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SOME MENON DESIGNS HAVING U(3,3) AS AN AUTOMORPHISM GROUP

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Dedicated to the memory of Reinhold Baer

Abstract. There exists a unique symmetric (36,15,6) design $\mathcal{D}$ having $G'(2,2) \cong U(3,3)$ as an automorphism group. There is an incidence matrix M of $\mathcal{D}$ which is symmetric with 1 everywhere on the main diagonal. Thus $\mathcal{D}$ admits a polarity for which all points are absolute. Therefore, M is an adjacency matrix of a strongly regular graph with parameters $(36, 14, 4, 6)$.

Using this design one can produce a series of symmetric designs with parameters $(4 \cdot (3 \cdot 2^k)^2, 2 \cdot (3 \cdot 2^k)^2 - 3 \cdot 2^k, (3 \cdot 2^k)^2 - 3 \cdot 2^k)$, $k \in N$, each of which admits an automorphism group isomorphic to the unitary group $U(3,3)$. There is an incidence matrix for each of these designs which is symmetric with constant diagonal. Therefore, these matrices correspond to adjacency matrices of strongly regular graphs.

1. Introduction and preliminaries

A symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$.
2. Every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$.
3. Every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Also, we insist that $k > \lambda$ and $1 < k < v - 1$. The elements of the set $\mathcal{P}$ are called points and the elements of the set $\mathcal{B}$ are called blocks.

Given two designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$, an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and blocks onto blocks preserving the incidence relation. An isomorphism from a symmetric

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design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of the design $\mathcal{D}$ forms a group; it is called the full automorphism group of $\mathcal{D}$ and denoted by $\operatorname{Aut}\mathcal{D}$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq \operatorname{Aut}\mathcal{D}$. The group action of G produces the same number of point and block orbits. We denote that number by t, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_r - 1}$, i.e., $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r-1}\}$. Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda)\,. \tag{2}$$

Definition 1. A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit structure for the parameters (v, k, λ) and orbit lengths distributions $(\omega_1, \ldots, \omega_t)$, $(\Omega_1, \ldots, \Omega_t)$.

The first step—when constructing designs for given parameters and orbit lengths distributions—is to find all compatible orbit structures (γ_{ir}). The next step, called indexing, consists in determining exactly which points from the point orbit $\mathcal{P}_r$ are incident with a fixed representative of the block orbit $\mathcal{B}_i$ for each number γ_{ir}. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Definition 2. The set of those indices of points of the orbit $\mathcal{P}_r$ which are incident with a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i, r) of the orbit structure and the given representative.

A Hadamard matrix of order m is an $(m \times m)$-matrix $H = (h_{i,j})$, $h_{i,,j} \in \{-1, 1\}$, satisfying $HH^T = H^TH = mI$, where I is the unit matrix. Suppose H is a Hadamard matrix of order v having constant row and column sum. Such a matrix is called regular Hadamard matrix. Replacing each entry -1 of the matrix H by 0, we obtain an incidence matrix of a symmetric (v, k, λ) design, where $v = 4(k - \lambda)$. Also, from each symmetric (v, k, λ) design with $v = 4(k - \lambda)$, one can recover a regular Hadamard matrix with constant row and column sums. Since v is even, $k - \lambda$ must be a square by Schutzenberger's theorem (see [7]). Therefore, (v, k, λ) is of type $(4u^2, 2u^2 \pm u, u^2 \pm u)$, for some positive integer u. Symmetric designs with $v = 4(k - \lambda)$ are called H-designs or Menon designs. See also [7].

Symmetric $(36, 15, 6)$ designs are Menon designs for $u = 3$. According to [9] more than 25634 symmetric $(36, 15, 6)$ designs have been constructed so far.

A lot of these designs admit only the trivial automorphism group (see [1]). Many symmetric $(36, 15, 6)$ designs with automorphisms of order 3, 4, 5, or 7 are described in [4]. For further basic definitions and construction procedures we refer the reader to [6] and [12].

