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THE SPECTRUM OF THE p-LAPLACIAN AND p-HARMONIC MORPHISMS ON GRAPHS

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Dedicated to Professor Takashi Sakai on his sixtieth birthday

Abstract. For a real number p with $1 < p < \infty$ we consider the spectrum of the p-Laplacian on graphs, p-harmonic morphisms between two graphs, and estimates for the solutions of p-Laplace equations on graphs. More precisely, we prove a Cheeger type inequality and a Brooks type inequality for infinite graphs. We also define p-harmonic morphisms and horizontally conformal maps between two graphs and prove that these two concepts are equivalent. Finally, we give some estimates for the solutions of p-Laplace equations, which coincide with Green kernels in the case $p = 2$.

1. Introduction

In the last decade there has been an increasing interest in the p-Laplacian, which plays an important role in geometry and partial differential equations. The p-Laplacian is a natural generalization of the Laplacian, which corresponds to $p = 2$. Although the Laplacian has been much studied, little is known about the nonlinear case $p \neq 2$. On the other hand, the discrete analogue of the Laplacian on Riemannian manifolds has recently been investigated. For the Laplacian on an infinite graph, Dodziuk and Kendall [3] proved a discrete analogue of a Cheeger type inequality, which gave a lower bound for the bottom of the spectrum in terms of an isoperimetric constant. Dodziuk and Karp [2], Ohno and Urakawa [6], and Fujiwara [4] gave an upper bound for the essential spectrum. Urakawa [9] introduced a discrete analogue of harmonic morphisms and horizontally conformal maps between two graphs and proved that those concepts are equivalent. He [8] also gave estimates for Green kernels of infinite graphs.

The purpose of this paper is to generalize these results to the p-Laplacian and p-harmonic morphisms for graphs, and to estimate the solutions of p-Laplace equations, which coincide with Green kernels in the case $p = 2$.

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This paper is organized as follows. In Section 2, we define the bottom of the spectrum of the p-Laplacian for an infinite graph. In Sections 3 and 4, we estimate the bottom of the spectrum in terms of an isoperimetric constant (Theorem 1), and the essential spectrum in terms of the exponential growth rate of the graph (Theorem 2). In Section 5, we define p-harmonic morphisms and horizontally conformal maps between two graphs, and we show that these two concepts are equivalent. In Section 7, we estimate the solutions of p-Laplace equations, which coincide with the Green kernel in the case $p = 2$.

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2. Preliminaries

Let $G = (V, E)$ be a connected simple graph, with V as the vertex set and E as the edge set. We give an orientation to each edge. Vertices x and y are called neighbors, denoted by $x \sim y$, if x and y are connected by an edge. $e{=}[x, y]$ denotes a directed edge from x to y. The degree of the vertex x, denoted by $m(x)$, is the number of edges incident with x. For any real-valued functions u, v on vertices, we define the inner product and the norm as follows:

$$(u, v) = \sum_{x \in V} m(x)u(x)v(x), \qquad \|u\|^2 = (u, u).$$

A vector field F is a map $F : E \to \mathbf{R}$ with the property

$$F([x, y]) = -F([y, x]) \qquad \text{for every } x \sim y.$$

Given vector fields F and G, the inner product and the norm are defined by

$$(F, G) = \sum_{e} F(e)G(e), \qquad \|F\|^2 = (F, F).$$

For $1 < p < \infty$ we define a Banach space ℓ_0^p of functions on vertices as

$$\ell_0^p = \left\{ u : V \to \mathbf{R} : \sum_{x} m(x)|u(x)|^p < \infty \right\},$$

and a Banach space ℓ_1^p of functions on the directed edges $e = [x, y]$ as

$$\ell_1^p = \left\{ F : E \to \mathbf{R} : F([x, y]) = -F([y, x]) \text{ for all } [x, y], \sum_{e} |F(e)|^p < \infty \right\}.$$

The linear operators $d : \ell_0^p \to \ell_1^p$ and $d^* : \ell_1^p \to \ell_0^p$ are defined by

$$(du)([x, y]) = u(y) - u(x), \tag{1}$$

$$(d^*F)(x) = -\frac{1}{m(x)} \sum_{x \sim y} F([x, y]). \tag{2}$$

The discrete p-Laplacian Δ_p acting on a real-valued function $u : V \to \mathbf{R}$ is defined by

$$\Delta_p u(x) = -d^*(|du|^{p-2}du)(x) = \frac{1}{m(x)} \sum_{x\sim y} |du([x,y])|^{p-2} du([x,y]) \tag{3}$$
$$= \frac{1}{m(x)} \sum_{x\sim y} |u(y)-u(x)|^{p-2}(u(y)-u(x)).$$

We define the bottom of the spectrum $\lambda_{0,p}$ of the p-Laplacian Δ_p as follows:

$$\lambda_{0,p} = \inf\left(\frac{\sum_{x\sim y} |du([x,y])|^p}{\sum_x m(x)|u(x)|^p} : u \in \ell_0^p \right) = \inf\left(\frac{\|du\|^p}{\sum_x m(x)|u(x)|^p} : u \in \ell_0^p \right).$$

For a finite subset F, denote by χ_F its characteristic function . Since

$$\frac{\sum_{x\sim y} |d\chi_z([x,y])|^p}{\sum_x m(x)|\chi_z(x)|^p} = 1 \text{ for } x,y,z \in V$$

we have $\lambda_{0,p} \le 1$.

