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CASTELNUOVO-MUMFORD REGULARITY AND DEGREES OF GENERATORS OF GRADED SUBMODULES

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Abstract. We extend the regularity criterion of Bayer-Stillman for a graded ideal $\mathfrak{a}$ of a polynomial ring $K[\underline{\mathbf{x}}] := K[\underline{\mathbf{x}}_0, \ldots, \mathbf{x}_r]$ over an infinite field K to the situation of a graded submodule M of a finitely generated graded module U over a Noetherian homogeneous ring $R = \oplus_{n\geq 0} R_n$, whose base ring R_0 has infinite residue fields. If R_0 is Artinian, we construct a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$, depending only on the Hilbert polynomial of U, such that $\operatorname{reg}(M) \leq \widetilde{P}(\max\{d(M), \operatorname{reg}(U)+1\})$, where $d(M)$ is the generating degree of M. This extends the regularity bound of Bayer-Mumford for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ over a field K to the pair $M \subseteq U$.

1. Introduction

Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian homogeneous ring (so that R is $\mathbb{N}_0$-graded with $R = R_0[R_1]$) and let $M \neq 0$ be a finitely generated graded R-module. For $i \in \mathbb{N}_0$ and $n \in \mathbb{N}$ let $H^i_{R_+}(M)_n$ denote the n-th graded component of the i-th local cohomology module $H^i_{R_+}(M)$ of M with respect to the irrelevant ideal $R_+ = \bigoplus_{n>0} R_n$ of R. The *(Castelnuovo-Mumford) regularity* $\operatorname{reg}(M)$ of M is defined by

$$\operatorname{reg}(M) := \inf\{m \in \mathbb{Z} \mid H^i_{R_+}(M)_{n-i} = 0 \quad \forall i \in \mathbb{N}_0 \quad \forall n > m\}. \tag{1.1}$$

The *generating degree* $d(M)$ of M is “the largest degree of a minimal homogeneous generator of M”; thus

$$d(M) = \inf\big\{m \in \mathbb{Z} \mid M \text{ is generated by homogeneous elements of degree } \leq m\big\}. \tag{1.2}$$

The principal aim of this paper is to derive a (polynomial) upper bound on $\operatorname{reg}(M)$ in terms of $d(M)$ and $\operatorname{reg}(U)$, where M is a graded submodule of a finitely generated graded R-module U.

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Upper bounds on $\operatorname{reg}(M)$ in terms of other invariants of M are of fundamental significance in algebraic geometry, commutative algebra and computational algebraic geometry (cf. [3]).

In the theory of Hilbert and Picard schemes one is led to bound the regularity of a graded submodule M of a graded free module F over a polynomial ring in terms of the Hilbert polynomial of M, the generating degree and the rank of F (cf. [13], [14], [15], [22]).

On the other hand, if the base ring R_0 is Artinian, $\operatorname{reg}(M)$ and various other cohomological invariants of M may be bounded in terms of the *diagonal values* $\operatorname{length}_{R_0}(H^i_{R_+}(M)_{-i})$ $(i = 0, 1, \dots)$ of cohomology (cf. [5], [6], [7]). Closely related to these bounds of diagonal type is the vanishing or nonvanishing of the graded components $H^i_{R_+}(M)_n$, which is completely governed by a few simple combinatorial conditions if R_0 is semilocal and of dimension ≤ 1 (cf. [4]).

If $R = K[\mathbf{x}_0, \dots, \mathbf{x}_r] =: K[\underline{\mathbf{x}}]$ is a polynomial ring over a field, $\operatorname{reg}(M)$ has a "*syzygetic*" *description*: namely, if $\cdots \to F_1 \to F_0 \to M \to 0$ is a minimal free resolution of M, then

$$\operatorname{reg}(M) = \sup\{d(F_i) - i \mid i \geq 0\}. \tag{1.3}$$

So, in the polynomial ring case, $\operatorname{reg}(M)$ gives an upper bound on the generating degrees of the syzygies of M and hence is of crucial significance for the classical *problem of "the finitely many steps"* (cf. [16], [17]). Expressed in modern terminology, $\operatorname{reg}(M)$ governs the computational complexity of calculating the syzygies of the finitely generated graded $K[\underline{\mathbf{x}}]$-module M (cf. [9]). In case R is not a polynomial ring, the "*syzygetic*" *regularity* (i.e., the term on the right hand side of equation (1.3)) may exceed the cohomological regularity $\operatorname{reg}(M)$ and in fact even become infinite.

Let us recall that the problem of "the finitely many steps" consists in constructing, in a predictable number of steps, a minimal graded free resolution of M from a minimal graded free presentation $F_1 \to F_0 \to M \to 0$. This problem can be solved as the regularity $\operatorname{reg}(M)$ of a graded submodule M of the free module $K[\underline{\mathbf{x}}]^{\oplus s}$ can be bounded in terms of r, s and the generating degree $d(M)$ of M. This was essentially shown by Hermann [17] using ideas of Hentzelt-Noether [16]. (Note that the bounds calculated by Hermann are not correct; for correctly calculated bounds see [19], for example.) In the same spirit, Bayer and Mumford [1] showed that for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ one has the bound

$$\operatorname{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{r!}. \tag{1.4}$$

In [5] we extended this bound by showing that for a graded submodule $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ one has

$$\operatorname{reg}(M) \leq s^{e_r} (2d(M))^{r!}, \tag{1.5}$$

where the numbers e_r are defined recursively by $e_0 = 0$, and $e_r := e_{r-1} \cdot r + 1$ if $r > 0$. In explicit form we have $e_r = r! \sum_{k=1,\dots,r} 1/k! = [r!(e-1)]$ (cf. [23, Sequence A002627]). It also should be noted that the bounds given in (1.4) and (1.5) still appear to be rather far from being sharp: namely, if $\mathrm{Char}(K) = 0$ one has $\mathrm{reg}(\mathfrak{a}) \le (2d(\mathfrak{a}))^{2^{r-1}}$ (cf. [11], [12]), and by the examples of Mayr and Meyer [21] this latter bound is close to being best possible.

One basic aim of this paper is to extend the regularity bounds of (1.4) and (1.5) to a much more general situation. Namely, we consider an arbitrary finitely generated graded module U over a Noetherian homogeneous ring $R = \bigoplus_{n\ge 0} R_n$ with Artinian base ring R_0. Then we show (cf. Theorem 5.7):

(1.6) *There is a polynomial* $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ (*of degree* $\dim(U)!$) *which depends only on the Hilbert polynomial* P *of* U, *such that for each graded submodule* $M \subseteq U$ *we have* $\mathrm{reg}(M) \le \widetilde{P}(\max\{d(M), \mathrm{reg}(U)+1\})$.

If in addition $\dim(U) = \dim(R)$ and $d(M) + \mathrm{reg}(M) \le \mathrm{reg}(U) + 1$, we may replace $\widetilde{P}$ by a polynomial $P^* \in \mathbb{Q}[\mathbf{x}]$ such that the bounds of (1.5) hold with $R = K[\underline{\mathbf{x}}]$ and $U = K[\underline{\mathbf{x}}]^{\oplus s}$.

In [1], the bound (1.4) is deduced using the *regularity criterion of Bayer-Stillman* (cf. [2]). In fact, it turns out that the bound (1.4), and its extension (1.5), may be deduced without using this criterion (cf. [5]). Nevertheless, our proof of the bound (1.5) (resp. its extension (1.6)) is closely related to the regularity criterion of Bayer-Stillman, as both rely on the technique of *(saturated) filter-regular sequences of linear forms.* In Section 3 we give a criterion—in terms of such sequences—for detecting whether a graded submodule M of a finitely generated graded module U over a homogeneous Noetherian ring $R = \bigoplus_{n\ge 0} R_n$ is m-regular (cf. Theorem 3.8). If the base ring R_0 has infinite residue fields, our criterion extends the corresponding criterion of Bayer-Stillman for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ to the case of a graded submodule $M \subseteq U$ (cf. Theorem 4.7).

