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ON THE CAUCHY PROBLEM FOR AN INTEGRABLE EQUATION WITH PEAKON SOLUTIONS

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ABSTRACT. We establish the local well-posedness for a new integrable equation. We prove that the equation has strong solutions that blow up in finite time and obtain the precise blow-up scenario for this equation. Moreover, we provide a framework of weak solutions for the study of soliton interaction.

1. Introduction

Recently, Degasperis and Procesi [14] studied the following family of third order dispersive PDE conservation laws, whose right-hand side is the derivative of a quadratic differential polynomial:

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.1}$$

where α, c_0, c_1, c_2, and c_3 are real constants. Applying the method of asymptotic integrability to the family (1.1), they found that there are only three equations that satisfy asymptotic integrability conditions within this family: the KdV equation, the Camassa-Holm equation, and one new equation.

With $\alpha = c_2 = c_3 = 0$, (1.1) becomes the well-known Korteweg-de Vries equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. Here $u(t,x)$ represents the wave height above a flat bottom, x is proportional to the distance in the direction of the propagation, and t is proportional to the elapsed time. The Cauchy problem of the KdV equation has been studied extensively (see [21], [22], [23], [24]). As soon as $u_0 \in H^1(\mathbb{R})$, the solutions of the KdV equation are global (cf. [24]). The equation is integrable (cf. [26]) and its solitary waves are solitons (cf. [15]).

For $c_1 = -(3/2)c_3/\alpha^2$ and $c_2 = c_3/2$, (1.1) becomes the Camassa-Holm equation, which models the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t,x)$ stands for the fluid velocity at time $t \geq 0$ in the spatial x direction. See [4] for the original derivation and [16] and [19]

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for new derivations of this equation. The Camassa-Holm equation has a bi-Hamiltonian structure and is completely integrable (see [4], [2], [6], [8]). Its solitary waves are smooth if $c_0 > 0$ and peaked in the limiting case $c_0 = 0$ (cf. [5]). They are orbitally stable and interact like solitons (see [4], [3], [11], [12], [20]). The equation has global strong solutions (cf. [7], [9]) and also solutions which blow up in finite time (cf. [7], [9], [10]). Families of integrable equations similar to the Camassa-Holm equation have long been known to be derivable from the theory of hereditary symmetries (see [17]). However, before [4], the Camassa-Holm equation had not been stated explicitly, nor had it been derived physically as a shallow water wave equation.

Taking $c_1 = -2c_3/\alpha^2$ and $c_2 = c_3$ in (1.1), rescaling, shifting the dependent variable and applying a Galilean boost (cf. [13]), we obtain the new equation

$$\begin{cases} u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

which has a form similar to the Camassa-Holm shallow water wave equation. The integrability of (1.2) was proved by Degasperis, Holm and Hone [13] by constructing a Lax pair. In the same paper [13], the bi-Hamiltonian structure and an infinite sequence of conserved quantities were obtained, and it was shown that the equation admits exact peakon solutions that are analogous to the Camassa-Holm peakons.

On the other hand, the Cauchy problem of (1.2) does not seem to have been discussed in the literature. The aim of this paper is to prove the local well-posedness of strong solutions to (1.2) for a large class of initial data, and to get a blow-up criterion for strong solutions to (1.2).

Main Result.

(a) Local well-posedness:

(i) *Given $u_0 \in H^s(\mathbb{R}),\ s > 3/2$, there exists a maximal $T = T(u_0) > 0$, and a unique solution u to* (1.2)*, such that*

$$u = u(.,u_0) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); L^2(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(.,u_0) : H^s(\mathbb{R}) \to C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); L^2(\mathbb{R}))$$

is continuous.

(ii) *T may be chosen independent of s in the following sense: If*

$$u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$$

and if $u_0 \in H^{s'}(\mathbb{R})$ for some $s' \neq s,\ s' > 3/2$, then

$$u \in C([0,T); H^{s'}(\mathbb{R})) \cap C^1([0,T); H^{s'-1}(\mathbb{R}))$$

with the same value of T.

(b) Blow-up:

(i) *Given* $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, *blow-up of the solution* $u{=}u(.,u_0)$ *in finite time* $T < +\infty$ *occurs if and only if*

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{R}}[u_x(t,x)]\} = -\infty.$$

(ii) *Assume that* $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, *is odd, and* $u_0' < 0$. *Then the corresponding strong solution to* (1.2) *blows up in finite time. The maximal time of existence is bounded above by* $-1/u_0'$.

By applying Kato's semigroup approach we will prove local well-posedness for (1.2), analogous to similar results for the Camassa-Holm equation (see [9], [18], [25], [28]).

We will also give an explosion criterion for (1.2) with odd initial data. The precise blow-up scenario of (b) for (1.2) is better than the blow-up scenario $\limsup_{t\to T}\|u_x\|_{L^\infty} = +\infty$, which is quite common for nonlinear hyperbolic PDE's (see [1], [29], and [30]).

Degasperis, Holm and Hone [13] also showed that there exist peakon solutions that interact like solitons. In Section 4, we introduce the notion of weak solutions to (1.2) as a suitable mathematical framework for the study of soliton interactions.

Our paper is organized as follows. In Section 2, we prove the local well-posedness of the initial value problem associated with (1.2). In Section 3, we obtain the precise blow-up scenario and give an explosion criterion for strong solutions to (1.2) with odd initial data. In Section 4 we introduce and study the notion of weak solutions and soliton interactions of (1.2). Section 5 is an appendix in which we collect some results from the literature that we shall need.

2. Local well-posedness

In this section, we apply Kato's theory to establish local well-posedness for the Cauchy problem of (1.2).