2. A Frobenius group of order 14 acting on a symmetric (36,15,6) design

Let $\mathcal{D}$ be a symmetric (36,15,6) design and G a Frobenius group of order 14. Since there is only one isomorphism class of nonabelian groups of order 14 we may write

$$G = \langle \rho, \sigma \mid \rho^7 = 1,\, \sigma^2 = 1,\, \rho^\sigma = \rho^6 \rangle.$$

LEMMA 1. *Let $\mathcal{D}$ be a symmetric $(36, 15, 6)$ design and let $\langle\rho\rangle$ be a subgroup of $\operatorname{Aut}\mathcal{D}$. If $|\langle\rho\rangle| = 7$, then $\langle\rho\rangle$ fixes precisely one point and one block of $\mathcal{D}$.*

Proof. Let $F(\rho)$ be the number of points fixed by ρ. From [7, Corollary 3.7] follows that $F(\rho) \leq k+\sqrt{k-\lambda}$; clearly, $F(\rho) \equiv v \pmod{|\langle\rho\rangle|}$. Therefore, $F(\rho) \in \{1, 8, 15\}$. If $F(\rho) = 8$, then the fixed structure must be a symmetric $(8, 8, 6)$ design. Such a design does not exist, so $F(\rho) \neq 8$. The case $F(\rho) = 15$ can be eliminated in a similar way. Solving equations (1) and (2), one gets the following two orbit structures for orbit lengths distribution $(1, 7, 7, 7, 7, 7)$ in case when $F(\rho) = 1$:

OS1	1	7	7	7	7	7
1	1	7	7	0	0	0
7	1	1	4	3	3	3
7	1	4	1	3	3	3
7	0	3	3	5	2	2
7	0	3	3	2	5	2
7	0	3	3	2	2	5

OS2	1	7	7	7	7	7
1	1	7	7	0	0	0
7	1	1	4	3	3	3
7	1	4	1	3	3	3
7	0	3	3	4	4	1
7	0	3	3	1	4	4
7	0	3	3	4	1	4

□

LEMMA 2. *Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design, with an involution ρ fixing F points and blocks. If $F \neq 0$, then*

$$F \geq \begin{cases} 1 + k/\lambda, & \textit{if } k \textit{ and } \lambda \textit{ are both even,} \\ 1 + (k-1)/\lambda, & \textit{otherwise.} \end{cases}$$

Proof. See Lander [7, Proposition 4.23, p. 155]. □

LEMMA 3. *Let $G = \langle \rho, \sigma \rangle$ be the Frobenius group of order 14 defined above, $\mathcal{D}$ a symmetric $(36, 15, 6)$ design, and $G \leq \operatorname{Aut}\mathcal{D}$. Then G acts semistandardly on $\mathcal{D}$ with orbit lengths distribution $(1, 7, 7, 7, 7, 7)$ or $(1, 7, 7, 7, 14)$.*

Proof. The Frobenius kernel $\langle\rho\rangle$ of order 7 acts on $\mathcal{D}$ with orbit lengths distribution $(1,7,7,7,7,7)$. Since $\langle\rho\rangle \lhd G$, the element σ of order 2 maps $\langle\rho\rangle$-orbits onto $\langle\rho\rangle$-orbits. Therefore, the possibilities for orbit lengths distributions are $(1,7,7,7,7,7)$, $(1,7,7,7,14)$ and $(1,7,14,14)$. Since an automorphism group of a symmetric design has the same number of orbits on the set of blocks and on the set of points, G must act semistandardly on $\mathcal{D}$. The case of the orbit lengths distribution $(1,7,14,14)$ cannot occur, because then the involution σ would act with two fixed points which contradicts Lemma 2.