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2. Some preliminaries

In this section we recall a few generalities on graded rings and graded modules. We use $\mathbb{N}_0$ (resp. $\mathbb{N}$) to denote the set of non-negative (resp. positive) integers.

2.1. Definition and Remark.

(A) By a *homogeneous ring* we mean a (commutative unitary) $\mathbb{N}_0$-graded ring $R = \bigoplus_{n\ge 0} R_n$, which is generated over its base ring R_0 by linear forms, so that $R = R_0[R_1]$. Keep in mind that the $\mathbb{N}_0$-graded ring $R = \bigoplus_{n\ge 0} R_n$

is homogeneous and Noetherian if and only if R_0 is Noetherian and there are finitely many linear forms $f_0, \dots, f_r \in R_1$ such that $R = R_0[f_0, \dots, f_r]$.

(B) If $R = \bigoplus_{n \geq 0} R_n$ is an $\mathbb{N}_0$-graded ring, we denote by R_+ the *irrelevant ideal* of R, i.e., $R_+ := \bigoplus_{n>0} R_n$. Recall that R is homogeneous if and only if R_+ is generated by linear forms, and thus if and only if $R_+ = R_1 \cdot R$.

(C) If $R = \bigoplus_{n \geq 0} R_n$ is an $\mathbb{N}_0$-graded ring, we use $\mathrm{Proj}(R)$ to denote the *projective spectrum* of R, i.e., the set of all graded primes $\mathfrak{p} \subseteq R$ with $R_+ \nsubseteq \mathfrak{p}$.

2.2. Definition.

(A) Let $R = \bigoplus_{n \geq 0} R_n$ be an $\mathbb{N}_0$-graded ring and let $T = \bigoplus_{n \in \mathbb{N}} T_n$ be a graded R-module. We define the *beginning* and the *end* of T, respectively, by

$$\mathrm{beg}(T) := \inf\{n \in \mathbb{Z} \mid T_n \neq 0\}, \quad \mathrm{end}(T) := \sup\{n \in \mathbb{Z} \mid T_n \neq 0\},$$

where "inf" and "sup" are formed in $\mathbb{Z} \cup \{\pm\infty\}$ with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

(B) Let R and T be as in part (A) and let $m \in \mathbb{Z}$. We define the m-th *left-truncation* and the m-th *right-truncation* of T, respectively, as the following R_0-submodules of T:

$$T_{\geq m} := \bigoplus_{n \geq m} T_n \ ; \quad T_{\leq m} := \bigoplus_{n \leq m} T_n.$$

As R is $\mathbb{N}_0$-graded, $T_{\geq m}$ is a (graded) R-submodule of T.

(C) Let R and T be as above. We denote the *generating degree* of T by $d(T)$, so that

$$d(T) := \inf\{m \in \mathbb{Z} \mid T = T_{\leq m} \cdot R\},$$

where "inf" is formed under the same convention as in part (A).

2.3. Definition and Remark (cf. [8]).

(A) Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R-module. Then, for each $i \in \mathbb{N}_0$, the i-th *local cohomology module* $H^i_{R_+}(M)$ of M with respect to the irrelevant ideal R_+ of R carries a natural grading. For all $n \in \mathbb{Z}$ we use $H^i_{R_+}(M)_n$ to denote the n-th *graded component* of $H^i_{R_+}(M)$.

(B) Let $R = \bigoplus_{n \geq 0} R_n$ and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be as in part (A), but assume in addition that the R-module M is finitely generated. Then, for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$, the R_0-module $H^i_{R_+}(M)_n$ is finitely generated and vanishes for all $n \gg 0$. Moreover, $H^i_{R_+}(M)$ vanishes for all $i > \dim(M)$. So, for each $k \in \mathbb{N}_0$ we may define the (*Castelnuovo-Mumford*) *regularity of M at and above level k* by

$$\mathrm{reg}^k(M) := \sup\{\mathrm{end}\left(H^i_{R_+}(M)\right) + i \mid i \geq k\},$$

and obtain $\mathrm{reg}^k(M) \in \mathbb{Z} \cup \{-\infty\}$.

(C) Let R and M be as in part (B). The (*Castelnuovo-Mumford*) *regularity of* M is defined as (cf. (1.1))

$$\operatorname{reg}(M) := \operatorname{reg}^0(M),$$

where $\operatorname{reg}^0(M)$ is defined as in part (B). It is important to keep in mind that the generating degree and the regularity of M are related by the inequality (cf. [8, 15.3.1])

$$d(M) \le \operatorname{reg}(M).$$

(D) Let R and M be as in part (B) and let $k \in \mathbb{N}, m \in \mathbb{Z}$. Then the following equivalence is known to hold (cf. [8, 15.2.5]):

$$\operatorname{reg}^k(M) \le m \Longleftrightarrow H^i_{R_+}(M)_{m-i+1} = 0 \quad \forall i \ge k.$$

If $\operatorname{reg}^k(M) \le m$ we say that M is *m-regular at and above level k*. If $\operatorname{reg}(M) \le m$, i.e., if M is m-regular at and above level 0, we say that M is *m-regular*.

2.4. Remark (Replacement argument). Let $R = \bigoplus_{n\ge 0} R_n$ be a homogeneous Noetherian ring and let R'_0 be a Noetherian faithfully flat R_0-algebra. Let M be a finitely generated graded R-module and $N \subseteq M$ a graded submodule. Then by faithful flatness and the graded flat base change property of local cohomology [8, 15.2.3]) we may replace M and N by $R'_0 \otimes_{R_0} M$ resp. $R'_0 \otimes_{R_0} N$ whenever we wish to prove a statement on regularities and generating degrees of M and N.

For notation and terminology from commutative algebra that has not been explained here we refer to [10] and [20].

3. Filter-regular sequences and regularity

Let $R = \bigoplus_{n\ge 0} R_n$ be a homogeneous Noetherian ring, let U be a finitely generated graded R-module and let $M \subseteq U$ be a graded submodule. Let $m \in \mathbb{Z}$ and let $f_1, \dots, f_r \in R_1$ be a sequence of linear forms. We prove a criterion for the property that M is m-regular and $f_1, \dots, f_r$ form a saturated filter-regular sequence with respect to U/M.

We briefly recall the notion of filter-regular sequence.

3.1. Reminder and Remark (cf. [8, Chapt. 18]).

(A) Let $R \bigoplus_{n\ge 0} R_n$ be a homogeneous Noetherian ring and let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a finitely generated and graded R-module. A homogeneous element $f \in R$ is said to be (R_+-) *filter-regular (or almost-regular) with respect to* T if it is a non-zero divisor with respect to $T/H^0_{R_+}(T)$. This is equivalent to saying that f avoids all elements $\mathfrak{p} \in \operatorname{Ass}_R(T) \cap \operatorname{Proj}(R)$. Clearly, f is filter-regular with respect to T if and only if the annihilator $0 \underset{T}{:} f$ of f in T is contained in $H^0_{R_+}(T)$, and thus if and only if $\operatorname{end}(0 \underset{T}{:} f) < \infty$.

(B) Let R and T be as in part (A). A sequence of homogeneous elements $f_1, \ldots, f_r \in R$ is called a *filter-regular (or almost-regular) sequence with respect to* T if f_i is filter-regular with respect to $T/\sum_{j=1}^{i-1} f_j T$ for all $i \in \{1, \ldots, r\}$. If in addition $f_1, \ldots, f_r \in R_1$, we call the sequence a *filter-regular sequence of linear forms*. If $W \subseteq H^0_{R_+}(T)$ is a graded submodule, a sequence $f_1, \ldots, f_r$ of homogeneous elements in R is filter-regular with respect to T if and only if it is filter-regular with respect to T/W.

3.2. Lemma. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring, let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a finitely generated graded R-module, let $f_1, \ldots, f_r \in R_1$ be a filter-regular sequence with respect to T and let $i \in \{0, \ldots, r\}$. Then:*

(a) $\operatorname{reg}\left(T/\sum_{j=1}^{i} f_j T\right) \leq \operatorname{reg}(T)$.

