 4.3. □

We finish with two questions that are still open to the best of our knowledge.

A result of Bourgain and Talagrand [3] states that if X is a Banach space and H is a norm bounded and $\sigma(X, \text{ext}B(X^*))$-countably compact subset of X, then H is weakly compact. (Rainwater's theorem is a weak version of this.) Therefore, a positive solution of the problem below would imply the Boundary Problem.

PROBLEM 4.10. Let X be a Banach space, $B \subset B(X^*)$ a boundary and $D = \text{conv}(B \cup \{-B\})$. Given $e^* \in \text{ext}B(X^*)$, $\varepsilon > 0$, and a sequence $(x_n)_n$, is there $d^* \in D$ such that

$$|d^*(x_n) - e^*(x_n)| < \varepsilon,$$

for every $n \in \mathbf{N}$?

PROBLEM 4.11. Is a Banach space X angelic in the topology $\sigma(X, \text{ext}B(X^*))$?

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