For convenience, we state here Kato's theorem in a form suitable for our purpose. Consider the abstract quasi-linear evolution equation

$$\frac{dv}{dt} + A(v)v = f(v), \qquad t \geq 0, \quad v(0) = v_0. \tag{2.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y,X)$ denote the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. We make the following assumptions, where μ_1, μ_2, μ_3, and μ_4 are constants depending only on $\max\{\|y\|_Y,\ \|z\|_Y\}$:

(i) $A(y) \in L(Y,X)$ for $y \in X$ with

$$\|(A(y) - A(z))w\|_X \leq \mu_1\|y - z\|_X\|w\|_Y, \qquad y,\ z,\ w \in Y,$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y) - B(z))w\|_X \leq \mu_2 \|y - z\|_Y \|w\|_X, \qquad y,\, z \in Y,\, w \in X.$$

(iii) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\|f(y) - f(z)\|_Y \leq \mu_3 \|y - z\|_Y, \qquad y,\, z \in Y,$$

$$\|f(y) - f(z)\|_X \leq \mu_4 \|y - z\|_X, \qquad y,\, z \in Y.$$

THEOREM 2.1 (Kato, [21]). *Assume that* (i)*,* (ii)*, and* (iii) *hold. Given* $v_0 \in Y$*, there is a maximal* $T > 0$ *depending only on* $\|v_0\|_Y$*, and a unique solution* v *to* (2.1) *such that*

$$v = v(., v_0) \in C([0, T); Y) \cap C^1([0, T); X).$$

Moreover, the map $v_0 \mapsto v(., v_0)$ *is a continuous map from* Y *to* $C([0, T); Y) \cap C^1([0, T); X)$.

We now provide a framework in which we shall reformulate the problem (1.2). We begin by fixing some notations. All spaces of functions are assumed to be over $\mathbb{R}$; for simplicity, we drop $\mathbb{R}$ in our notation for function spaces if there is no ambiguity. If A is an unbounded operator, we denote by $D(A)$ the domain of A. $[A, B]$ denotes the commutator of two linear operators A and B. $\|.\|_X$ denotes the norm of Banach space X. We denote the norm and the inner product of H^s, $s \in \mathbb{R}_+$, by $\|.\|_s$ and $(.,.)_s$, respectively.

Next, because of the bi-Hamiltonian structure of (1.2) (see [13]), in analogy to the case of the Camassa-Holm equation [4], (1.2) can be written in Hamiltonian form and has the invariant

$$E(u) = -\frac{1}{6} \int_{\mathbb{R}} u^3 dx. \tag{2.2}$$

With $y := u - u_{xx}$, (1.2) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} y_t + u y_x + 3 u_x y = 0, & t > 0,\, x \in \mathbb{R}, \\ y(0, x) = u_0(x) - \partial_x^2 u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

We also note that if $p(x) := (1/2) e^{-|x|}$, $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2$, and $p * y = u$, where $*$ denotes convolution. Using this identity, we can rewrite (2.3) as

$$\begin{cases} u_t + u u_x = -\partial_x p * (\frac{3}{2} u^2), & t > 0,\, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.4}$$

or, equivalently, as

$$
\text{(2.5)} \qquad \begin{cases} u_t + uu_x = -\partial_x(1-\partial_x^2)^{-1}(\frac{3}{2}u^2), & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases}
$$

THEOREM 2.2. *Given $u_0 \in H^s,\ s > 3/2$, there exists a maximal value $T = T(u_0) > 0$, and a unique solution u to* (1.2) (*or* (2.5)), *such that*

$$
u = u(., u_0) \in C([0,T); H^s) \cap C^1([0,T); L^2).
$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$
u_0 \to u(., u_0) : H^s \to C([0,T); H^s) \cap C^1([0,T); L^2)
$$

is continuous.

To prove this result, we will apply Theorem 2.1 with $A(u) = u\partial_x$, $f(u) = -\partial_x(1-\partial_x^2)^{-1}(\frac{3}{2}u^2)$, $Y = H^s$, $X = L^2$, $\Lambda = (1-\partial_x^2)^{1/2}$, and $Q = \Lambda^s$. Obviously, Q is an isomorphism of H^s onto L^2. Thus, in order to derive Theorem 2.2 from Theorem 2.1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions (i)–(iii).

We break the argument into several lemmas.

LEMMA 2.1. *The operator $A(u) = u\partial_x$, with $u \in H^s,\ s > 3/2$, belongs to $G(L^2, 1, \beta)$.*

Proof. Since L^2 is a Hilbert space, we have $A(u) \in G(L^2, 1, \beta)$ for some real number β if and only if the following conditions hold (cf. [23]):

(a) $(A(u)y, y)_0 \geq -\beta\|y\|_0^2$.
(b) The range of $A + \lambda$ is all of X, for some (or all) $\lambda > \beta$.

We first prove (a). Since $u \in H^s,\ s > 3/2$, u and u_x belong to L^∞. Note that $\|u_x\|_{L^\infty} \leq \|u\|_s$. Thus

$$
\begin{aligned}
(A(u)y, y)_0 = (u\partial_x y, y)_0 &= -\frac{1}{2}(u_x y, y)_0 \\
&\leq \frac{1}{2}\|u_x\|_{L^\infty}\|y\|_0^2 \leq c\|u\|_s\|y\|_0^2.
\end{aligned}
$$

Setting $\beta = c\|u\|_s$, we obtain $(A(u)y, y)_0 \geq -\beta\|y\|_0^2$.

Next, we prove (b). Because $A(u)$ is a closed operator and satisfies (a), $(\lambda I + A)$ has closed range in L^2 for all $\lambda > \beta$. Therefore, it suffices to show that $(\lambda I + A)$ has dense range in L^2 for all $\lambda > \beta$.

Given $u \in H^s,\ s > 3/2$, and $y \in L^2$, we have the generalized Leibnitz formula

$$
\partial_x(uy) = u_x y + u\partial_x y \quad \text{in } H^{-1}.
$$

Since $u_x \in L^\infty$, we have

$$\begin{aligned} D(A) &= D(u\partial_x) = \{y \in L^2,\ u\partial_x y \in L^2\} \\ &= \{z \in L^2, -\partial_x(uz) \in L^2\} = D((u\partial_x)^*) = D(A^*). \end{aligned}$$