3. A Cheeger type inequality

If $N \subset V$ is a finite subset, we define ∂N, the edge boundary of N, to be the set of edges of the graph G joining a vertex of N with a vertex in the complement of N, i.e.,

$$\partial N = \{e = [x,y] \in E : (x \in N \text{ and } y \notin N) \text{ or } (x \notin N \text{ and } y \in N)\}.$$

We define

$$A(N) = \sum_{x\in N} m(x), \tag{4}$$

$$L(\partial N) = \#(\partial N). \tag{5}$$

The isoperimetric constant $h(G)$ of the graph G is defined by

$$h(G) = \inf\left\{ \frac{L(\partial N)}{A(N)} : N \text{ is a finite subset of } V \right\}.$$

Theorem 1. *For every infinite graph G we have*

$$\left(\frac{h(G)}{2p}\right)^p \le \lambda_{0,p} \le h(G).$$

Proof. We use an argument similar to that in [3]. Taking $u \in \ell_0^p$ of finite support and $\sum m(x)|u(x)|^p = 1$, we have

$$\sum_{x\sim y} |du([x,y])|^p = \sum_{x\sim y} |u(y)-u(x)|^{p-2}(u(y)-u(x))^2$$
$$\ge \sum_{x\sim y} |u(y)-u(x)|^{p-2}(|u(y)|-|u(x)|)^2.$$

Thus we may assume $u \geq 0$. For such a function u, set

$$A = \sum_{x \sim y} |u(x)^p - u(y)^p|.$$

Using Hölder inequality, we get

(6)

$$\begin{aligned} A &\leq \sum_{x \sim y} p(u(x)^{p-1} + u(y)^{p-1})|u(x) - u(y)| \\ &\leq p \left(\sum_{x \sim y} \left\{ u(x)^{p-1} + u(y)^{p-1} \right\}^{p/(p-1)} \right)^{(p-1)/p} \cdot \left(\sum_{x \sim y} \{|u(x) - u(y)|\}^p \right)^{1/p} \\ &\leq 2p \left(\sum_{x \sim y} \{(|u(x)|^p) + (|u(y)|^p)\} \right)^{(p-1)/p} \left(\sum_{x \sim y} \{|u(x) - u(y)|\}^p \right)^{1/p} \\ &\leq 2p \left(\sum_{x} m(x)u(x)^p \right)^{(p-1)/p} \left(\sum_{x \sim y} |du([x,y])|^p \right)^{1/p}. \end{aligned}$$

Next, we estimate A from below. Let $0 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_M$ be the sequence of all values of u, and set

$$K_i = \{x \in V : u(x) \geq \beta_i\}.$$

Then

$$\partial K_i = \{e \in E : e = [x,y], u(x) \geq \beta_i, u(y) < \beta_i\},$$

and

$$A = \sum_{i=1}^{M} \sum_{u(x)=\beta_i} \sum_{y,(y \sim x), u(y) < \beta_i} |u(x)^p - u(y)^p|.$$

If $x \sim y, u(x) = \beta_i$ and $u(y) = \beta_{i-k} < \beta_i$, then

$$[x,y] \in \partial K_i \cap \partial K_{i-1} \cap \cdots \cap \partial K_{i-k+1}$$

and

$$u(x)^p - u(y)^p = (\beta_i^p - \beta_{i-1}^p) + (\beta_{i-1}^p - \beta_{i-2}^p) + \cdots + (\beta_{i-k+1}^p - \beta_{i-k}^p).$$

Thus

$$
\begin{aligned}
A &= \sum_{i=1}^{M} \sum_{[x,y]\in\partial K_i} (\beta_i^p - \beta_{i-1}^p) \\
&= \sum_{i=1}^{M} L(\partial K_i)(\beta_i^p - \beta_{i-1}^p) \geq h(G) \sum_{i=1}^{M} A(K_i)(\beta_i^p - \beta_{i-1}^p) \\
&= h(G) \sum_{i=1}^{M} \sum_{x\in K_i} m(x)(\beta_i^p - \beta_{i-1}^p) \\
&= h(G) \Bigg(\beta_1^p \sum_{x\in K_1} m(x) + \cdots + \beta_{M-1}^p \sum_{x\in K_{M-1}} m(x) + \beta_M^p \sum_{x\in K_M} m(x) \\
&\qquad - \beta_1^p \sum_{x\in K_2} m(x) - \cdots - \beta_{M-1}^p \sum_{x\in K_M} m(x) \Bigg) \\
&= h(G) \left(\sum_{i=1}^{M-1} \sum_{x\in K_i\setminus K_{i+1}} m(x)\beta_i^p + \sum_{x\in K_M} m(x)\beta_M^p \right).
\end{aligned}
$$

Since

$$K_i \setminus K_{i-1} = \{x \in V : \beta_i \leq u(x) < \beta_{i+1}\} = \{x \in V : u(x) = \beta_i\},$$

we get

$$\sum_{x\in K_i\setminus K_{i+1}} m(x)\beta_i^p = \sum_{i=1}^{M-1} m(x)|u(x)|^p.$$

It follows that

$$A \geq h(G) \sum_{x} m(x)|u(x)|^p. \tag{7}$$

Combining (6) and (7), we obtain

$$
\begin{aligned}
h(G) \sum_{x} m(x)|u(x)|^p \leq A \leq 2p &\left(\sum_{x} m(x)|u(x)|^p \right)^{(p-1)/p} \\
&\times \left(\sum_{x\sim y} |du([x,y])|^p \right)^{1/p}.
\end{aligned}
$$

It follows that

$$\left(\frac{h(G)}{2p}\right)^p \leq \frac{\sum_e |du(e)|^p}{\sum_x m(x)|u(x)|^p}.$$

Taking the infimum over all u on the right hand side, we get

$$\left(\frac{h(G)}{2p}\right)^p \le \lambda_{0,p},$$

which is the first inequality of the theorem.