(b) $\operatorname{end}\left(H^i_{R_+}(T)\right) + i \leq \operatorname{end}\left(H^0_{R_+}\left(T/\sum_{j=1}^{i} f_j T\right)\right)$.

Proof. (a) This follows from [8, (18.3.11)].

(b) The case $i = 0$ is obvious. So, let $i > 0$. As $f_2, \ldots, f_r$ is a filter-regular sequence with respect to T/f_1T, by induction

$$\operatorname{end}\left(H^{i-1}_{R_+}(T/f_1T)\right) + i - 1 \leq \operatorname{end}\left(H^0_{R_+}\left(T/\sum_{j=1}^{i} f_j T\right)\right) =: e.$$

Let $\overline{T} := T/H^0_{R_+}(T)$. Then the graded epimorphism

$$H^{i-1}_{R_+}(T/f_1T) \twoheadrightarrow H^{i-1}_{R_+}(\overline{T}/f_1\overline{T})$$

shows that $\operatorname{end}(H^{i-1}_{R_+}(\overline{T}/f_1\overline{T})) + i - 1 \leq e$. But now the exact sequences

$$H^{i-1}_{R_+}(\overline{T}/f_1\overline{T})_{n+1} \longrightarrow H^i_{R_+}(\overline{T})_n \xrightarrow{f_1} H^i_{R_+}(\overline{T})_{n+1}$$

and the vanishing of $H^i_{R_+}(\overline{T})_n$ for all $n \gg 0$ imply

$$\operatorname{end}\left(H^i_{R_+}(\overline{T})\right) \leq \operatorname{end}\left(H^{i-1}_{R_+}(\overline{T}/f_1\overline{T}))\right) - 1 \leq e - i.$$

In view of the graded isomorphism $H^i_{R_+}(T) \cong H^i_{R_+}(\overline{T})$ we get our claim. □

In order to formulate and prove the announced regularity criterion we introduce the notion of a saturated filter-regular sequence.

3.3. Definition and Remark.

(A) Let $R = \bigoplus_{n\geq 0} R_n$ and $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be as in 3.1. A filter-regular sequence $f_1, \ldots, f_r$ with respect to T is *saturated* if $f_1, \ldots, f_r \in R_+$ and

$T/\sum_{j=1}^{r} f_jT$ is an R_+-torsion module. This is equivalent to saying that

$$\sum_{j=1}^{r} f_jR \subseteq R_+ \subseteq \sqrt{0 \underset{R}{:} T/\sum_{j=1}^{r} f_jT}$$

or that

$$\sqrt{(0 \underset{R}{:} T) + R_+} = \sqrt{(0 \underset{R}{:} T) + \sum_{j=1}^{r} f_jR}.$$

(B) As a consequence of this definition (cf. [8, 2.1.9]), if $f_1, \ldots, f_r \in R$ is a saturated filter-regular sequence with respect to T, then there are natural isomorphisms $H^i_{R_+}(T) \cong H^i_{(f_1,\ldots,f_r)}(T)$ for all $i \in \mathbb{N}_0$. Hence, in this situation we have $H^i_{R_+}(T) = 0$ for all $i > r$.

3.4. Proposition. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring, let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a finitely generated graded R-module, let $f_1, \ldots, f_r \in R_1$ and let $m \in \mathbb{Z}$. Then the following statements are equivalent:*

(i) *$\operatorname{reg}(T) < m$ and $f_1, \ldots, f_r$ is a saturated filter-regular sequence with respect to T.*

(ii) *$\operatorname{end}\left(0 \underset{T/\sum_{j=1}^{i-1} f_jT}{:} f_i\right) < m$ for all $i \in \{1, \ldots, r\}$ and $\operatorname{end}\left(T/\sum_{j=1}^{r} f_jT\right) < m$.*

Proof. "(i) $\Longrightarrow$ (ii)": Assume that condition (i) holds. Then 3.2(a) shows that

$$\operatorname{end}\left(H^0_{R_+}\left(T/\sum_{j=1}^{k} f_jT\right)\right) \leq \operatorname{reg}\left(T/\sum_{j=1}^{k} f_jT\right) \leq \operatorname{reg}(T) < m$$

for all $k \in \{1, \ldots, r\}$. As f_i is filter-regular with respect to $T/\sum_{j=1}^{i-1} f_jT$, we obtain

$$\operatorname{end}\left(0 \underset{T/\sum_{j=1}^{i-1} f_jT}{:} f_i\right) \leq \operatorname{end}\left(H^0_{R_+}\left(T/\sum_{j=1}^{i-1} f_jT\right)\right) < m, \quad \forall i \in \{1, \ldots, r\}.$$

As the sequence $f_1, \ldots, f_r$ is saturated, we have

$$T/\sum_{j=1}^{r} f_jT = H^0_{R_+}\left(T/\sum_{j=1}^{r} f_jT\right)$$

and hence obtain $\operatorname{end}(T/\sum_{j=1}^{r} f_jT) < m$.

"(ii) $\Longrightarrow$ (i)": Assume that condition (ii) holds. As $\mathrm{end}(0 \underset{T/\sum_{j=1}^{i-1} f_jT}{:} f_i) < \infty$ for $i = 1, \dots, r$, it follows that the sequence $f_1, \dots, f_r$ is filter-regular with respect to T. As $\mathrm{end}(T/\sum_{j=1}^{r} f_jT) < \infty$, this sequence is saturated. In particular, we have $H^i_{R_+}(T) = 0$ for all $i > r$ (cf. 3.3(B)). If we apply 3.2(b) with $i = 1, \dots, r$ we obtain $\mathrm{reg}(T) < m$. □

3.5. Corollary. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring, let $m \in \mathbb{Z}$ and let U be a finitely generated graded R-module such that* $\mathrm{reg}(U) < m$. *Let $M \subseteq U$ be a graded submodule and let $f_1, \dots, f_r \in R_1$. Then the following statements are equivalent:*

(i) $\mathrm{reg}(M) \leq m$ *and $f_1, \dots, f_r$ is a saturated filter-regular sequence with respect to U/M.*

(ii) $\left(\left(M + \sum_{j=1}^{i-1} f_jU\right) \underset{U}{:} f_i\right)_{\geq m} = \left(M + \sum_{j=1}^{i-1} f_jU\right)_{\geq m}$ *for all* $i \in \{1, \dots, r\}$ *and* $\left(M + \sum_{j=1}^{r} f_jU\right)_{\geq m} = U_{\geq m}$.

Proof. Let $T := U/M$. Then the graded exact sequence $0 \to M \to U \to T \to 0$ shows that $\mathrm{reg}(M) \leq \max\{\mathrm{reg}(U), \mathrm{reg}(T) + 1\}$ and $\mathrm{reg}(T) \leq \max\{\mathrm{reg}(U), \mathrm{reg}(M) - 1\}$ (cf. [8, 15.2.15]). So, 3.4(i) is equivalent to 3.5(i). The equivalence of 3.4(ii) and 3.5(ii) is immediate. □

The announced regularity criterion turns the criterion 3.5 into a "persistency result", in which the comparison of graded components in all degrees $\geq m$ which appears in statement 3.5 (ii) is replaced by a comparison in degree m. To prove this, we use the following lemma:

3.6. Lemma. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring. Let U be a finitely generated graded R-module, let $m \in \mathbb{Z}$, and let $M, N \subseteq U$ be two graded submodules such that $d(M), d(N) \leq m$ and* $\mathrm{reg}(M + N) < m$. *Then $d(M \cap N) \leq m$.*