Assume that the range of $(A+\lambda)$ is not all of L^2. Then there exists $z \in L^2,\ z \neq 0$ such that $((\lambda I + A)y, z)_0 = 0$ for all $y \in D(A)$. Since $H^1 \subset D(A)$, $D(A)$ is dense in L^2. Hence it follows that $z \in D(A^*)$ and $\lambda z + A^* = 0$ in L^2. Since $D(A) = D(A^*)$, multiplying by z and integrating by parts, we obtain

$$0 = ((\lambda I + A^*)z, z)_0 = (\lambda z, z) + (z, Az) \geq (\lambda - \beta)\|z\|_0^2, \quad \forall\, \lambda > \beta,$$

and thus $z = 0$, which contradicts our assumption $z \neq 0$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $A(u) = u\partial_x$ with $u \in H^s,\ s > 3/2$. Then $A(u) \in L(H^s, L^2)$ for all $u \in H^s$. Moreover,*

$$\|(A(u) - A(z))w\|_0 \leq \mu_1 \|u - z\|_0 \|w\|_s, \qquad u, z, w \in H^s.$$

Proof. Let $u,\ z,\ w \in H^s$, $s > 3/2$. Then we have

$$\begin{aligned} \|(A(u) - A(z))w\|_0 &\leq c\|u - z\|_0 \|\partial_x w\|_{L^\infty} \\ &\leq \mu_1 \|u - z\|_0 \|w\|_s. \end{aligned}$$

Taking $z = 0$ in the above inequality, we obtain $A(u) \in L(H^s, L^2)$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *We have $B(u) = [\Lambda^s,\ u\partial_x]\Lambda^{-s} \in L(L^2)$, for $u \in H^s$. Moveover,*

$$\|(B(u) - B(z))w\|_0 \leq \mu_2 \|u - z\|_s \|w\|_0.$$

Proof. Let $u, z \in H^s,\ s > 3/2$, and $w \in L^2$. Then

$$\begin{aligned} \|(B(u) - B(z))w\|_0 &= \|[\Lambda^s, (u - v)\partial_x]\Lambda^{-s} w\|_0 \\ &\leq \|[\Lambda^s, (u - v)]\Lambda^{1-s}\|_{L(L^2)} \|\Lambda^{-1}\partial_x w\|_0 \\ &\leq \mu_2 \|y - z\|_s \|w\|_0, \end{aligned}$$

where we applied Lemma 5.1 (see Section 5) with $r = 0,\ t = s - 1$. Taking $z = 0$ in the above inequality, we obtain $B(u) \in L(L^2)$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $f(u) = -\partial_x(1 - \partial_x^2)^{-1}(\frac{3}{2}u^2)$. Then f is bounded on bounded sets in H^s, and satisfies*

(a) $\|f(y) - f(z)\|_s \leq \mu_3 \|y - z\|_s, \quad y, z \in H^s,$
(b) $\|f(y) - f(z)\|_0 \leq \mu_4 \|y - z\|_0, \quad y, z \in H^s.$

Proof. Let $y, z \in H^s$, $s > 3/2$, and note that H^{s-1} is a Banach algebra. Then we have

$$\begin{aligned}\|f(y) - f(z)\|_s &\le \frac{3}{2}\|(y-z)(y+z)\|_{s-1}\\ &\le \frac{3}{2}\|y-z\|_{s-1}\|y+z\|_{s-1}\\ &\le \frac{3}{2}(\|y\|_s + \|z\|_s)\|y-z\|_s.\end{aligned}$$

This proves (a).

Taking $z = 0$ in the above inequality, we obtain that f is bounded on bounded sets in H^s.

Next, we prove (b). Let y, $z \in H^s$, $s > 3/2$, and note that $\|y\|_{L^\infty} \le \|y\|_s$. Then we have

$$\begin{aligned}\|f(y) - f(z)\|_0 &= \| - \partial_x(1-\alpha^2\partial_x^2)^{-1}\frac{3}{2}(y^2 - z^2)\|_0\\ &\le \frac{3}{2}\|(y-z)(y+z)\|_0\\ &\le \frac{3}{2}\|y-z\|_0\|y+z\|_{L^\infty}\\ &\le \frac{3}{2}(\|y\|_s + \|z\|_s)\|y-z\|_0,\end{aligned}$$

which proves (b). This completes the proof of Lemma 2.4. □

Proof of Theorem 2.2. The result follows by combining Theorem 2.1 and Lemmas 2.1–2.4. □

Remark 2.1. In Theorem 2.2, the function u actually is an element of $C([0,T); H^s) \cap C^1([0,T); H^{s-1})$ because (2.5) implies that $du/dt \in H^{s-1}$.

Theorem 2.3. *The maximal T in Theorem 2.2 may be chosen independent of s in the following sense. If*

$$u = u(., u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$$

is a solution to (1.2) (*or* (2.5)), *and if $u_0 \in H^{s'}$ for some $s' \neq s$, $s' > 3/2$, then*

$$u \in C([0,T); H^{s'}) \cap C^1([0,T); H^{s'-1}),$$

with the same value of T. In particular, if $u_0 \in H^\infty = \bigcap_{x\ge 0} H^s$, then $u \in C([0,T); H^\infty)$.

Proof. If $s' < s$, the result follows the uniqueness of the solution guaranteed by Theorem 2.2, so it suffices to consider the case $s' > s$. To this end, we return to (2.3). Setting $y(t) = \Lambda^2 u(t)$, we have

$$\frac{dy}{dt} + A(t)y + B(t)y = 0, \qquad y(0) = \Lambda^2 u(0), \tag{2.6}$$

where $A(t)y = \partial_x(uy)$ and $B(t)y = 2u_x y$.

Because $u \in C([0,T); H^s)$ and $u_0 \in H^{s'}$, we have $y \in C([0,T); H^{s-2})$ and $y(0) = (1-\partial_x^2)u(0) \in C([0,T); H^{s'})$. We will show that $y \in C([0,T); H^{s'-2})$, which implies $u \in C([0,T); H^{s'})$ since $(1-\partial_x^2)$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$. This will complete the proof of Theorem 2.3.

Since $u \in C([0,T); H^s)$, $u_x \in H^{s-1}$, and H^{s-1} is a Banach algebra, we have $B(t) \in L(H^{s-1})$.

Following the arguments in Lemmas 3.1–3.3 in [22], we first prove that the family $A(t)$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^h$ and $Y = H^k$, where $-s \le h \le s-2$, $1-s \le k \le s-1$, and $k \ge h+1$. To this end, as in the proof of Lemma 3.1 in [22], we need to verify the following three conditions:

(i) $A(t) \in G(H^h, 1, \beta)$ for all $y \in H^s$.
(ii) $\Lambda^h \partial_x [\Lambda^{k-h}, u]\Lambda^{-k}$ is uniformly bounded on L^2.
(iii) $A(t) \in L(H^k, H^h)$ is strongly continuous in t.