Next, we prove the second inequality in the theorem. For any subset $S \subset V$ with $S \neq \emptyset$ and $\#S < \infty$ set

$$u(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$

We have

$$\sum_{x \in V} m(x)|u(x)|^p = \sum_{x \in S} m(x) = A(S),$$

and

$$\|du\|^p = \sum_{x \sim y} |du([x,y])|^p = \sum_{x \sim y} |u(x) - u(y)|^p \le L(\partial S) = \#(\partial S).$$

Therefore,

$$\lambda_{0,p} \le \frac{\sum_{x \sim y} |du([x,y])|^p}{\sum_x m(x)u(x)^p} \le \frac{L(\partial S)}{A(S)} = h(G).$$

This completes the proof. □

4. A Brooks type inequality

We assign the path metric 1 to every edge of G. Let $B(r) = B_{x_0}(r)$ denote the ball of radius r and with center $x_0 \in V$. Put $V(r) = \sum_{x \in B(r)} m(x)$. The exponential growth rate of G is defined as $\mu = \limsup_{r \to \infty} (\log V(r))/r$. For a finite subgraph K of G, the p-Laplacian $\Delta_{p,G \setminus K}$ on $G \setminus K$ with the Dirichlet boundary condition is given by

$$\Delta_{p,G \setminus K} f(x) = \begin{cases} \Delta_{p,G} f(x) & \text{on } G \setminus K, \\ 0 & \text{on } K, \end{cases}$$

for $f \in \ell^p(G \setminus K) = \{f \in \ell^p(G) : f|_K = 0\}$. Let $\lambda_{0,p}(G \setminus K)$ denote the infimum of the spectrum of $\Delta_{p,G \setminus K}$. We define $\lambda_{0,p}^{\mathrm{ess}} = \lim_K \lambda_{0,p}(G \setminus K)$, where K runs through all finite subgraphs of G. Set $\rho(x) = \rho_{x_0}(x) = d(x_0, x)$, where $d(x_0, x)$ is the distance between two vertices x_0 and x. The following theorem generalizes a result proved in [4] for the case $p = 2$.

THEOREM 2. *If G is an infinite graph, then*

$$\lambda_{0,p}^{\mathrm{ess}} \le \frac{(\exp(\mu/p) - 1)^p}{1 + \exp(\mu)}.$$

For the proof we need the following lemmas, which can be proved in the same way as the corresponding results in [4] by replacing 2 by p.

LEMMA 1. *If $\mu < p\alpha$ for some $\alpha > 0$, then $\sum_x m(x)\exp(-p\alpha\rho(x)) < \infty$.*

For any integer $j \in \mathbf{N}$ define functions h_j and f_j by

$$h_j(x) = \begin{cases} \alpha\rho(x) & \text{if } \rho(x) \le j, \\ 2\alpha j - \alpha\rho(x) & \text{if } \rho(x) > j, \end{cases}$$
$$f_j(x) = \exp(h_j(x)).$$

LEMMA 2. *If $x \sim y$, then for all j,*

$$(f_j(x) - f_j(y))^p \le \frac{(\exp(\alpha) - 1)^p}{1 + \exp(p\alpha)}(f_j(x)^p + f_j(y)^p).$$

LEMMA 3. *For all j,*

$$\|df_j\|^p = \sum_{x\sim y} |df_j([x, y])|^p \le \frac{(\exp(\alpha) - 1)^p}{1 + \exp(p\alpha)} \sum_x m_j(x)|f_j(x)|^p.$$

Let K be a finite subset of V, let χ be its characteristic function, and set

$$g_j = f_j(1 - \chi_K).$$

LEMMA 4. *If $\sum_x m(x)\exp(-p\alpha\rho(x)) < \infty$, then for all j,*

$$\sum_x m(x)|g_j(x)|^p < \infty, \quad \lim_j \sum_x m(x)|g_j(x)|^p = \infty.$$

LEMMA 5. *Suppose k satisfies $K \subset B(k)$. Then for all j,*

$$\|dg_j\|^p = \sum_{x\sim y} |dg_j([x, y])|^p \le C_1(k) + \frac{(\exp(\alpha) - 1)^p}{1 + \exp(p\alpha)} \sum_x m(x)|g_j(x)|^p,$$

where

$$C_1(k) = \exp(p\alpha(k + 1))V(k + 1)\left\{2^{p-1} + \frac{(\exp(\alpha) - 1)^p}{1 + \exp(p\alpha)}\right\}.$$