The orbit structures for the orbit lengths distribution $(1,7,7,7,7,7)$ are OS1 and OS2 from Lemma 1. For the orbit lengths distribution $(1,7,7,7,14)$, we obtain the following four orbit structures by solving equations (1) and (2):

OS3	1	7	7	7	14
1	1	7	7	0	0
7	1	1	4	3	6
7	1	4	1	3	6
7	0	3	3	5	4
14	0	3	3	2	7

OS4	1	7	7	7	14
1	1	7	7	0	0
7	1	1	4	3	6
7	1	4	1	3	6
7	0	3	3	1	8
14	0	3	3	4	5

OS5	1	7	7	7	14
1	1	0	0	0	14
7	0	5	2	2	6
7	0	2	5	2	6
7	0	2	2	5	6
14	1	3	3	3	5

OS6	1	7	7	7	14
1	1	0	0	0	14
7	0	4	4	1	6
7	0	4	1	4	6
7	0	1	4	4	6
14	1	3	3	3	5

□

THEOREM 1. *Up to isomorphisms there is only one symmetric* $(36,15,6)$ *design admitting an automorphism group isomorphic to a Frobenius group of order* 14. *Its full automorphism group is isomorphic to* $U(3,3):Z_2$. *The unitary group* $U(3,3)$ *acts transitively on the set of points and blocks of the design. The design possesses an incidence matrix IM which is symmetric and has* 1 *everywhere on the main diagonal.*

Proof. We denote the points by $1_0, 2_i, \ldots, 6_i$, $i = 0,1,\ldots,6$, and put $G = \langle\rho,\sigma\rangle$, where the generator ρ is a permutation defined by

$$\rho = (1_0)(I_0,\ldots,I_6),\ I = 2,3,4,5,6.$$

We shall deal first with the case when the group G acts with an orbit lengths distribution $(1,7,7,7,14)$. Indexing the fixed part of an orbit structure is a trivial task. Therefore, we shall consider only the right lower part of order 4 of the orbit structures OS3, OS4, OS5, and OS6. In order to perform the indexing of the row and column that correspond to orbits of length 14, we shall decompose these orbits in two $\langle\rho\rangle$-orbits of length 7. That decomposition leads to orbit structures with respect to the normal subgroup $\langle\rho\rangle$. It is obvious that the orbit structures OS3 and OS5 correspond to OS1, and the orbit structures OS4 and OS6 correspond to OS2. We shall proceed with indexing the orbit structures OS1 and OS2 making use of the action of the involutary permutation σ on the sets of points and blocks.

Indexing the orbit structures OS3, OS4, and OS6 does not produce designs, whereas OS5 leads to precisely one symmetric design. Therefore, we shall explain only the indexing of the orbit structure OS5. In fact, we shall proceed with the indexing process of the orbit structure OS1, keeping in mind that the permutation σ acts on the set of points in the following way:

$$\sigma = (1_0)(2_0,3_0)(2_1,3_6)(2_2,3_5)(2_3,3_4)(2_4,3_3)(2_5,3_2)(2_6,3_1)$$
$$(I_0)(I_1,I_6)(I_2,I_5)(I_3,I_4), \quad I=4,5,6.$$

The involution σ acts on the six $\langle\rho\rangle$−orbits of points and blocks as the transposition $(1)(2,3)(4)(5)(6)$. For the indexing of the second and third block orbit it is sufficient to determine the index sets for the block orbit $\mathcal{B}_2$ followed by an application of the permutation σ. As possible representatives for the block orbit $\mathcal{B}_2$ we use the first lines of all appropriate $\langle\rho\rangle$-orbits with respect to the lexicographical order. As representatives of the fourth, fifth, and sixth block orbit of OS1 we use all appropriate blocks fixed by $\langle\sigma\rangle$. Therefore, the index sets for the positions (i,r), $4 \le i,r \le 6$, of the orbit structure OS1 are unions of the sets $\{0\},\{1,6\},\{2,5\}$, and $\{3,4\}$. Once we have found the index sets for positions $(i,2)$, $i=4,5,6$, and because we know the action of σ on $\langle\rho\rangle$-orbits, we have determined at the same time the index sets for the positions $(i,3)$, $i=4,5,6$. For example, if the index set for the position $(4,2)$ is $\{0,1,2\}$, then the index set for the position $(4,3)$ is $\{0,5,6\}$.