Let us first show (i). Since H^h is a Hilbert space, we have $A(t) \in G(H^h, 1, \beta)$ for some real number β if and only if the following conditions hold (cf. [23]):

(a) $(A(t)y, y)_h \ge -\beta \|y\|_h^2$.
(b) $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h, for some (or all) $\lambda > \beta$.

To prove (a), take $y \in H^h$ and note that

$$\begin{aligned}\Lambda^h \partial_x(uy) &= \Lambda^h \partial_x(-[\Lambda^{-h}, u]\Lambda^h y + \Lambda^{-h}(u\Lambda^h y))\\ &= -\Lambda^h \partial_x [\Lambda^{-h}, u]\Lambda^h y + \partial_x(u\Lambda^h y).\end{aligned}$$

Thus

$$\begin{aligned}(A(t)y, y)_h &= (-\Lambda^h \partial_x[\Lambda^{-h}, u]\Lambda^h y + \partial_x(u\Lambda^h y),\ \Lambda^h y)_0\\ &= (\Lambda^{h+1}[\Lambda^{-h}, u]\Lambda^h y, \partial_x \Lambda^{h-1} y)_0 + \frac{1}{2}(u_x \Lambda^h y, \Lambda^h y)_0\\ &\le \|\Lambda^{h+1}[\Lambda^{-h}, u]\|_{L(L^2)} \|\Lambda^h y\|_0^2 + \frac{1}{2}\|u_x\|_{L^\infty} \|\Lambda^h y\|_0^2\\ &\le c\|u\|_s \|y\|_h^2,\end{aligned}$$

where we have used Lemma 5.1 with $r = -(h+1)$ and $t = 0$. Setting $\beta = c\|u\|_s$, we obtain $(A(t)y, y)_h \ge -\beta\|y\|_h^2$, as claimed.

Next, we prove (b). Let $S = \Lambda^{s-1-h}$, and note that S is an isomorphism of H^{s-1} onto H^h and that H^{s-1} is continuously and densely embedded in H^h as $-s \le h \le s-2$. Define

$$\begin{aligned}A_1(t) &:= SA(t)S^{-1} = \Lambda^{s-1-h} A(t) \Lambda^{h+1-s},\\ B_1(t) &:= A_1(t) - A(t) = [S, A(t)]S^{-1}.\end{aligned}$$

Let $y \in H^h$ and $u \in H^s,\ s > 3/2$. Then

$$\begin{aligned}\|B_1(t)y\|_h &= \|\Lambda^h \partial_x[\Lambda^{s-1-h}, u]\Lambda^{h+1-s}y\|_0 \\ &\le \|\Lambda^h \partial_x[\Lambda^{s-1-h}, u]\Lambda^{1-s}\|_{L(L^2)}\|\Lambda^h y\|_0 \\ &\le c\|u\|_s\|y\|_h,\end{aligned}$$

on applying Lemma 5.1 with $r = -(h+1),\ t = s-1$. Therefore, we have $B_1(t) \in L(H^h)$. Since

$$A(t)y = \partial_x(uy) = u_x y + u\partial_x y \quad \text{and} \quad u_x \in L(H^{s-1}),$$

by applying Lemma 5.4 and a perturbation theorem for semigroups, we see that H^{s-1} is $A(t)$-admissible. Further, applying Lemma 5.3 with $Y = H^{s-1}$, $X = H^h$ and $S = \Lambda^{s-1-h}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. Since $A_1(t) = A(t)+B_1(t)$ and $B_1(t) \in L(H^h)$, by a perturbation theorem for semigroups it follows that $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. This proves (b).

Next, we verify (ii). For $y \in L^2$ we have

$$\|\Lambda^h \partial_x[\Lambda^{k-h}, u]\Lambda^{-k}y\|_0 \le c\|u\|_s\|y\|_0,$$

by Lemma 5.1 with $r = -(h+1)$, $t = k$. This proves (ii).

Finally, we verify (iii). Take $y \in H^k$. Then

$$\begin{aligned}\|(A(t+\tau) - A(t))y\|_h &= \|\partial_x((u(t+\tau) - u(t))y)\|_h \\ &\le \|(u(t+\tau) - u(t))y\|_{h+1} \\ &\le c\|u(t+\tau) - u(t)\|_{s-1}\|y\|_{h+1} \\ &\le c\|u(t+\tau) - u(t)\|_s\|y\|_k,\end{aligned}$$

by Lemma 5.1 with $r = s-1$, $t = h+1$. The continuity of u now yields (iii).

Thus, the above conditions (i)–(iii) imply the existence and uniqueness of an evolution operator $U(t,\tau)$ for the family $A(t)$. In particular, for $-s \le r \le s-1$, $U(t,\tau)$ maps H^r into itself.

Next, take $Y = H^{s-2}$, $X = H^{s-3}$, and note that

$$y \in C([0,T); H^{s-1}) \cap C^1([0,T); H^{s-2}).$$

Using the properties of the evolution operator $U(t,\tau)$, we obtain

$$\frac{d}{d\tau}(U(t,\tau)y(\tau)) = U(t,\tau)(-B(\tau)y(\tau)).$$

An integration over $\tau \in [0,t]$ gives

$$y(t) = U(t,0)y(0) - \int_0^t U(t,\tau)B(\tau)y(\tau)d\tau. \tag{2.7}$$

If $s < s' \le s+1$, then $B(t) = u_x(t) \in L(H^{s'-2})$ is strongly continuous on $[0,t)$, and $H^{s-1}H^{s'-2} \subset H^{s'-2}$ since $s-1 > 1/2$. Since $-s < s-2 < s'-2 \le s-1$, the family $\{U(t,\tau)\}$ is a strongly continuous map from the space $H^{s'-2}$ into itself. Noting that $y(0) \in H^{s'-2}$, and regarding (2.7) as an integral equation of Volterra type that can be solved for y by successive approximation, we then obtain the assertion of Theorem 2.3 for the case $s < s' \leq s+1$.

In the case $s' > s+1$, the result follows by a repeated application of the above argument. This completes the proof of Theorem 2.3. □

3. Blow-up

In this section we address the question of the formation of singularities for solutions to (1.2). We first derive the precise blow-up scenario for solutions to (1.2), and then show that there are smooth initial data for which the corresponding solution to (1.2) does not exist globally in time.