Proof of Theorem 2. Suppose

$$\frac{(\exp(\mu/p) - 1)^p}{1 + \exp(\mu)} < \lambda_{0,p}^{\mathrm{ess}}(G).$$

Then there exists a finite subset K with

$$\frac{(\exp(\mu/p) - 1)^p}{1 + \exp(\mu)} < \lambda_{0,p}(G \setminus K).$$

Fix K and define $g_j = f_j(1 - \chi_K)$. Since $(\exp(\mu/p) - 1)^p/(1 + \exp(\mu))$ is a monotone increasing function of μ, we can take α such that

$$\mu < p\alpha, \quad \frac{(\exp(\alpha) - 1)^p}{1 + \exp(p\alpha)} < \lambda_{0,p}(G \setminus K).$$

From Lemmas 1 and 4 we get

$$\sum_x m(x)|g_j(x)|^p < \infty, \quad \lim_{j\to\infty} \sum_x m(x)|g_j(x)|^p = \infty.$$

From Lemma 5 we obtain

$$\frac{\|dg_j\|^p}{\sum_x m(x)|g_j(x)|^p} \leq \frac{C_1(k)}{\sum_x m(x)|g_j(x)|^p} + \frac{(\exp(\alpha)-1)^p}{1+\exp(p\alpha)}.$$

Since the first term of the right hand side goes to 0, it follows that for all sufficiently large j

$$\frac{\|dg_j\|^p}{\sum_x m(x)g_j(x)^p} < \lambda_{0,p}(G \setminus K).$$

This contradicts the definition of $\lambda_{0,p}^{\mathrm{ess}}(G \setminus K)$ and thus completes the proof of Theorem 2. □

5. A p-harmonic morphism

In this section we define the notion of p-harmonic morphisms and horizontally conformal maps between two graphs. We show that these concepts are equivalent. Urakawa [9] introduced a discrete analogue of a harmonic morphism and characterized it in terms of horizontal conformality. Here we further generalize these results to p-harmonic morphisms. Let $G_i = (V_i, E_i)$ $(i = 1, 2)$ be two graphs and $\varphi : V_1 \to V_2$ an onto mapping.

Definition 1.

(1) φ is said to be a p-harmonic morphism from G_1 to G_2 if, for any p-harmonic function f at $y = \varphi(x) \in V_2$, the composition $\varphi^* f = f \circ \varphi$ is p-harmonic at $x \in V_1$.

(2) φ is said to be horizontally conformal if the following two conditions hold:
   - (a) For all $z, x \in V_1$ such that $z \sim x$, we have either $\varphi(x) = \varphi(z)$ or $\varphi(z) \sim \varphi(x)$.
   - (b) For all $y \in V_2$ and $x \in \varphi^{-1}(y)$, the number of elements in $\varphi^{-1}(y')$ connected to x, $\#\{z \in \varphi^{-1}(y') : z \sim x\}$, is the same for all choices of $y' \in V_2$ with $y' \sim y$.

We have the following theorem.

Theorem 3. *Let $G_i = (V_i, E_i)$ $(i = 1, 2)$ be two graphs. Let $\varphi : V_1 \to V_2$ be an onto mapping. Then φ is a p-harmonic morphism if and only if it is horizontally conformal.*

Proof. Let φ be a p-harmonic morphism and let $y = \varphi(x)$. Consider a function f on V_2 defined by

$$f(w) = \begin{cases} 1, & w = y, \\ 1, & w \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$(\Delta_p f)(y) = \frac{1}{m(y)} \sum_{w \sim y} |f(w) - f(y)|^{p-2}(f(w) - f(y)) = 0,$$

the function f is p-harmonic at y. Since φ is a p-harmonic morphism, we get

$$\begin{aligned} 0 &= (\Delta_p f)(y) = \Delta_p(f \circ \varphi)(x) \\ &= \frac{1}{m(x)} \sum_{x \sim z} |f(\varphi(z)) - f(\varphi(x))|^{p-2}(f(\varphi(z)) - f(\varphi(x)) \\ &= \frac{1}{m(x)} \sum_{z \in \varphi^{-1}(y), x \sim z} |f(\varphi(z)) - f(\varphi(x))|^{p-2}(f(\varphi(z)) - f(\varphi(x)) \\ &\quad + \frac{1}{m(x)} \sum_{z \notin \varphi^{-1}(y), x \sim z} |f(\varphi(z)) - f(\varphi(x))|^{p-2}(f(\varphi(z)) - f(\varphi(x)) \\ &= \frac{1}{m(x)} \sum_{z \notin \varphi^{-1}(y), x \sim z} |f(\varphi(z)) - 1|^{p-2}(f(\varphi(z)) - 1) \\ &= \frac{1}{m(x)} \sum_{z \notin \varphi^{-1}(y), x \sim z} (1 - f(\varphi(z)))^{p-1}. \end{aligned}$$

Thus when $z \notin \varphi^{-1}(y)$ and $x \sim z$, the value of $f(\varphi(z))$ is necessarily 1. By the definition of f and since $\varphi(z) \neq y$, we obtain $\varphi(z) \sim y = \varphi(x)$. This proves condition (a) of horizontal conformality.