The index sets—numbered from 0 to 82—which can occur in the designs constructed from OS1 are among the following:

$$\begin{aligned}
0 &= \{0\},\ \ldots,\ 6=\{6\},\\
7 &= \{1,6\},\ 8=\{2,5\},\ 9=\{3,4\},\\
10 &= \{0,1,2\},\ \ldots,\ 44=\{4,5,6\},\\
45 &= \{0,1,2,3\},\ \ldots,\ 79=\{3,4,5,6\},\\
80 &= \{0,1,2,5,6\},\ 81=\{0,1,3,4,6\},\ 82=\{0,2,3,4,5\}.
\end{aligned}$$

We present the unique symmetric design $\mathcal{D}$ obtained from OS1 by the (5×5)-matrix of index sets as follows:

$$\begin{pmatrix}
0 & 46 & 13 & 15 & 23\\
63 & 0 & 18 & 22 & 11\\
18 & 13 & 82 & 9 & 8\\
22 & 15 & 9 & 81 & 7\\
11 & 23 & 8 & 7 & 80
\end{pmatrix}$$

The index sets needed and not listed above are:

$$11 = \{0,1,3\},\ 13 = \{0,1,5\},\ 15 = \{0,2,3\},\ 18 = \{0,2,6\},\ 22 = \{0,4,5\},$$
$$23 = \{0,4,6\},\ 46 = \{0,1,2,4\},\ 63 = \{0,3,5,6\}.$$

It is an easy task to expand the above matrix to an incidence matrix for $\mathcal{D}$ which we denote by IM. We get that IM is symmetric with 1 everywhere on the main diagonal. In particular, $\mathcal{D}$ is self-dual.

To eliminate isomorphic structures during the indexing process we have used the permutations which—on each $\langle\rho\rangle$-orbit of length 7 of points—act as $x \mapsto 2x \pmod 7$ or $x \mapsto 3x \pmod 7$, and—in addition—automorphisms of our orbit structure OS1 which commute with σ.

A computer program by Vladimir D. Tonchev [10] computes the order as well as generators of the full automorphism group of the design. The structure of the full automorphism group is determined using GAP [5]. The transitive action of $U(3,3)$ follows from the orders of the maximal subgroups of $U(3,3)$; see, for instance, [3].

In the case that the group G is acting with an orbit lengths distribution $(1,7,7,7,7,7)$, the permutation σ is defined as follows:

$$\sigma = (1_0)(I_0)(I_1,I_6)(I_2,I_5)(I_3,I_4), \quad I = 2,3,4,5,6.$$

The indexing of the orbit structures OS1 and OS2 under that assumption does not produce designs. $\square$

3. A series of Menon designs admitting U(3,3) as an automorphism group

By taking the complement of the incidence relation in a symmetric (v,k,λ) design $\mathcal{D}$, we obtain the complement of $\mathcal{D}$, denoted by $\mathcal{D}'$. The incidence structure $\mathcal{D}'$ is a symmetric $(v, v-k, v-2k+\lambda)$ design.

LEMMA 4. *Let M be an incidence matrix of a Menon design with parameters $(4u^2, 2u^2-u, u^2-u)$ and M' its complement . Then*

$$\begin{bmatrix} M' & M & M & M \\ M & M' & M & M \\ M & M & M' & M \\ M & M & M & M' \end{bmatrix}$$

is an incidence matrix of a Menon design with parameters $(16u^2, 8u^2-2u, 4u^2-2u)$.

Proof. The assertion follows by verifying properties 1, 2, and 3 of the definition of symmetric designs. $\square$

LEMMA 5. *Let $\mathcal{D}$ be a Menon design with parameters $(4u^2, 2u^2-u, u^2-u)$ and let $\mathcal{D}_1$ be the Menon design with parameters $(16u^2, 8u^2-2u, 4u^2-$*

$2u$) obtained from $\mathcal{D}$ using the procedure described in Lemma 4. If G is an automorphism group of the design $\mathcal{D}$, then $G \times S_4$ is an automorphism group of the design $\mathcal{D}_1$.