Proof. Write R as a graded homomorphic image of a polynomial ring $R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_0, \dots, \mathbf{x}_r]$ and observe that neither the generating degree nor the regularity of a finitely generated graded R-module V change their values if we consider V as an $R_0[\underline{\mathbf{x}}]$-module. Therefore we may assume that $R = R_0[\underline{\mathbf{x}}]$ is a polynomial ring. We can now proceed as in the proof of [5, 2.4], where our result was shown for the special case when R is a polynomial ring over a field. Namely, as $d(M), d(N) \leq m$, there are graded epimorphisms $\pi : F \to M \to 0$ and $\varrho : G \to N \to 0$ in which F and G are graded free R-modules of finite rank with $d(F), d(G) \leq m$. As $\mathrm{reg}(R) = 0$ we thus obtain $\mathrm{reg}(F \bigoplus G) \leq m$. The graded short exact sequence

$$0 \to \mathrm{Ker}(\pi + \varrho) \to F \bigoplus G \xrightarrow{\pi+\varrho} M + N \to 0$$

yields that $\operatorname{reg}(\operatorname{Ker}(\pi+\varrho)) \leq m$ and thus $d(\operatorname{Ker}(\pi+\varrho)) \leq m$ (cf. 2.3(C)). Now the commutative diagram

$$\begin{array}{ccc} M \bigoplus N & \xrightarrow{\sigma:=\mathrm{id}_M+\mathrm{id}_N} & M+N \\ \uparrow{\scriptstyle \pi\bigoplus\varrho} & & \uparrow{\scriptstyle \pi+\varrho} \\ F\bigoplus G & = & F\bigoplus G \end{array}$$

shows that $(\pi \bigoplus \varrho)(\operatorname{Ker}(\pi+\varrho)) = \operatorname{Ker}(\sigma)$ and thus $d(\operatorname{Ker}(\sigma)) \leq m$. In view of the graded isomorphism $M \cap N \cong \operatorname{Ker}(\sigma)$ our claim follows. □

3.7. Lemma. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring and let $m \in \mathbb{Z}$. Let U be a finitely generated graded R-module, let $M \subseteq U$ be a graded submodule and let $f \in R_1$ be filter-regular with respect to U. Assume that $d(M), \operatorname{reg}(U), \operatorname{reg}(M+fU) \leq m$. Then $d(M \underset{U}{:} f) \leq m$.*

Proof. As $d(fU) \leq d(U)+1 \leq \operatorname{reg}(U)+1 \leq m+1$, Lemma 3.6 implies that $d(M\cap fU) \leq m+1$. Since $M \cap fU = f(M \underset{U}{:} f)$, we have a graded short exact sequence

$$0 \to (0 \underset{U}{:} f) \to (M \underset{U}{:} f) \to (M\cap fU)(1) \to 0.$$

As f is filter-regular with respect to U, we have $(0 \underset{U}{:} f) \subseteq H^0_{R_+}(U)$ and hence

$$d(0 \underset{U}{:} f) \leq \operatorname{end}(0 \underset{U}{:} f) \leq \operatorname{end}\left(H^0_{R_+}(U)\right) \leq \operatorname{reg}(U) \leq m.$$

Now, the above exact sequence yields $d(M \underset{U}{:} f) \leq m$. □

We are now ready to formulate and prove the main result of this section.

3.8. Theorem. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring and let $m \in \mathbb{Z}$. Let U be a finitely generated graded R-module, let $M \subseteq U$ be a graded submodule, let $f_1, \ldots, f_r \in R_1$ be filter-regular elements with respect to U and assume that $\operatorname{reg}(U) < m$ and $d(M) \leq m$. Then the following statements are equivalent:*

(i) $\operatorname{reg}(M) \leq m$ *and $f_1, \ldots, f_r$ is a saturated filter-regular sequence with respect to U/M.*

(ii) $\left(\left(M + \sum_{j=1}^{i-1} f_j U\right) \underset{U}{:} f_i\right)_m = \left(M + \sum_{j=1}^{i-1} f_j U\right)_m$ *for all $i \in \{1, \ldots, r\}$ and* $\left(M + \sum_{j=1}^{r} f_j U\right)_m = U_m$.

Proof. "(i) $\Longrightarrow$ (ii)": This is clear by 3.5.

"(ii) $\Longrightarrow$ (i)": We proceed by induction on r. First, let $r = 1$. By statement (ii) we have $(M + f_1U)_m = U_m$. As $d(U) \leq \operatorname{reg}(U) \leq m$, it follows that $(M + f_1U)_{\geq m} = U_{\geq m}$, and hence $\operatorname{end}(U/(M + f_1U)) < m$. In view of the graded short exact sequence $0 \to (M + f_1U) \to U \to U/(M + f_1U) \to 0$ it follows that $\operatorname{reg}(M + f_1U) \leq m$. By Lemma 3.7 we get $d(M \underset{U}{:} f_1) \leq m$. By statement (ii), we have $(M \underset{U}{:} f_1)_m = M_m$; it follows that $(M \underset{U}{:} f_1)_{\geq m} = M_{\geq m}$. From the implication "(ii) $\Longrightarrow$ (i)" of Corollary 3.5 we get $\operatorname{reg}(M) \leq m$ and that f_1 constitutes a saturated filter-regular sequence with respect to U/M.

Now, let $r > 1$ and assume that statement (ii) holds. As $d(f_1U) \leq d(U) + 1 \leq \operatorname{reg}(U) + 1 \leq m$, we have $d(M + f_1U) \leq m$. Applying induction to the graded submodule $M + f_1U \subseteq U$ and the sequence $f_2, \ldots, f_r \in R_1$, we see that $\operatorname{reg}(M + f_1U) \leq m$ and that $f_2, \ldots, f_r$ is a saturated filter-regular sequence with respect to $U/(M + f_1U)$. Hence, by 3.5 we have

$$\left(\left(M + \sum_{j=1}^{i-1} f_jU\right) \underset{U}{:} f_i\right)_{\geq m} = \left(M + \sum_{j=1}^{i-1} f_jU\right)_{\geq m}$$

for all $i \in \{2, \ldots, r\}$ and $(M + \sum_{j=1}^{r} f_jU)_{\geq m} = U_{\geq m}$. By 3.7 we also have $d(M \underset{U}{:} f_1) \leq m$. As $(M \underset{U}{:} f_1)_m = M_m$ and $d(M) \leq m$, it follows that $(M \underset{U}{:} f_1)_{\geq m} = M_{\geq m}$. Now, another application of 3.5 gives statement (i). □

4. Extending the regularity criterion of Bayer-Stillman

Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \ldots, \mathbf{x}_t]$ be a polynomial ring over an infinite field K and let $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ be a graded ideal. Let $m \in \mathbb{N}$. In [2, 1.10] Bayer and Stillman proved that $\mathfrak{a}$ is m-regular if and only if there is a sequence of linear forms $f_1, \ldots, f_r \in K[\underline{\mathbf{x}}]_1$ such that statement (ii) of Theorem 3.8 holds with $M = \mathfrak{a}$ and $U = K[\underline{\mathbf{x}}]$. The aim of this section is to extend this regularity criterion of Bayer-Stillman to a situation nearly as general as that in 3.8. To do so, we obviously need the existence of saturated filter-regular sequences of linear forms with respect to arbitrary finitely generated modules over the considered homogeneous Noetherian ring $R = \bigoplus_{n \geq 0} R_n$. To ensure that such sequences exist, we shall subject the base ring R_0 to an appropriate condition.

4.1. Definition and Remark.

(A) A Ring R_0 is said to have *infinite residue fields* if the field $R_0/\mathfrak{m}_0$ is infinite for each $\mathfrak{m}_0 \in \operatorname{Max}(R_0)$ or, equivalently, if $R_0/\mathfrak{p}_0$ is an infinite domain for each $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$.

(B) Clearly, if $f : R_0 \to R_0'$ is a homomorphism of rings and R_0 has infinite residue fields, then R_0' also has infinite residue fields. In particular, R_0 has infinite residue fields if it contains an infinite field.