THEOREM 3.1. *Given $u_0 \in H^s$, $s > 3/2$, blow-up of the solution $u=u(.,u_0)$ in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{R}}[u_x(t,x)]\} = -\infty.$$

Proof. Let us first assume that $u_0 \in H^s$, for some $s \in N,\ s \geq 4$. Multiplying (2.3) by $y = u - u_{xx}$ and integrating by parts, we get

$$\frac{d}{dt}\int_{\mathbb{R}} y^2 dx = -3\int_{\mathbb{R}} y^2 u_x dx - \int_{\mathbb{R}} uyy_x dx = -\frac{5}{2}\int_{\mathbb{R}} y^2 u_x dx. \tag{3.1}$$

Next, we differentiate (2.3) with respect to the spatial variable x, and then multiply this equation by y_x. Using the identity $y = u - u_{xx}$ and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} y_x^2 dx &= -\int_{\mathbb{R}} y_x y_{xx} u\, dx - 4\int_{\mathbb{R}} y_x^2 u_x dx - 3\int_{\mathbb{R}} yy_x u_{xx} dx \\ &= -\frac{7}{2}\int_{\mathbb{R}} y_x^2 u_x dx + \frac{3}{2}\int_{\mathbb{R}} y^2 u_x dx.\end{aligned} \tag{3.2}$$

Adding the equations (3.1) and (3.2), we obtain

$$\frac{d}{dt}\left(\int_{\mathbb{R}} y^2 dx + \int_{\mathbb{R}} y_x^2 dx\right) = -\frac{7}{2}\int_{\mathbb{R}} y_x^2 u_x dx - \int_{\mathbb{R}} y^2 u_x dx. \tag{3.3}$$

If u_x is bounded from below on $[0,T)\times\mathbb{R}$, i.e., if there exists $M > 0$ such that

$$-u_x(t,x) \leq M \quad \text{on } [0,T)\times\mathbb{R},$$

(3.3) implies

$$\frac{d}{dt}\left(\int_{\mathbb{R}} y^2 dx + \int_{\mathbb{R}} y_x^2 dx\right) \leq \frac{7}{2}M\left(\int_{\mathbb{R}} y^2 dx + \int_{\mathbb{R}} y_x^2 dx\right). \tag{3.4}$$

Using Gronwall's inequality, we see that the H^3 norm of the solution to (1.2) does not blow up in finite time. Moreover, an analogous inductive argument shows that the H^k-norm of this solution does not blow up in finite time for all $k \in N$, $4 \leq k \leq s$. Applying Theorem 2.2 and Theorem 2.3, we obtain the assertion of Theorem 3.1 for all $s > 3/2$. $\square$

THEOREM 3.2. *Assume that $u_0 \in H^s$, $s > 3/2$, is odd, and $u_0' < 0$. Then the corresponding strong solution to* (1.2) *blows up in finite time. The maximal time of existence is bounded above by $-1/u_0'$.*

Proof. Let T be the maximal time of existence of the solution

$$u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}),$$

of (1.2) (or (2.5)). The existence of T is guaranteed by Theorem 2.2 and Remark 2.1. Note that, because of the symmetry $(u,x) \to (-u,-x)$ of (1.2) (or (2.5)), if $u_0(x)$ is odd, then $u(t,x)$ is odd for any $t \in [0,T)$. In particular, if $s \geq 3$ and the functions u and u_{xx} are continuous in x, we have

$$u(t,0) = u_{xx}(t,0) = 0, \qquad t \in [0,T). \tag{3.5}$$

Differentiating (2.5) with respect to x, and noting that $\partial_x^2 p * f = p * f - f$, we obtain

$$u_{tx} = -u_x^2 - uu_{xx} + \frac{3}{2}u^2 - p * \left(\frac{3}{2}u^2\right). \tag{3.6}$$

Setting $h(t) = u_x(t,0)$ for $t \in [0,T)$, noting that $p * (\frac{3}{2}u^2) \geq 0$, and using (3.5), we get

$$\frac{dh}{dt}(t) \leq -h^2(t), \qquad t \in [0,T). \tag{3.7}$$

It follows that

$$0 > \frac{1}{h(t)} \geq \frac{1}{h(0)} + t, \qquad t \in [0,T), \tag{3.8}$$

and hence $T \leq -1/h(0)$. This proves the assertion of the theorem for the case $s \geq 3$.

To complete the proof we use a simple density argument. Let $u_0 \in H^s$ with $s > 3/2$. Set $u_0^n = e^{(1/n)(1-\partial_x^2)}u_0$. If $u_0(x)$ is odd, then $u_0^n(x)$ is also odd. Since $u_0^n(x) \in H^3$, the above argument yields

$$T(u_0^n) \leq -\frac{1}{h^n(0)},$$

where $h^n(0) = e^{(1/n)(1-\partial_x^2)}u_0'$. Letting $n \to \infty$, and applying Theorem 2.3, we conclude that $T(u_0) \leq -1/h(0)$. This completes the proof of Theorem 3.2. $\square$

REMARK 3.1. Because (1.2) is invariant with respect to the transformation $(x,t) \mapsto (x+k,t)$, where $k \in \mathbb{R}$, Theorem 3.2 remains true for all initial data $u_0 \in H^s$, $s > 3/2$, which are point-symmetric.

Corollary 3.2. *The only equilibrium point of* (1.2) (*or* (2.4)) *in* H^s, $s > 3/2$, *is the point* 0. *It is unstable.*

Proof. Note that (2.4) can be written in the form

$$u_t = -\left(\frac{u^2}{2} + p * \left(\frac{3}{2}u^2\right)\right)_x. \tag{3.9}$$

Since $H^s \in C_0^1(\mathbb{R})$, $s > 3/2$, an equilibrium solution $u \in H^s$, $s > 3/2$, satisfies

$$\frac{u^2}{2} + p * \left(\frac{3}{2}u^2\right) = 0, \qquad x \in \mathbb{R}.$$

Since $p * (\frac{3}{2}u^2) \geq 0$, it follows that $u \equiv 0$. Because in every neighborhood of 0 in H^s, $s > 3/2$, there are odd functions u_0 with $u_0'(0) < 0$, we can apply Theorem 3.2 to these functions, and then obtain the result of Corollary 3.2. □

4. Weak solutions and soliton interactions

In this section we define strong solutions and weak solutions for (1.2). We also prove that there are no traveling waves for (1.2) which are strong solutions, and that its peakon solutions are weak solutions.