Proof. The assertion follows directly from the construction of $\mathcal{D}_1$. □

THEOREM 2. *There exists a Menon design $\Delta = \Delta(k)$ with parameters $(4 \cdot (3 \cdot 2^k)^2, 2 \cdot (3 \cdot 2^k)^2 - 3 \cdot 2^k, (3 \cdot 2^k)^2 - 3 \cdot 2^k)$, for each $k \in N$, which possesses the following properties:*

(a) *Δ admits an automorphism group isomorphic to $U(3,3)$.*
(b) *An incidence matrix of Δ is symmetric with constant diagonal.*

Proof. This is a direct consequence of Theorem 1, Lemma 4, and Lemma 5. □

4. Strongly regular graphs

DEFINITION 3. Let $G = (V, E, I)$ be a finite incidence structure. Then G is called a graph if each element of E is incident with exactly two elements of V. The elements of V are called vertices and the elements of E edges.

If two vertices u and v are incident with the same edge, then u and v are called adjacent; alternatively, we may call them neighbours.

An automorphism of a graph is any permutation of the vertices preserving adjacency. The set of all automorphisms forms the full automorphism group of the graph. A subgroup of the full automorphism group of the graph is simply called an automorphism group.

DEFINITION 4. Let G be a graph. Define a square $\{0,1\}-$matrix $A = (a_{uv})$ labelled with the vertices of G in such a way that $a_{uv} = 1$ if and only if the vertices u and v are adjacent. The matrix A is called the adjacency matrix of G.

If v is any vertex of a graph $G = (V, E, I)$, we put $\langle v \rangle = \{e \in E \mid v \ I \ e\}$. The number $|\langle v \rangle|$ is called the degree of v.

DEFINITION 5. Let G be a graph. If the degree of every vertex of G equals k, then G is called $k-$regular.

DEFINITION 6. Let G be a $k-$regular graph with n vertices. If any two adjacent vertices have λ common neighbours while any two non-adjacent vertices have μ common neighbours, G is called a strongly regular graph with parameters (n, k, λ, μ).

A polarity Φ of a symmetric design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ is an isomorphism from $\mathcal{D}$ onto its dual, whose square is the identity. Thus, if Φ is a polarity of $\mathcal{D}$, then for any point P and block x, one has $P\ I\ x$ if and only if $x\Phi\ I\ P\Phi$. A point P is absolute with respect to Φ if $P\ I\ P\Phi$.

THEOREM 3. *A symmetric (v, k, λ) design admitting a polarity without absolute points exists if and only if there is a strongly regular graph with parameters (v, k, λ, λ).*

Proof. See Beth, Jungnickel, Lenz [2, II, Theorem 9.19, p. 143]. □

THEOREM 4. *A symmetric (v, k, λ) design admitting a polarity for which all points are absolute exists if and only if there is a strongly regular graph with parameters $(v, k-1, \lambda-2, \lambda)$.*

Proof. See Beth, Jungnickel, Lenz [2, II, Theorem 9.21, p. 144]. □

A symmetric (v, k, λ) design $\mathcal{D}$ admits a polarity without absolute points if and only if $\mathcal{D}$ possesses a symmetric incidence matrix with 0 everywhere on the diagonal. This incidence matrix is an adjacency matrix of a strongly regular graph with parameters (v, k, λ, λ). Also, $\mathcal{D}$ admits a polarity for which all points are absolute if and only if it possesses a symmetric incidence matrix with 1 everywhere on the diagonal. Let X be such an incidence matrix. Then $X - I$ is an adjacency matrix of a strongly regular graph with parameters $(v, k-1, \lambda-2, \lambda)$. The above statements follow directly from the definitions, see also [11].

THEOREM 5. *There exists a strongly regular graph with parameters $(36, 14, 4, 6)$ whose full automorphism group is isomorphic to $U(3,3) : Z_2$. The unitary group $U(3,3)$ acts transitively on the set of vertices of this graph.*

Proof. The incidence matrix IM of the symmetric $(36, 15, 6)$ design $\mathcal{D}$ having $U(3,3) : Z_2$ as full automorphism group is symmetric with 1 everywhere on the diagonal. Therefore, $\mathcal{D}$ admits a polarity for which all points are absolute and $IM - I$ is an adjacency matrix of a strongly regular graph G with parameters $(36, 14, 4, 6)$. The full automorphism group of G is determined using Nauty [8]. For the transitive action see [3]. □