4.2. Lemma. *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 has infinite residue fields and let $\mathfrak{Q} \subseteq \operatorname{Proj}(R)$ be a finite set. Then $R_1 \nsubseteq \bigcup_{\mathfrak{q} \in \mathfrak{Q}} \mathfrak{q}$.*

Proof. We may assume that $\mathfrak{Q} \neq \emptyset$. For $\mathfrak{m}_0 \in \mathrm{Max}(R_0)$ set $\mathfrak{Q}(\mathfrak{m}_0) := \{\mathfrak{q} \in \mathfrak{Q} \mid \mathfrak{q} \cap R_0 \subseteq \mathfrak{m}_0\}$. Clearly, there is a finite set $\mathbb{M} \subseteq \mathrm{Max}(R_0)$ such that $\mathfrak{Q}(\mathfrak{m}_0) \neq \emptyset$ for each $\mathfrak{m}_0 \in \mathbb{M}$ and $\mathfrak{Q} = \bigcup_{\mathfrak{m}_0 \in \mathbb{M}} \mathfrak{Q}(\mathfrak{m}_0)$. For each $\mathfrak{m}_0 \in \mathbb{M}$ and each $\mathfrak{q} \in \mathfrak{Q}(\mathfrak{m}_0)$ it follows by Nakayama that $\mathfrak{q} \cap R_1 + \mathfrak{m}_0 R_1 \subsetneqq R_1$. So, as $\mathfrak{Q}(\mathfrak{m}_0)$ is finite and $R_0/\mathfrak{m}_0$ is infinite, there is some $v_{\mathfrak{m}_0} \in R_1 \setminus \bigcup_{\mathfrak{q} \in \mathfrak{Q}(\mathfrak{m}_0)} (\mathfrak{q}_1 + \mathfrak{m}_0 R_1)$. For each $\mathfrak{m}_0 \in \mathbb{M}$ we find some element $a_{\mathfrak{m}_0} \in (\bigcap_{\mathfrak{n}_0 \in \mathbb{M} \setminus \{\mathfrak{m}_0\}} \mathfrak{n}_0) \setminus \mathfrak{m}_0$. With $v := \sum_{\mathfrak{m}_0 \in \mathbb{M}} a_{\mathfrak{m}_0} v_{\mathfrak{m}_0}$ it follows that

$$v \in R_1 \setminus \bigcup_{\mathfrak{m}_0 \in \mathbb{M}} \bigcup_{\mathfrak{q} \in \mathfrak{Q}(\mathfrak{m}_0)} (\mathfrak{q}_1 + \mathfrak{m}_0 R_1) = R_1 \setminus \bigcup_{\mathfrak{q} \in \mathfrak{Q}} \mathfrak{q}. \qquad \square$$

4.3. Lemma. *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 has infinite residue fields and let $\mathcal{P} \subseteq \mathrm{Proj}(R)$ be a finite set. Let $r \in \mathbb{N}$ and let $T = \bigoplus_{n \in \mathbb{Z}} T_n$ be a finitely generated graded R-module. Then there is a sequence $(f_i)_{i \in \mathbb{N}} \subseteq R_1 \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$ such that $f_1, \ldots, f_r$ is a filter-regular sequence with respect to T for each $r \in \mathbb{N}$.*

Proof. If we apply 4.2 with $\mathfrak{Q} := \mathcal{P} \cap \mathrm{Ass}(T) \cap \mathrm{Proj}(R)$ we get an element $f_1 \in R_1 \setminus \bigcup_{\mathfrak{q} \in \mathcal{P}} \mathfrak{p}$ which is filter-regular with respect to T. Using this observation, a sequence $(f_i)_{i \in \mathbb{N}}$ of the requested type is easily constructed by induction. $\square$

Hence, if the base ring R_0 has infinite residue fields, filter-regular sequence of arbitrary length and consisting of linear forms exist. The existence of saturated filter-regular sequences now follows easily.

4.4. Lemma. *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring and let T be a finitely generated graded R-module. Let $(f_i)_{i \in \mathbb{N}} \subseteq R_+$ be a sequence such that $f_1, \ldots, f_r$ is a filter-regular sequence with respect to T for each $r \in \mathbb{N}$. Then there is some $r_0 \in \mathbb{N}$ such that the filter-regular sequence $f_1, \ldots, f_r$ is saturated for each $r \geq r_0$.*

Proof. If, for some $r \in \mathbb{N}$, the filter-regular sequence $f_1, \ldots, f_r$ is non-saturated, f_{r+1} avoids some member of $\mathrm{Ass}_R(T/\sum_{i=1}^r f_i T)$, so that $f_{r+1} \notin \sum_{i=1}^r f_i R$, and hence $\sum_{i=1}^r f_i R \subsetneqq \sum_{i=1}^{r+1} f_i R$. As R is Noetherian, we obtain our claim. $\square$

The possible values of the number r_0 in Lemma 4.4 can easily be bounded. In order to do so, let us recall some notion.

4.5. Definition. The *arithmetic rank* $\mathrm{ara}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a Noetherian ring R is defined as the minimal number of elements in R which generate an ideal that is radically equal to $\mathfrak{a}$; thus

$$\operatorname{ara}(\mathfrak{a}) := \min\left\{ r \in \mathbb{N}_0 \mid \exists a_1, \ldots, a_r \in R : \sqrt{\sum_{i=1}^{r} a_i R} = \sqrt{\mathfrak{a}} \right\}.$$

4.6. Lemma. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring, let T be a finitely generated graded R-module and let $f_1, \ldots, f_r \in R_+$ be a filter-regular sequence with respect to T. Then:*

(a) *If the filter-regular sequence $f_1, \ldots, f_r$ is saturated, then $r \geq \operatorname{ara}((R/(0 \underset{R}{:} T))_+)$.*
(b) *If $r \geq \dim(T)$, the filter-regular sequence $f_1, \ldots, f_r$ is saturated.*
(c) *If R_0 is Artinian, then the filter-regular sequence $f_1, \ldots, f_r$ is saturated if and only if $r \geq \dim(T)$.*

Proof. (a) This is clear by 3.3(A).

(b) Assume that the sequence $f_1, \ldots, f_r$ is not saturated, so that

$$\sqrt{(0 \underset{R}{:} T) + R_+} \underset{\neq}{\supseteq} \sqrt{(0 \underset{R}{:} T) + \sum_{j=1}^{r} f_j R}.$$

Then there is a prime $\mathfrak{p} \in \operatorname{Var}((0 \underset{R}{:} T) + \sum_{j=1}^{r} f_j R) \setminus \operatorname{Var}(R_+)$. Thus $f_1/1, \ldots, f_r/1 \in \mathfrak{p}R_{\mathfrak{p}}$ is a regular sequence with respect to $T_{\mathfrak{p}}$ (cf. [8, 18.3.8]), so that $r \leq \operatorname{depth}(T_{\mathfrak{p}}) \leq \dim(T_{\mathfrak{p}})$. As $\mathfrak{p} \underset{\neq}{\subseteq} \mathfrak{p}_0 + R_+ \in \operatorname{Spec}(R)$, we have $\dim(T_{\mathfrak{p}}) < \dim(T)$ and hence get $r < \dim(T)$.

(c) As R_0 is Artinian, we have $\dim(R/(0 \underset{R}{:} T)) = \operatorname{ara}((R/(0 \underset{R}{:} T))_+)$. Now, the result follows by statements (a) and (b). □

Next, we give the announced extension of the regularity criterion of Bayer-Stillman.

4.7. Theorem. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 has infinite residue fields. Let $m \in \mathbb{Z}$, let U be a finitely generated graded R-module and let $M \subseteq U$ be a graded submodule. Assume that $\operatorname{reg}(U) < m$ and $d(M) \leq m$. Then the following statements are equivalent:*

(i) *$\operatorname{reg}(M) \leq m$.*
(ii) *There are elements $f_1, \ldots, f_r \in R_1$ which are filter-regular with respect to U and such that*

$$\left(\left(M + \sum_{j=1}^{i-1} f_j U\right) \underset{U}{:} f_i\right)_m = \left(M + \sum_{j=1}^{i-1} f_j U\right)_m \quad \forall i \in \{1, \ldots, r\}$$

and

$$\left(M + \sum_{j=1}^{r} f_j U\right)_m = U_m.$$

Proof. "(ii) $\Longrightarrow$ (i)": This is clear by Theorem 3.8.