Definition 4.1. If

$$u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$$

with $s > 3/2$ is a solution to (2.4), then $u(t,x)$ is called a strong solution to (2.4) (or (1.2)).

Note that (1.2) has the soliton interaction property of solitary waves with corners at their peaks, discovered in [13] and [14]. Obviously, such solutions are not strong solutions to (2.4). In order to provide a mathematical framework for the study of soliton interaction, we define the notion of weak solutions to (2.4).

Let us return to (3.9). Setting

$$F(u) = \left(\frac{u^2}{2} + p * \left(\frac{3}{2}u^2\right)\right),$$

(2.4) can be rewritten as the conservation law

$$u_t + F(u)_x = 0, \qquad u(0,x) = u_0. \tag{4.1}$$

Definition 4.2. Let $u_0 \in H^\alpha$, $\alpha \in [1, 3/2]$. If u belongs to the space $L^\infty_{\mathrm{loc}}([0,T); H^\alpha)$ and satisfies

$$\int_0^T \int_{\mathbb{R}} (u\psi_t + F(u)\psi_x)dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0$$

for all functions $\psi \in C_0^\infty([0,T)\times\mathbb{R})$ that are restrictions to $[0,T)\times\mathbb{R}$ of a function having continuous derivatives of arbitrary positive integer order on $\mathbb{R}^2$ with compact support contained in $(-T,T)\times\mathbb{R}$, then u is called a weak solution to (2.4). If u is a weak solution on $[0,T)$ for every $T>0$, it is called a global weak solution to (2.4) (or (1.2)).

Proposition 4.1.

(i) *Every strong solution is a weak solution.*

(ii) *If u is a weak solution and*

$$u \in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$$

with $s>3/2$, then it is a strong solution.

(iii) *All nontrivial traveling wave solutions of* (1.2) *are not strong solutions.*

(iv) *There exist peakon solitons of* (1.2) *which are weak solutions.*

Proof. (i) Let

$$u \in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$$

with $s>3/2$ be a strong solution to (2.4). Obviously, $u\in L^\infty_{\mathrm{loc}}([0,T);H^\alpha)$ and $u_0\in H^\alpha$ with $\alpha\in[1,3/2]$, and $F(u)\in H^s,\ s>3/2$. Therefore, the equation $u_t+F(u)_x=0$ holds in $C([0,T);H^{s-1})$ with $s>3/2$. Integrating by parts in $C([0,T);L^2)$, we obtain that u is a weak solution of (2.4).

(ii) Let

$$u \in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$$

with $s>3/2$, be a weak solution. Then $u_0\in H^s,\ s>3/2$. By Theorem 2.2 there exists a unique strong solution v with initial data u_0, and by (i), v is also a weak solution to (2.4). Thus we have

$$\int_0^T\int_{\mathbb{R}}(u\psi_t+F(u)\psi_x)dxdt=\int_0^T\int_{\mathbb{R}}(v\psi_t+F(v)\psi_x)dxdt.$$

Integration by parts yields

$$\int_0^T\int_{\mathbb{R}}(u_t+F(u)_x)\psi dxdt=\int_0^T\int_{\mathbb{R}}(v_t+F(v)_x)\psi dxdt.$$

Because $C_0^\infty([0,T)\times\mathbb{R})$ is dense in $L^2([0,T)\times\mathbb{R})$, it follows that

$$u_t+F(u)_x=v_t+F(v)_x \quad \text{in } L^2([0,T)\times\mathbb{R}). \tag{4.2}$$

Since

$$u \in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$$

with $s>3/2$, (4.2) holds in $C([0,T);H^{s-1})$ with $s>3/2$. Since v is a strong solution of (2.4), we have

$$u_t+F(u)_x=0 \quad \text{in } C([0,T);H^{s-1})$$

Hence u is a strong solution to (2.4).

(iii) Assume first that there exists a nontrivial traveling wave $u(t,x) = \varphi_c(x-ct) \in H^3$, which is a strong solution to (1.2). Then

$$-c\varphi_c' + c\varphi_c^{'''} + 4\varphi_c'\varphi_c - 3\varphi_c'\varphi_c^{''} - \varphi_c\varphi_c^{'''} = 0 \quad \text{in } L^2. \tag{4.3}$$

It follows that

$$(-c\varphi_c + c\varphi_c^{''} + 2\varphi_c^2 - (\varphi_c')^2 - \varphi_c\varphi_c^{''})' = 0 \quad \text{in } L^2. \tag{4.4}$$

Therefore, we have

$$((\varphi_c - c)^2)^{''} - 4\varphi_c^2 + 2c\varphi_c = 0 \quad \text{in } H^1. \tag{4.5}$$

Now note that $H^3 \subset C_0^2$ and that the solutions of (4.5) in C^2, except for the trivial solutions $\varphi_c = 0$ and $\varphi_c = c/2$, are all unbounded. In particular, taking $c = 0$ and solving (4.5) gives

$$u^2 = k_1 e^{-2x} + k_2 e^{2x},$$

where k_1, k_2 are two arbitrary nonnegative constants. Since $u \in C_0^2$, this is impossible.

If $u(t,x) = \varphi_c(x-ct) \in H^s$ with $s > 3/2$, then applying a simple density argument and using the continuous dependence on initial data guaranteed by Theorem 2.2, we obtain that there exists no nontrivial traveling wave solution to (1.2) that is a strong solution to this equation.

(iv) Take the initial data $u_0(x) = ce^{-|x|}$. By computation one can check that the traveling wave $u(x,t) = ce^{-|x-ct|}$ is a global weak solution to (2.4) for any $c \in \mathbb{R}$. In addition, the functions $u(x,t) = ce^{-|x-ct|}$ are peakon solitons (see [13]). $\square$

Next, we use the framework of weak solution to describe the soliton interaction for (1.2), as presented in the work of Degasperis, Holm and Hone [13]. Multi-peakon solutions of the (1.2) take the form [13]

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{4.6}$$

where N is the number of peakons and $p_j, q_j \in W^{1,\infty}(\mathbb{R})$, $j = 1, \ldots, N$.