Next we prove condition (b). Let $\{y_1, y_2, \cdots, y_n\}$ be the set $\{y' \in V_2 : y' \sim y\}$ with $n = m(y)$, $y = \varphi(x)$. Set $m_i = \#\{z \in \varphi^{-1}(y_i) : z \sim x\}$ $(i = 1, \ldots, n)$. We will show that $m_1 = m_2 = \cdots = m_n$, which is condition (b). We first prove that $m_1 = m_2$. Take $0 < d_1 < d_2$ and define a function f on V_2 by

$$f(y) = \begin{cases} d_1, & w = y_1, \\ d_2, & w = y_2, \\ (d_1 + d_2)/2, & \text{otherwise.} \end{cases}$$

Then f is a p-harmonic. In fact,

$$\begin{aligned}
m(y)(\Delta_p f)(y) &= \sum_{y'\sim y} |f(y') - f(y)|^{p-2}(f(y') - f(y)) \\
&= |f(y_1) - f(y)|^{p-2}(f(y_1) - f(y)) + |f(y_2) - f(y)|^{p-2}(f(y_2) - f(y)) \\
&\qquad + \sum_{i=3}^{n} |f(y_i) - f(y)|^{p-2}(f(y_i) - f(y_1)) \\
&= |d_1 - f(y)|^{p-2}(d_1 - f(y)) + |d_2 - f(y)|^{p-2}(d_2 - f(y)) \\
&= \left|d_1 - \frac{d_1+d_2}{2}\right|^{p-2}\left(d_1 - \frac{d_1+d_2}{2}\right) + \left|d_2 - \frac{d_1+d_2}{2}\right|^{p-2}\left(d_2 - \frac{d_1+d_2}{2}\right) \\
&= -\left(\frac{d_2-d_1}{2}\right)^{p-1} + \left(\frac{d_2-d_1}{2}\right)^{p-1} = 0.
\end{aligned}$$

Since φ is a p-harmonic morphism, the function $\varphi^* f(x) = f \circ \varphi(x) = f(y)$ on V_2 is a p-harmonic function at $y = \varphi(x)$ for the p-harmonic function f at $x \in V_1$. Thus we have

$$\begin{aligned}
0 = m(x)\Delta_p(f\circ\varphi)(x) &= \sum_{z\sim x} |f(\varphi(z)) - f(\varphi(x))|^{p-2}(f(\varphi(z)) - f(\varphi(x)) \\
&= \sum_{z\sim x, \varphi(z)=y_1} |f(y_1) - f(y)|^{p-2}(f(y_1) - f(y) \\
&\qquad + \sum_{z\sim x, \varphi(z)=y_2} |f(y_2) - f(y)|^{p-2}(f(y_2) - f(y) \\
&\qquad + \sum_{z\sim x, \varphi(z)\neq y_1, y_2} |f(\varphi(z)) - f(y)|^{p-2}(f(\varphi(z)) - f(y) \\
&= m_1\left\{\left|d_1 - \frac{d_1+d_2}{2}\right|^{p-2}\left(d_1 - \frac{d_1+d_2}{2}\right)\right\} \\
&\qquad + m_2\left\{\left|d_2 - \frac{d_1+d_2}{2}\right|^{p-2}\left(d_2 - \frac{d_1+d_2}{2}\right)\right\} \\
&= -m_1\left(\frac{d_2-d_1}{2}\right)^{p-1} + m_2\left(\frac{d_2-d_1}{2}\right)^{p-1} \\
&= \left(\frac{d_2-d_1}{2}\right)^{p-1}(m_2 - m_1).
\end{aligned}$$

This implies that $m_1 = m_2$. In a similar fashion we get $m_i = m_{i+1}$ for $i = 2, \ldots, n-1$. Thus $m_1 = m_2 = \cdots = m_n$.

Next assume (a) and (b). For all $x \in V_1$ and $y = \varphi(x)$ we have

$$\{z \notin \varphi^{-1}(y) : z \sim x\} = \bigcup_{y' \sim y} \{z \in \varphi^{-1}(y') : z \sim x\},$$

and by (b)

$$\#\{z \notin \varphi^{-1}(y) : z \sim x\} = m(y)\#\{z \in \varphi^{-1}(y') : z \sim x\}.$$

Assume that a function f on V_2 is p-harmonic at $y = \varphi(x)$. Then

$$\begin{aligned}
m(x)\Delta_p(f \circ \varphi)(x) = &\sum_{\substack{z \in \varphi^{-1}(y) \\ z \sim x}} |f(\varphi(z)) - f(\varphi(x))|^{p-2} f((\varphi(z)) - f(\varphi(x)) \\
&+ \sum_{z \notin \varphi^{-1}(y), z \sim x} |f(\varphi(z)) - f(\varphi(x))|^{p-2} f((\varphi(z)) - f(\varphi(x)) \\
= &\sum_{y' \sim y} \sum_{z \in \varphi^{-1}(y'), z \sim x} |f(\varphi(z)) - f(y)|^{p-2}(f(\varphi(z)) - f(y)) \\
= &\frac{1}{m(y)} \#\{z \in \varphi^{-1}(y') : z \sim x\} \sum_{y' \sim y} |f(y') - f(y)|^{p-2}(f(y') - f(y)) \\
= &\, 0,
\end{aligned}$$

because f is p-harmonic. This completes the proof. $\square$

6. Green's formula

In this section we prove a discrete analogue of Green's formula for the p-Laplacian on a graph.