"(i) $\Longrightarrow$ (ii)": Applying 4.3 with $\mathcal{P} = \mathrm{Ass}_R(U) \cap \mathrm{Proj}(R)$ and keeping in mind 4.4, we get a saturated filter-regular sequence $f_1, \ldots, f_r \in R_1$ with respect to U/M such that each f_i is filter-regular with respect to U. The result now follows by Theorem 3.8. □

4.8. Remark. Let $K[\underline{\mathbf{x}}] = K[\underline{\mathbf{x}}_0, \ldots, \mathbf{x}_t]$ be a polynomial ring over an infinite field K, let $m, s \in \mathbb{N}$, let $U := K[\underline{\mathbf{x}}]^{\bigoplus s}$ and let $M \subseteq U$ be a graded submodule with $d(M) \leq m$. As $\mathrm{reg}(U) = 0$ and U is torsion-free, it follows from 4.7 that $\mathrm{reg}(M) \leq m$ if and only there are generic linear forms $f_1, \ldots, f_r \in K[\underline{\mathbf{x}}]_1 \backslash \{0\}$ such that the conditions 4.7 (ii) hold. This is precisely what is shown in [18, 1.10]. Choosing $s = 1$, we get the regularity criterion of Bayer-Stillman.

5. Extending the regularity bound of Bayer-Mumford

Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \ldots, \mathbf{x}_t]$ be a polynomial ring over a field K and let $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ be a graded ideal. In [1, 3.8] Bayer and Mumford showed that $\mathrm{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{n!}$. Our aim is to extend this bound to the case where $K[\underline{\mathbf{x}}]$ is replaced by an arbitrary finitely generated graded module U over a homogeneous Noetherian ring $R = \bigoplus_{n \geq 0} R_n$ with Artinian base ring R_0 and $\mathfrak{a}$ is replaced by a graded submodule M of U.

5.1. Notation and Remark.

(A) Let R_0 be an Artinian ring and let V be a finitely generated R_0-module. We use $\ell(V) = \ell_{R_0}(V)$ to denote the length of V.

(B) Let R_0 and V be as in part (A). Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the different maximal ideals of R_0, let $\mathbf{x}$ be an indeterminate and set

$$R_0' := \left(R_0[\mathbf{x}] \backslash \bigcup_{i=1}^{t} \mathfrak{m}_i R_0[\mathbf{x}]\right)^{-1} R_0[\mathbf{x}].$$

Then clearly R_0' is a faithfully flat Artinian extension ring of R_0 with the different maximal ideals $\mathfrak{m}_i' = \mathfrak{m}_i R_0'$ $(i = 1, \ldots, t)$. Moreover, we have $\ell_{R_0'}(R_0' \otimes_{R_0} V) = \ell_{R_0}(V)$. As $R_0'/\mathfrak{m}_i' \cong R_0/\mathfrak{m}_i(\mathbf{x})$ for all $i \in \{1, \ldots, t\}$, the ring R_0' has infinite residue fields.

5.2. Lemma. *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 is Artinian, let U be a finitely generated graded R-module, let $M \subseteq U$ be a graded submodule and let $f \in R_1$ be filter-regular with respect to U and U/M. Let $k \in \mathbb{Z}$ be such that $d(M), \mathrm{reg}(M + fU), \mathrm{reg}(U) + 1 \leq k$. Then*

(a) $\operatorname{end}(H^i_{R_+}(M)) + i \leq k$ *for all* $i \neq 1$.
(b) $\operatorname{end}(H^1_{R_+}(M)) \leq \ell(U_k) + k - 1$.

Proof. Let $T := U/M$. The short exact sequence $0 \to (M + fU) \to U \to T/fT \to 0$ shows that $\operatorname{reg}(T/fT) \leq \max\{\operatorname{reg}(U), \operatorname{reg}(M + fU) - 1\} \leq k - 1$. As $f \in R_1$ is filter-regular with respect to T, it follows that $\operatorname{reg}^1(T) \leq \operatorname{reg}(T/fT) \leq k-1$ (cf. [8, 18.3.11]), and the graded short exact sequence $0 \to M \to U \to T \to 0$ implies $\operatorname{reg}^2(M) \leq \max\{\operatorname{reg}^2(U), \operatorname{reg}^1(T) + 1\} \leq k$ (cf. [8, 15.2.15]) and hence $\operatorname{end}(H^i_{R_+}(M)) + i \leq k$ for all $i \geq 2$. As $\operatorname{end}(H^0_{R_+}(M)) \leq \operatorname{end}(H^0_{R_+}(U)) \leq \operatorname{reg}(U) \leq k$, we obtain statement (a).

It remains to prove statement (b). In view of the graded short exact sequence $0 \to M \to U \to T \to 0$ and since $\operatorname{end}(H^1_{R_+}(U)) \leq \operatorname{reg}(U) - 1 \leq k - 1$, it suffices to show that $\operatorname{end}(H^0_{R_+}(T)) \leq \ell(U_k) + k - 1$. We have seen above that $\operatorname{reg}(T/fT) \leq k-1$. So, if we apply cohomology to the graded short exact sequence $0 \to T/(0 \underset{T}{:} f) \xrightarrow{f} T(1) \to (T/fT)(1) \to 0$ we get isomorphisms

$$H^0_{R_+}(T/(0 \underset{T}{:} f))_n \cong H^0_{R_+}(T)_{n+1}, \quad \forall n \geq k - 1.$$

If we apply cohomology to the graded short exact sequence $0 \to (0 \underset{T}{:} f) \to T \to T/(0 \underset{T}{:} f) \to 0$ and keep in mind that $(0 \underset{T}{:} f) \subseteq H^0_{R_+}(T)$ (cf. 3.1(A)), we thus get exact sequences

$$0 \to (0 \underset{T}{:} f)_n \to H^0_{R_+}(T)_n \xrightarrow{\pi_n} H^0_{R_+}(T)_{n+1} \to 0, \quad \forall n \geq k - 1.$$

By 3.7 we have $d(0 \underset{T}{:} f) \leq d(M \underset{U}{:} f) \leq k$, so that π_m becomes an isomorphism for all $m \geq n$, provided π_n is an isomorphism for some $n \geq k$. From this it follows that the length $\ell(H^0_{R_+}(T)_n)$ of the R_0-module $H^0_{R_+}(T)_n$ is strictly decreasing as a function of n in the range $n \geq k$ until its value becomes 0. This implies that $\operatorname{end}(H^0_{R_+}(T)) \leq \ell(H^0_{R_+}(T)_k) + k - 1$. As $H^0_{R_+}(T)_k$ is a subquotient of the R_0-module U_k we get $\operatorname{end}(H^0_{R_+}(T)) \leq \ell(U_k) + k - 1$. □

5.3. Lemma. *Let* $R = \bigoplus_{n \geq 0} R_n$ *be a homogeneous Noetherian ring such that* R_0 *is Artinian and* $\dim(R) = 1$. *Let* U *be a finitely generated and graded* R*-module and let* $M \subseteq U$ *be a graded submodule. Let* $k \in \mathbb{Z}$ *be such that* $d(M) + \operatorname{reg}(R)$ *and* $\operatorname{reg}(U) + 1 \leq k$. *Then* $\operatorname{reg}(M) \leq k$.