Regarding a solution $u(x,t)$ of (4.6) as a global weak solution of (1.2), applying Definition 2.4, then fixing $x \in \mathbb{R}$ and splitting the spatial integral of $F(u)$ over $\mathbb{R}$ according to the order of magnitude of x, p_j, and q_j $j = 1, \ldots, N$, we find, by a routine computation, that the functions p_j and q_j satisfy the following system of ordinary differential equations with discontinuous right-hand side:

$$\begin{cases} p_j' = 2\sum_{k=1}^N p_j p_k \operatorname{sgn}(q_j - q_k) e^{-|q_j - q_k|}, \\ q_j' = \sum_{k=1}^N p_k e^{-|q_j - q_k|}. \end{cases} \tag{4.7}$$

In the case of 2-peakon interaction, i.e., (4.7) with $N = 2$, we consider two peakons which are initially well-separated with asymptotic speeds $c_1 > c_2 > 0$ as $t \longrightarrow -\infty$ so that they eventually collide. In this case, we have two conserved quantities,

$$\begin{aligned} P &:= p_1 + p_2 = c_1 + c_2, \\ H &:= p_1^2 + p_2^2 + 2p_1p_2e^{-|q_1-q_2|}(2 - e^{-|q_1-q_2|}) = c_1^2 + c_2^2. \end{aligned}$$

Let $N = 2$, $P = p_1 + p_2$, $Q = q_1 + q_2$, $p = p_1 - p_2$ and $q = q_1 - q_2$. Then (4.7) becomes

$$\begin{cases} P' = 0, & Q' = P(1 + e^{-|q|}), \\ p' = (p^2 - P^2)e^{-|q|}, & q' = p(1 - e^{-|q|}), \end{cases} \tag{4.8}$$

with two conserved quantities,

$$\begin{aligned} P &= c_1 + c_2, \\ H &= \frac{1}{2}(P^2 + p^2) + \frac{1}{2}(P^2 - p^2)e^{-|q|}(2 - e^{-|q|}) = c_1^2 + c_2^2. \end{aligned}$$

If at some instant t_0 the peakons overlap, i.e., $q(t_0) = 0$, we would have

$$c_1^2 + c_2^2 = H = P^2 = (c_1 + c_2)^2$$

However, this is impossible since $c_1 > c_2 > 0$, so the peakons do not overlap at any instant t_0. Since the faster wave starts to the left of the slower one, we have $q < 0$. Thus, from the two conserved quantities we obtain

$$(c_1 + c_2)^2 - p^2 = \frac{4c_1c_2}{(1 - e^q)^2}.$$

This relation can be used to solve (4.8) and obtain explicit formulas for p_1, p_2, q_1, and q_2 (see [13]), and hence also the phase shift for the faster soliton

$$\triangle q_f = q_2(+\infty) - q_1(-\infty) = \log\left[\frac{c_1(c_1 + c_2)}{(c_1 - c_2)^2}\right]$$

and the phase shift for the slower soliton

$$\triangle q_s = q_1(+\infty) - q_2(-\infty) = \log\left[\frac{(c_1 - c_2)^2}{c_2(c_1 + c_2)}\right].$$

Therefore, we infer that:

(a) If $c_1 > 3c_2$, both waves experience a forward shift.
(b) If $c_1 = 3c_2$, no shift occurs for the slower wave, while the faster one is shifted forward.
(c) If $c_1 < 3c_2$, the faster wave is shifted forward while the slower one is shifted backward.

The phenomenon of 2-peakon interaction observed here for (1.2) is the same as that of 2-peakon interaction for the Camassa-Holm equation, except that the constant 3 above is replaced by 2 (see [4], [9]).

5. Appendix

Lemma 5.1 ([22]). *Let $f \in H^s$, $s > 3/2$. Then,*

$$\|\Lambda^{-r}[\Lambda^{r+t+1}, M_f]\Lambda^{-t}\|_{L(L^2)} \leq c\|f\|_s, \qquad |r|,\ |t| \leq s-1,$$

where M_f is the operator of multiplication by f, and c is a constant depending only on r and t.

Lemma 5.2 ([21]). *Let r, t be real numbers such that $-r < t \leq r$. Then,*

$$\|fg\|_t \leq c\|f\|_r\|g\|_t, \qquad \text{if } r > 1/2,$$
$$\|fg\|_{r+t-1/2} \leq c\|f\|_r\|g\|_t, \qquad \text{if } r < 1/2,$$

where c is a positive constant depending on r and t.

Lemma 5.3 ([27, §4.5, Theorems 5.5 and 5.8]). *Let X and Y be two Banach spaces and Y be continuously and densely embedded in X. Let $-A$ be the infinitesimal generator of the C_0-semigroup $T(t)$ on X and let S be an isomorphism from Y onto X. Then Y is $-A$-admissible (i.e., $T(t)Y \subset Y$ for all $t \geq 0$ and the restriction of $T(t)$ to Y is a C_0-semigroup on Y) if and only if $-A_1 = -SAS^{-1}$ is the infinitesimal generator of the C_0-semigroup $T_1(t) = ST(t)S^{-1}$ on X. Moreover, if Y is $-A$-admissible, then the part of $-A$ in Y is the infinitesimal generator of the restriction of $T(t)$ to Y.*

Lemma 5.4. *The operator $A(u) = u\partial_x$, with $u \in H^s$, $s > 3/2$, belongs to $G(H^{s-1}, 1, \beta)$.*

Proof. Since H^{s-1} is a Hilbert space, $A(u)$ belongs to $G(H^{s-1}, 1, \beta)$ for some real number β if and only if the following conditions hold (cf. [23]):

(a) $(A(u)y, y)_{s-1} \geq -\beta\|y\|^2_{s-1}$.
(b) $-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}, for some (or all) $\lambda > \beta$.