PROPOSITION 1. *For every finite subset $N \subset V$ and any functions $u, v : V \to \mathbf{R}$ we have*

$$\begin{aligned}
&\frac{1}{2} \sum_{x,y \in N, x \sim y} |u(x) - u(y)|^{p-2}(u(y) - u(x))(v(y) - v(x)) \\
&= - \sum_{x \in N} m(x)v(x)\Delta_p u(x) + \sum_{x \in N} \sum_{\substack{y \in V \setminus N \\ x \sim y}} v(x)|u(y) - u(x)|^{p-2}(u(y) - u(x)).
\end{aligned}$$

Proof. By the definition of Δ_p, the first term of the right hand side is

$$
\begin{aligned}
(8)\qquad & -\sum_{x\in N} v(x)m(x)\Delta_p u(x) \\
&= -\sum_{x\in N} v(x)\sum_{x\sim y}|u(y)-u(x)|^{p-2}(u(y)-u(x)) \\
&= -\sum_{x\in N} v(x)\sum_{y\in N, x\sim y}|u(y)-u(x)|^{p-2}(u(y)-u(x)) \\
&\quad -\sum_{x\in N} v(x)\sum_{\substack{y\in V\setminus N \\ x\sim y}}|u(y)-u(x)|^{p-2}(u(y)-u(x)).
\end{aligned}
$$

Since the last term is equal to the second term of the right hand side of the asserted formula, these two terms cancel. On the other hand, the left hand side of the formula to be proved is

$$
\begin{aligned}
&\frac{1}{2}\sum_{y\in N} v(y)\sum_{x\in N, y\sim x}|u(y)-u(x)|^{p-2}(u(y)-u(x)) \\
&\qquad -\frac{1}{2}\sum_{x\in N} v(x)\sum_{y\in N, y\sim x}|u(y)-u(x)|^{p-2}(u(y)-u(x)).
\end{aligned}
$$

By interchanging x and y in the first term, we see that this term is equal to the second one. Therefore the sum of these terms becomes

$$
-\sum_{x\in N} v(x)\sum_{y\in N, y\sim x}|u(y)-u(x)|^{p-2}(u(y)-u(x)),
$$

which is equal to the first term of the right hand side of (8). This completes the proof. $\square$

7. Estimates

Let (V,E) be an infinite graph. For a subset $D\subset V$ we define the vertex boundary ∂D to be the set of all vertices y that are not in D, but connected to some vertex in D, i.e., $\partial D=\{y\notin D : x\sim y, x\in D\}$. We consider the following equation on a finite graph with boundary ∂D:

$$
\begin{aligned}
\triangle_p g(x,y) &= \begin{cases} -1 & \text{if} \quad x=y \\ 0 & \text{if} \quad x\neq y \end{cases} && (x,y\in D\cup\partial D), \\
g(x,y) &= 0 && (x\in\partial D \text{ or } y\in\partial D).
\end{aligned}
$$

Denote by $G_D(x,y)$ the solutions of the above equation, which become Green kernels in the case $p=2$. We define the function $G(x,y)$ on an infinite graph (V,E) by

$$
G(x,y)=\lim_{n\to\infty} G_{B_n(x_0)}(x,y), \tag{9}
$$

where, for $x_0 \in V$, $B_n(x_0) = \{x \in V : \rho_{x_0}(x) = d(x, x_0) < n\}$ is a ball with center x_0 and radius n. For a graph G=(V, E) and $x \in V$, we consider functions R_x and K_x on the natural numbers $\mathbf{N}$ satisfying

$$R_x(t) \leq 1 - \frac{m_x^+(\sigma(t))}{m_x^-(\sigma(t))} \leq K_x(t) < 1 \tag{10}$$

for any geodesic $\sigma(t)$ starting at $\sigma(0) = x$. Here a path σ from $x \in V$ to $y \in V$ is called a geodesic if its length, the number of edges in the path, is the minimum of the lengths of all paths from x to y, and we define for $y \in V$,

$$\begin{aligned} m_x^-(y) &= \#\{z \in V : z \sim y, \rho_x(z) = \rho_x(y) - 1\}, \\ m_x^+(y) &= \#\{z \in V, z \sim y, \rho_x(z) = \rho_x(y) + 1\}. \end{aligned}$$

Given such a function R_x, we let $f_x : \mathbf{N} \to \mathbf{R}$ be the unique solution of the difference equation

$$f_x(t+1) - f_x(t) = -R_x(t) f_x(t), \tag{11}$$

for all $t = 1, 2, \ldots, l$, with the initial condition

$$f_x(1) = 1, \tag{12}$$

i.e.,

$$f_x(t+1) = (1 - R_x(t)) f_x(t), \qquad t = 1, 2, \ldots, l.$$

Similarly, given a function K_x, we let $F_x : \mathbf{N} \to \mathbf{R}$ be the unique solution of the difference equation

$$F_x(t+1) - F_x(t) = -K_x(t) F_x(t), \tag{13}$$

for all $t = 1, 2, \ldots, l$, with the initial condition

$$F_x(1) = 1, \tag{14}$$

i.e.,

$$F_x(t+1) = (1 - K_x(t)) F_x(t), \qquad t = 1, 2, \ldots, l.$$

LEMMA 6.