Proof. Applying the replacement argument 2.4 with R'_0 defined as in 5.1(B), we may assume that R_0 has infinite residue fields. As $\operatorname{end}(H^0_{R_+}(M)) \leq \operatorname{end}(H^0_{R_+}(U)) < k$ and $H^i_{R_+}(M) = 0$ for all $i > 1$, it remains to show that $\operatorname{end}(H^1_{R_+}(M)) \leq k - 1$. Choosing $\mathcal{P} = \operatorname{Ass}_R(R) \cap \operatorname{Proj}(R)$ we conclude by 4.3 that there is a linear form $f \in R_1$ which is at the same time filter-regular with respect to U and with respect to R. As f is filter-regular with respect to U, we have $\operatorname{end}(0 \underset{U}{:} f) \leq \operatorname{end}(H^0_{R_+}(U)) < k$. Therefore, the multiplication map $f : U_n \to U_{n+1}$ is injective for all $n \geq k$. As $\dim(R) = 1$ and $f \in R_1$ avoids all minimal primes of R, we have $R_+ \subseteq \sqrt{Rf}$ and R is a finitely generated graded module over its subring $R_0[f]$. In particular, by the graded base ring independence of local cohomology, $\operatorname{reg}(R)$ does not change if we consider R as an $R_0[f]$-module. We then obtain $d(R) \leq \operatorname{reg}(R) \leq k - d(M)$, so that $R_{n+1} = fR_n$ for all $n \geq k - d(M)$. Hence for each $n \geq k$ we obtain $M_{n+1} = R_{n-d(M)+1}M_{d(M)} = fR_{n-d(M)}M_{d(M)} = fM_n$. As $f : U_n \to U_{n+1}$ is injective for all $n \geq k$, it follows that $(M_{n+1} \underset{U_n}{:} f) = M_n$ for all such n. From this we see that $\operatorname{end}(0 \underset{U/M}{:} f) < k$. As $f \in R_1$, it follows that $\operatorname{end}(H^0_{R_+}(U/M)) < k$. If we apply cohomology to the graded exact sequence $0 \to M \to U \to U/M \to 0$ and keep in mind that $\operatorname{end}(H^1_{R_+}(U)) < \operatorname{reg}(U) < k$, we obtain indeed $\operatorname{end}(H^1_{R_+}(M)) < k$. $\square$

In order to formulate our main result, we introduce some notation.

5.4. Definition and Remark.

(A) Let $\mathbb{P}$ be the set of all polynomials $P \in \mathbb{Q}[\mathbf{x}]$ with the property that $P(n) \in \mathbb{N}_0$ for all integers $n \gg 0$. For $P \in \mathbb{P}$, let $\Delta P \in \mathbb{P}$ denote the difference polynomial $P(\mathbf{x}) - P(\mathbf{x}-1)$ of P.

(B) For $P \in \mathbb{P}$ we define a polynomial $P^* = P^*(\mathbf{x})$ recursively by

$$P^*(\mathbf{x}) := \begin{cases} \mathbf{x}, & \text{if } \deg(P) \leq 0, \\ (\Delta P)^*(\mathbf{x}) + P((\Delta P)^*(\mathbf{x})), & \text{if } \deg(P) > 0. \end{cases}$$

It is easy to see that $P^* \in \mathbb{P}$ whenever $P \in \mathbb{P}$.

(C) Now, let $s \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Then clearly $s\binom{\mathbf{x}+r}{r} \in \mathbb{P}$ and $\Delta\left[s\binom{\mathbf{x}+r}{r}\right] = s\binom{\mathbf{x}+r-1}{r-1}$. We write $F_r(s,\mathbf{x}) := \left[s\binom{\mathbf{x}+r}{r}\right]^*$, so that $F_0(s,\mathbf{x}) = \mathbf{x}$ and $F_r(s,\mathbf{x}) = F_{r-1}(s,\mathbf{x}) + s\binom{F_{r-1}(s,\mathbf{x})+r}{r}$ for all $r > 0$. This means that $F_r(s,\mathbf{x})$ is as in [5, 2.5 (A)]. In particular, we have (cf. [5, 2.5 (B)])

$$F_r(s,t) < s^{e_r}(2t)^{r!}, \quad \forall s,t \in \mathbb{N},$$

where the numbers e_r are defined inductively by

$$e_0 := 0 \text{ and } e_r := r \cdot e_{r-1} + 1 \text{ for } r > 0.$$

(D) Also, for each $P \in \mathbb{P}$ we recursively define a polynomial $\widetilde{P} \in \mathbb{P}$ by

$$\widetilde{P}(\mathbf{x}) := \begin{cases} \mathbf{x}, & \text{if } P = 0, \\ (\widetilde{\Delta P})(\mathbf{x}) + P((\widetilde{\Delta P})(\mathbf{x})), & \text{if } P \neq 0. \end{cases}$$

It is easy to see that $\widetilde{P}(k) \geq P^*(k)$ for all $k \gg 0$.

Finally let us recall a few facts about Hilbert polynomials.

5.5. Reminder.

(A) Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 is Artinian and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module. We denote the Hilbert polynomial of M by P_M, so that (cf. [8, Chap. 17])

$$P_M(n) = \ell(M_n) \quad \forall n > \operatorname{reg}(M).$$

(B) Also, if $f \in R_1$ is filter regular with respect to M, we have short exact sequences $0 \to M_{n-1} \xrightarrow{f} M_n \to (M/fM)_n \to 0$ for all $n \gg 0$ and these yield $P_{M/fM} = \Delta P_M$.

If R_0' is defined as in 5.1(B), then in the notation of 2.4(B) we have

$$P_{R_0'\otimes_{R_0} M} = P_M.$$

5.6. Lemma. *Let $R \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 is Artinian. Let U be a finitely generated graded R-module with Hilbert polynomial $P_U =: P$ and let $k \in \mathbb{Z}$ be such that* $\operatorname{reg}(U) < k$. *Then:*

(a) $k \leq (\Delta P)^*(k) \leq P^*(k)$.

(b) $k \leq \widetilde{(\Delta P)}(k) \leq \widetilde{P}(k)$.

Proof. In view of 2.4 and 5.5 (B) we may assume that R_0 has infinite residue fields. We now proceed by induction on $\deg(P)$. If $P = 0$, we have $P^* = \widetilde{P} = (\Delta P)^* = \widetilde{(\Delta P)} = \mathbf{x}$, and our claims are obvious. If $\deg(P) = 0$, we have $P^* = (\Delta P)^* = \widetilde{(\Delta P)} = \mathbf{x}$ and $\widetilde{P} = \mathbf{x}+P(\mathbf{x})$. As P is a positive constant, our claims follow. Let $\deg(P) > 0$. As R_0 has infinite residue fields, there is a linear form $f \in R_1$ which is filter regular with respect to U. In particular, we have $\Delta P = P_{U/fU}$ (cf. 5.5(B)) and $\operatorname{reg}(U/fU) < k$ (cf. 3.2(a)). So, by induction we have $k \leq (\Delta P)^*(k)$ and $k \leq \widetilde{(\Delta P)}(k)$. In particular (cf. 5.5(A)), $P((\Delta P)^*(k)) = \ell(U_{(\Delta P)^*(k)}) \geq 0$ and $P(\widetilde{(\Delta P)}(k)) = \ell(U_{\widetilde{(\Delta P)}(k)}) \geq 0$. Now, both claims follow from the definitions of P^* and $\widetilde{P}$. □

We now prove the main result of this section.

5.7. Theorem. *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring such that R_0 is Artinian . Let U be a finitely generated graded R-module with Hilbert polynomial $P_U =: P$ and let $M \subseteq U$ be a graded submodule. Let $k \in \mathbb{Z}$ and assume that* $\operatorname{reg}(U) < k$.

(a) *If $d(M) \leq k$, then* $\operatorname{reg}(M) \leq \widetilde{P}(k)$.

(b) *If* $\dim(R) = \dim(U)$ *and* $d(M) + \operatorname{reg}(R) \leq k$, *then* $\operatorname{reg}(M) \leq P^*(k)$.

Proof. In view of 2.4 and the last observation made in 5.5(B), we may assume that R_0 has infinite residue fields. We proceed by induction on $\dim(U)$. If $\dim(U) \leq 0$ we have $P = 0$ and $\operatorname{reg}(M) = \operatorname{end}(H^0_{R_+}(M)) \leq \operatorname{end}(H^0_{R_+}(U)) = \operatorname{reg}(U) < k = 0^*(k) = \tilde{0}(k)$, which proves both claims in this case. Now, let $\dim(U) > 0$. For the remainder of the proof, we treat the two claims separately.