We first prove (a). Since $u \in H^s$, $s > 3/2$, it follows that u and u_x belong to L^∞ and $\|u_x\|_{L^\infty} \leq \|u\|_s$. Note that

$$\Lambda^{s-1}(u\partial_x y) = [\Lambda^{s-1}, u]\partial_x y + u\Lambda^{s-1}(\partial_x y) = [\Lambda^{s-1}, u]\partial_x y + u\partial_x\Lambda^{s-1}y.$$

Thus

$$\begin{aligned}
(A(u)y, y)_{s-1} &= (\Lambda^{s-1}(u\partial_x y), \Lambda^{s-1}y)_0 \\
&= ([\Lambda^{s-1}, u]\partial_x y, \Lambda^{s-1}y)_0 - \frac{1}{2}(u_x\Lambda^{s-1}y, \Lambda^{s-1}y)_0 \\
&\leq \|[\Lambda^{s-1}, u]\Lambda^{2-s}\|_{L(L^2)}\|\Lambda^{s-1}y\|_0^2 + \|u_x\|_{L^\infty}\|\Lambda^{s-1}y\|_0^2 \\
&\leq c\|u\|_s\|y\|^2_{s-1},
\end{aligned}$$

by Lemma 5.1 with $r = 0$, $t = s-2$. Setting $\beta = c\|u\|_s$, we obtain $(A(u)y, y)_{s-1} \geq -\beta\|y\|^2_{s-1}$, as claimed.

Next, we prove (b). Let $S = \Lambda^{s-1}$, and note that S is an isomorphism of H^{s-1} onto L^2 and that H^{s-1} is continuously and densely embedded in L^2 since $s > 3/2$. Define

$$A_1(u) := SA(u)S^{-1} = \Lambda^{s-1}A(u)\Lambda^{1-s}, \quad B_1(u) = A_1(u) - A(u).$$

Let $y \in L^2$ and $u \in H^s,\ s > 3/2$. Then we have

$$\begin{aligned}\|B_1(u)y\|_0 &= \|[\Lambda^{s-1}, u\partial_x]\Lambda^{1-s}y\|_0\\ &\le \|[\Lambda^{s-1}, u]\Lambda^{2-s}\|_{L(L^2)}\|\Lambda^{-1}\partial_x y\|_0\\ &\le c\|u\|_s\|y\|_0,\end{aligned}$$

by Lemma 5.1 with $r = 0,\ t = s - 2$. Hence $B_1(u) \in L(L^2)$.

Note that $A_1(u) = A(u) + B_1(u)$ and $A(u) \in G(L^2, 1, \beta)$ in Lemma 2.1. By a perturbation theorem for semigroups (cf. [27, §5.2, Theorem 2.3]) we obtain $A_1(u) \in G(L^2, 1, \beta')$. Applying Lemma 5.3 with $Y = H^{s-1}$, $X = L^2$, and $S = \Lambda^{s-1}$, we conclude that H^{s-1} is A-admissible. Hence $-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}. This completes the proof of Lemma 5.4. □

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References

[1] S. Alinhac, *Blowup for nonlinear hyperbolic equation*, Progress in nonlinear differential equations and their applications, vol.17, Birkhäuser, Boston, 1995.

[2] R. Beals, D. Sattinger, and J. Szmigielski, *Acoustic scattering and the extended Korteweg-de Vries hierarchy*, Adv. Math. **140** (1998), 190–206.

[3] ———, *Multipeakons and a theorem of Stieltjes*, Inverse Problems **15** (1999), 1–4.

[4] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Letters **71** (1993), 1661–1664.

[5] R. Camassa, D. Holm and J. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech. **31** (1994), 1–33.

[6] A. Constantin, *The Hamiltonian structure of the Camassa-Holm equation*, Expo.Math. **15** (1997), 53–85.

[7] ———, *Global existence of solutions and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble) **50** (2000), 321–362.

[8] ———, *On the scattering problem for the Camassa-Holm equation*, R. Soc. Lond. Proc. Ser. A **457** (2001), 953–970.

[9] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Ann. Scuola Norm. Sup. Pisa **26** (1998), 303–328.

[10] ———, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math. **181** (1998), 229–243.

[11] A. Constantin and W. A. Strauss, *Stability of peakons*, Comm. Pure Appl. Math. **53** (2000), 603–610.

[12] ———, *Stability of the Camassa-Holm solitons*, J. Nonlinear Science **12** (2002), 415–422.

[13] A. Degasperis, D. D. Holm, and A. N. W. Hone, *A new integral equation with peakon solutions*, Theoret. Math. Phys. **133** (2002), 1463–1474.

[14] A. Degasperis and M. Procesi, *Asymptotic integrability*, Symmetry and Perturbation Theory (A. Degasperis and G. Gaeta, eds.), World Scientific, Singapore, 1999, pp. 23–37.

[15] P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, Cambridge University Press, Cambridge, 1989.

[16] H. R. Dullin, G. A. Gottwald, and D. D. Holm, *An integrable shallow water equation with linear and nonlinear dispersion*, Phys. Rev. Letters **87** (2001), 4501–4504.

[17] A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäcklund transformation and hereditary symmetries*, Physica D **4** (1981), 47–66.

[18] A. A. Himonas and G. Misiolek, *The Cauchy problem for an integrable shallow water equation*, Differential and Integral Equations **14** (2001), 821–831.

[19] R. S. Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid. Mech. **457** (2002), 63–82.

[20] ———, *On solutions of the Camassa-Holm equation*, R. Soc. Lond. Proc. Ser. A, to appear

[21] T. Kato, *Quasi-linear equations of evolution with applications to partial differential equations*, Spectral Theory and Differential Equations, Lecture Notes in Math., vol. 448, Springer Verlag, Berlin, 1975, pp. 25–70.

[22] T. Kato, *On the Korteweg-de Vries equation*, Manuscripta Math. **28**, (1979), 89–99.

[23] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in Applied Mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128.

[24] C. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Veris equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[25] Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, J. Differential Equations **162** (2000), 27–63.

[26] H. P. McKean, *Integrable systems and algebraic curves*, Global Analysis, Lecture Notes in Mathematics, vol. 755, Springer-Verlag, Berlin, 1979, pp. 83–200.

[27] A. Pazy, *Semigroup of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[28] G. Rodriguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Anal. **46** (2001), 309–327.

[29] W. A. Strauss, *Nonlinear wave equations*, CBMS Regional Conference Series in Mathematics, vol. 73, American Math. Soc., Providence, R. I., 1989.

[30] G. B. Whitham, *Linear and nonlinear waves*, J. Wiley & Sons, New York, 1980.

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