(1) *Assume that* R_x *satisfies* (10). *Then the solution* $f_x(t)$ *of* (11) *and* (12) *is positive for* $t = 1, 2, \ldots$
(2) *Assume that* K_x *satisfies* (10). *Then the solution* $F_x(t)$ *of* (13) *and* (14) *is positive for* $t = 1, 2, \ldots$

Proof. See [8]. □

For each $n = 1, 2, \ldots$ define functions $\psi_n(t)$ and $\Psi_n(t)$ on $\{0, 1, 2, \ldots, n\}$ by

$$\psi_n(t) = \begin{cases} \sum_{s=t+1}^{n} (1/f_x(s))^{1/(p-1)} & t = 0, 1, 2, \ldots, n-1, \\ 0 & t = n, \end{cases}$$

$$\Psi_n(t) = \begin{cases} \sum_{s=t+1}^{n} (1/F_x(s))^{1/(p-1)} & t = 0, 1, 2, \ldots, n-1, \\ 0 & t = n. \end{cases}$$

Also, define functions $\psi_n \circ \rho_x$ and $\Psi_n \circ \rho_x$ on the closed ball $B_n(x) \cup \partial B_n(x)$ of radius n centered at $x \in V$ by

$$(\psi_n \circ \rho_x)(y) = \sum_{s=\rho_x(y)+1}^{n} (1/f_x(s))^{1/(p-1)},$$

$$(\Psi_n \circ \rho_x)(y) = \sum_{s=\rho_x(y)+1}^{n} (1/F_x(s))^{1/(p-1)},$$

for $y \in B_n(x) \cup \partial B_n(x)$, where $\partial B_n(x) = \{z \in V : \rho_x(z) = \rho(x,z) = n\}$. The functions $\psi_n \circ \rho_x$ and $\Psi_n \circ \rho_x$ decrease in ρ_x on $B_n(x) \cup \partial B_n(x)$ and vanish on the boundary $\partial B_n(x)$.

Proposition 2.

(1) *The function $\psi_n \circ \rho_x$ is a p-subharmonic function on $B_n(x) \setminus \{x\}$, i.e., it satisfies*

$$\triangle_p(\psi_n \circ \rho_x) \geq 0 \qquad \textit{for} \quad 0 < \rho(x,y) < n,$$

and

$$\triangle_p(\psi_n \circ \rho_x) = -1 \qquad \textit{at} \quad y = x.$$

(2) *The function $\Psi_n \circ \rho_x$ is a p-superharmonic function on $B_n(x) \setminus \{x\}$, i.e., it satisfies*

$$\triangle_p(\Psi_n \circ \rho_x) \leq 0 \qquad \textit{for} \quad 0 < \rho(x,y) < n,$$

and

$$\triangle_p(\psi_n \circ \rho_x) = -1 \qquad \textit{at} \quad y = x.$$

Proof. We prove only (1), since the proof of (2) is similar. Take $y \in V$ with $0 < \rho(x,y) < n$, a geodesic $\sigma \in B_n(x) \cup \partial B_n(x)$ starting at x, and let $y = \sigma(t)$. For $0 < \rho(x,y) < n$, we calculate

$$
\begin{aligned}
&m(y)\triangle_p(\psi_n\circ\rho_x)(y) = \sum_{z\sim y}|d(\psi_n\circ\rho_x)([y,z])|^{p-2}d(\psi_n\circ\rho_x)([y,z])\\
&= \sum_{z\sim y}|(\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y)|^{p-2}\left(\psi_n\circ\rho_x(z)-\psi_n\circ\rho_x(y)\right)\\
&= \sum_{\substack{z\sim y\\ \rho_x(y)=\rho_x(z)}}|(\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y)|^{p-2}((\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y))\\
&\quad+\sum_{\substack{z\sim y\\ \rho_x(z)=\rho_x(y)-1}}|(\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y)|^{p-2}((\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y))\\
&\quad+\sum_{\substack{z\sim y\\ \rho_x(z)=\rho_x(y)+1}}|(\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y)|^{p-2}((\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y))\\
&= \frac{m^-(y)}{f_x(\rho_x(y))}-\frac{m^+(y)}{f_x(\rho_x(y)+1)}.
\end{aligned}
$$

From the assumption on R_x we get

$$\frac{f_x(\rho_x(y)+1)}{f_x(\rho_x(y))} = 1-R_x(\rho_x(y)) \geq \frac{m_x^+(y)}{m_x^-(y)}.$$

Hence we obtain

$$\triangle_p(\psi_n\circ\rho_x)\geq 0.$$

At $y=x$ we have

$$
\begin{aligned}
&m(x)\triangle_p(\psi_n\circ\rho_x)(y)\\
&\qquad= \sum_{z\sim y}|(\psi_n\circ\rho_x)(z)-(\psi_n\circ\rho_x)(y)|^{p-2}\left(\psi_n\circ\rho_x(z)-\psi_n\circ\rho_x(y)\right)\\
&\qquad= m^+(x)[|\psi_n(\rho_x(x)+1)-\psi_n(\rho_x(x))|^{p-2}(\psi_n(\rho_x(x)+1)-\psi_n(\rho_x(x)))]\\
&\qquad= m^+(x)[|\psi_n(1)-\psi_n(0)|^{p-2}(\psi_n(1)-\psi_n(0))]\\
&\qquad= -\frac{m^+(x)}{f_x(1)} = -m^+(x).
\end{aligned}
$$