(a) If we apply 4.3 with $\mathcal{P} := \mathrm{Ass}_R(U/M) \cap \mathrm{Proj}(R)$, we find a linear form $f \in R_1$ which is filter-regular with respect to U and U/M. As $\dim(U) > 0$, f avoids all minimal members of $\mathrm{Ass}_R(U)$, so that $\dim(U/fU) = \dim(U) - 1$. By 3.2(a) we have $\mathrm{reg}(U/fU) \leq \mathrm{reg}(U) < k$. Clearly, $d((M + fU)/fU) \leq d(M) \leq k$. By 5.5(B) we also have $\Delta P = P_{U/fU}$. Now, by induction we have $\mathrm{reg}((M + fU)/fU) \leq (\widetilde{\Delta P})(k)$. As $(0 \underset{f}{:} U) \subseteq H^0_{R_+}(U)$ and in view of the graded isomorphism $fU \cong (U/(0 \underset{U}{:} f))(-1)$ we get $\mathrm{reg}(fU) = \mathrm{reg}(U/(0 \underset{U}{:} f)) + 1 \leq \mathrm{reg}(U) + 1 \leq k$, and hence $\mathrm{reg}(fU) \leq (\widetilde{\Delta P})(k)$ (cf. 5.6(b)). The exact sequence $0 \to fU \to (M + fU) \to (M + fU)/fU \to 0$ yields $\mathrm{reg}(M + fU) \leq (\widetilde{\Delta P})(k) =: m$. If we keep in mind that $k \leq m$ we get $m \leq \widetilde{P}(m)$ (cf. 5.6(b)) and $\ell(U_m) = P(m)$ (cf. 5.5(A)). So, applying 5.2 with m instead of k and observing 5.6(b), we get $\mathrm{end}(H^i_{R_+}(M)) + i \leq m = (\Delta\widetilde{P})(k) \leq \widetilde{P}(k)$ for all $i \neq 1$ and $\mathrm{end}(H^1_{R_+}(M)) + 1 \leq P(m) + m = P((\widetilde{\Delta P})(k)) + (\widetilde{\Delta P})(k) = \widetilde{P}(k)$. Therefore $\mathrm{reg}(M) \leq \widetilde{P}(k)$.

(b) Assume first that $\dim(U) = 1$ and hence $\dim(R) = 1$. Then 5.3 and 5.6(a) show that $\mathrm{reg}(M) \leq k \leq P^*(k)$. So, let $\dim(U) > 1$. Now apply 4.3 with $\mathcal{P} = (\mathrm{Ass}_R(U/M) \cup \mathrm{Ass}_R(R)) \cap \mathrm{Proj}(R)$ to obtain a linear form $f \in R_1$ which is filter-regular with respect to each of $U, U/M$ and R. As in the proof of statement (a) we now get $\dim(R/fR) = \dim(U/fU) = \dim(U) - 1, \mathrm{reg}(U/fU) < k$ and $d((M + fU)/fU) + \mathrm{reg}(R/fR) \leq k$. Again, by 5.5(B) we have $\Delta P = P_{U/fU}$. Thus, by induction we obtain $\mathrm{reg}((M + fU)/fU) \leq (\Delta P)^*(k)$. We can now complete the proof literally in the same way as that of statement (a) if we replace $(\widetilde{\Delta P})$ by $(\Delta P)^*$ and $\widetilde{P}$ by P^*. □

5.8. Corollary. *Let $R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ be a polynomial ring over an Artinian ring R_0. Let $w \in \mathbb{N}$ and let $M \subseteq R_0[\underline{\mathbf{x}}]^{\oplus w}$ be a graded submodule. Then*

$$\mathrm{reg}(M) \leq (\ell(R_0)w)^{e_r}(2d(M))^{r!},$$

where e_r is defined as in 5.4(C).

Proof. If $d(M) = 0$, there is a graded isomorphism $M \cong M_0 \otimes_{R_0} R_0[\underline{\mathbf{x}}]$, so that $\mathrm{reg}(M) = 0$. Therefore we may assume that $d(M) > 0$. Let $R := R_0[\underline{\mathbf{x}}]$, $U := R_0[\underline{\mathbf{x}}]^{\oplus w}$. Then $\mathrm{reg}(U) = \mathrm{reg}(R) = 0$, $\dim(R) = \dim(U) = r$ and the fact that $P_U = \ell(R_0)w\binom{\mathbf{x}+r}{r}$ yield the result, in view of 5.7(b) and 5.4(C). □

5.9. Remark. If in 5.8 we let $R_0 = K$ be a field, we obtain the bound given in [5, 2.7]. If we assume in addition $w = 1$, we get the bound of Bayer-Mumford [1, 3.8].

References

[1] D. Bayer and D. Mumford, *What can be computed in algebraic geometry?*, Computational algebraic geometry and commutative algebra (Proc. Cortona 1991, D. Eisenbud and L. Robbiano, eds.), Cambridge University Press, Cambridge, 1993, pp. 1–48.

[2] D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. **87** (1987), 1–11.

[3] M. Brodmann, *Cohomological invariants of coherent sheaves over projective schemes: a survey*, Local cohomology and its applications (G. Lyubeznik, ed.), Lecture Notes in Pure and Appl. Math., vol. 226, Marcel Dekker, New York, 2002, pp. 91–120.

[4] M. Brodmann and M. Hellus, *Cohomological patterns of coherent sheaves over projective schemes*, J. Pure Appl. Algebra **172** (2002), 165–182.

[5] M. Brodmann and A. Lashgari, *A diagonal bound for cohomological postulation numbers of projective schemes*, J. Algebra **265** (2003), 631–650.

[6] M. Brodmann, C. Matteotti, and N.D. Minh, *Bounds for cohomological Hilbert-functions of projective schemes over Artinian rings*, Vietnam J. Math. **28** (2000), 345–384.

[7] ______, *Bounds for cohomological deficiency functions of projective schemes over Artinian rings*, Vietnam J. Math. **31** (2003), 71–113.

[8] M. Brodmann and R.Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, vol. 60, Cambridge University Press, Cambridge, 1998.

[9] B. Buchberger, *A note on the complexity of constructing Gröbner-Bases*, Computer algebra (London, 1983), Lecture Notes in Comput. Sci., vol. 162, Springer-Verlag, Berlin, 1983, pp. 137–145.

[10] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Springer-Verlag, New York, 1996.

[11] A. Galligo, *Théorème de division et stabilité en géométrie analytique locale*, Ann. Inst. Fourier **29** (1979), 107–184.

[12] M. Giusti, *Some effectivity problems in polynomial ideal theory*, Eurosam 84 (Cambridge, 1984), Lecture Notes in Comput. Sci., vol. 174, Springer-Verlag, Berlin, 1984, pp. 159–171.

[13] G. Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines gradierten Ringes*, Math. Z. **158** (1978), 61–70.

[14] ______, *Durch Hilbertfunktionen definierte Unterschemata des Hilbert-Schemas*, Comment. Math. Helvetici **63** (1988), 114–149.

[15] A. Grothendieck, *Séminaire de géométrie algébrique VI*, Lecture Notes in Mathematics, vol. 225, Springer-Verlag, Berlin, 1971.

[16] K. Hentzelt and E. Noether, *Zur Theorie der Polynomideale und Resultanten*, Math. Ann. **88** (1923), 53–79.

[17] G. Hermann, *Über die Frage der endlich vielen Schritte in der Theorie der Polynomideale*, Math. Ann. **95** (1926), 736–788.

[18] A.F. Lashgari, *The cohomology diagonal bounds the postulation numbers of a coherent sheaf over a projective scheme*, Dissertation, University of Zürich, 2000.

[19] D. Masser and G. Wüstholz, *Fields of large transcendence degree generated by values of elliptic functions*, Invent. Math. **72** (1983), 407–464.

[20] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986.

[21] E.W. Mayr and A.R. Meyer, *The complexity of the word problem for commutative semigroups and polynomial ideals*, Adv. in Math. **46** (1982), 305–329.

[22] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Math. Studies, vol. 59, Princeton University Press, Princeton, 1966.

[23] *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/

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