Since $m(x)=m^+(x)$ and $f_x(1)=1$, we obtain the result. □

THEOREM 4. *Assume $G=(V,E)$ is an infinite graph such that the functions $G(x,y)$ defined in* (9) *exist. Assume further that R_x satisfies* (10). *Then for all $n=1,2,\dots$ we have*

$$G_{B_n(x)}(y,x)\geq\psi_n\circ\rho_x(y),$$

for all $y\in B_n(x)\cup\partial B_n(x)$. In particular, we have

$$G(y,x)\geq\sum_{s=\rho_x(y)+1}^{\infty}\left(1/f_x(s)\right)^{1/(p-1)}.$$

LEMMA 7. *Let u be a p-superharmonic function (i.e., $\triangle_p u \leq 0$) and v be a p-subharmonic function (i.e., $\triangle_p v \geq 0$) defined on a finite set D such that $u \geq v$ on ∂D. Then $u \geq v$ in D.*

Proof. See [5]. □

From Lemma 7, we have

$$G_{B_{n'}(x)}(x,y) \geq G_{B_n(x)}(x,y) \qquad \text{for} \quad n' \geq n.$$

In fact, putting $w = G_{B_{n'}} - G_{B_n}$, we have

$$\Delta_p w = 0 \quad \text{on} \quad B_n,$$

and for $x \in \partial B_n$, we have

$$G_{B_{n'}}(x,y) - G_{B_n}(x,y) = G_{B_{n'}}(x,y) \geq 0.$$

Thus $G_{B_{n'}} \geq G_{B_n}$ on B_n.

Proof of Theorem 4. By part (1) of Proposition 2, the function $\psi_n \circ \rho_x(y)$ is p-subharmonic. Also, $G(y,x)$ is p-harmonic by definition, and thus, in particular, p-superharmonic. Moreover, we have $G_{B_n}(y,x) \geq \psi_n \circ \rho_x(y)(= 0)$ on $\partial B_n(x)$. By Lemma 7 it follows that $G_{B_n(x)}(y,x) \geq \psi_n \circ \rho_x(y)$ on $B_n(x) \cup \partial B_n(x)$. □

We get similarly the following result.

THEOREM 5. *Assume $G = (V,E)$ is an infinite graph such that the functions $G(x,y)$ defined in (9) exist. Assume further that K_x satisfies (10). Then for all $n = 1, 2, \ldots$ we have*

$$G_{B_n(x)}(y,x) \leq \Psi_n \circ \rho_x(y),$$

for all $y \in B_n(x) \cup \partial B_n(x)$. In particular, we have

$$G(y,x) \leq \sum_{s=\rho_x(y)+1}^{\infty} \left(1/F_x(s)\right)^{1/(p-1)}.$$

We have the following corollaries; for a proof, see [8].

COROLLARY 1. *Let $G = (V,E)$ be a tree satisfying*

$$3 \leq \ell \leq m(x) \leq k \qquad \textit{for all} \quad x \in V.$$

Then $G(y,x)$ can be estimated as follows:

$$G(y,x) \geq \frac{(k-1)^{1/(p-1)}}{(k-1)^{1/(p-1)} - 1} \left(\frac{1}{k-1}\right)^{\rho_x(y)/(p-1)},$$

$$G(y,x) \leq \frac{(\ell-1)^{1/(p-1)}}{(\ell-1)^{1/(p-1)} - 1} \left(\frac{1}{\ell-1}\right)^{\rho_x(y)/(p-1)}.$$

COROLLARY 2. *Let (V, E) be the homogeneous regular tree T_d $(d \geq 3)$. Then we have*

$$G(y, x) = \frac{(d-1)^{1/(p-1)}}{(d-1)^{1/(p-1)} - 1} \left(\frac{1}{d-1} \right)^{\rho_x(y)/(p-1)}.$$

REFERENCES

[1] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), 787–794.

[2] J. Dodziuk and L. Karp, *Spectral and function theory for combinatorial Laplacian*, Geometry of random motion (Ithaca, NY, 1987), Contemp. Math., vol. 73, Amer. Math. Soc., Providence, RI, 1988, pp. 25–40.

[3] J. Dodziuk and W.S. Kendall, *Combinatorial Laplacians and isoperimetric inequality*, From local times to global geometry, control and physics (Elworthy K. D. ed.), Pitman Reseach Notes in Math., vol. 150, Longman Sci. & Technical, Harlow, 1986, pp. 68–74.

[4] K. Fujiwara, *Growth and the spectrum of the Laplacian of an infinite graph*, Tôhoku Math. J. **48** (1996), 293–302.

[5] I. Holopainen and P. Soardi, *P–harmonic functions on graphs and manifolds*, Manuscripta Math. **94** (1997), 95–110.

[6] Y. Ohno and H. Urakawa, *On the first eigenvalue of the combinatorial Laplacian for a graph*, Interdisp. Inform. Sci. **1** (1994), 33–46.

[7] H. Urakawa, *Eigenvalue comparison theorems of the discrete Laplacians for a graph*, Geom. Dedicata **74** (1999), 95–112.

[8] ———, *Heat kernel and Green kernel comparison theorem for infinite graphs*, J. Funct. Anal. **146** (1997), 206–235.

[9] ———, *A discrete analogue of harmonic morphism*, Harmonic morphisms, harmonic maps, and related topics (Anand, C.K., Baird, P., Loubeau, E., Wood, J.C. ed.), Pitman Reseach Notes in Math., vol. 413, Longman Sci. & Technical, Harlow, 2000, pp. 97–108.

[10] ———, *A discrete analogue of the harmonic morphism and Green kernel comparison theorem*, Glasgow Math. J. **42** (2000), 319–334.

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