THEOREM 6. *For each even positive integer k there exists a strongly regular graph with parameters $(4 \cdot (3 \cdot 2^k)^2, 2 \cdot (3 \cdot 2^k)^2 - 3 \cdot 2^k - 1, (3 \cdot 2^k)^2 - 3 \cdot 2^k - 2, (3 \cdot 2^k)^2 - 3 \cdot 2^k)$ admitting $U(3,3)$ as an automorphism group.*

Proof. For each of the Menon designs of Theorem 2 we have a symmetric incidence matrix with constant diagonal. If k is even, these incidence matrices have entries 1 everywhere on the diagonal, so each design admits a polarity for which all points are absolute. The assertion about the automorphism groups of the graphs is a consequence of the shape of the incidence matrices. □

THEOREM 7. *For each odd positive integer k there exists a strongly regular graph with parameters* $(4\cdot(3\cdot2^k)^2, 2\cdot(3\cdot2^k)^2-3\cdot2^k, (3\cdot2^k)^2-3\cdot2^k, (3\cdot2^k)^2-3\cdot2^k)$ *admitting* $U(3,3)$ *as an automorphism group.*

Proof. If k is odd, then, using again Theorem 2, we have incidence matrices for the Menon designs which are symmetric with 0 everywhere on the diagonal. Therefore, these designs admit polarities without absolute points and the incidence matrices of the Menon designs are adjacency matrices of strongly regular graphs. □

5. Hadamard designs and strongly regular graphs

From each Hadamard matrix of order m with $m \equiv 0 \pmod 4$, one can obtain a symmetric $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design by normalizing and deleting the first row and column and changing all entries -1 to 0; see [7]. Also, from any symmetric $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design we can recover a Hadamard matrix. Symmetric designs with parameters $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ are called Hadamard designs.

Let H be a Hadamard matrix and $\mathcal{D}$ the corresponding Hadamard design. If H is a symmetric matrix with constant diagonal, then $\mathcal{D}$ possesses a symmetric incidence matrix with constant diagonal.

THEOREM 8. *There exists a* $(35, 17, 8)$ *Hadamard design having* S_8*—the symmetric group of degree* 8*—as full automorphism group. The group* S_8 *acts transitively on the set of points as well as on the set of blocks of this design.*

Proof. Let H be the Hadamard matrix constructed from the incidence matrix IM of Theorem 1; see the remarks below Definition 2. Then from H, by normalizing and deleting the first row and column and changing all entries -1 to 0, one obtains a $(35, 17, 8)$ design having S_8 as full automorphism group. To study the automorphism group, we have again used [5]. □

THEOREM 9. *There exists a strongly regular graph with parameters* $(35, 16, 6, 8)$ *having* S_8 *as full automorphism group. The group* S_8 *acts transitively on the set of vertices of this graph.*

Proof. This strongly regular graph is constructed from the design described in the previous theorem. An incidence matrix of that design is symmetric with 1 everywhere on the diagonal. □

THEOREM 10. *Let* $\mathcal{D}$ *be a Menon design with parameters* $(4u^2, 2u^2-u, u^2-u)$ *and* $\mathcal{D}_1$ *be a Menon design with parameters* $(16u^2, 8u^2-2u, 4u^2-2u)$ *obtained from* $\mathcal{D}$ *using the procedure described in Lemma* 4*. Further, let* $\mathcal{D}_2$ *be a Hadamard design with parameters* $(4u^2-1, 2u^2-1, u^2-1)$ *obtained from a Hadamard matrix corresponding to* $\mathcal{D}$ *by normalizing and deleting the* *first row and column and by changing the entries -1 to 0, and let $\mathcal{D}_3$ be a Hadamard design with parameters $(16u^2-1, 8u^2-1, 4u^2-1)$ obtained from a Hadamard matrix corresponding to $\mathcal{D}_1$. If G is an automorphism group of the design $\mathcal{D}_2$, then $G \times S_3$ is an automorphism group of the design $\mathcal{D}_3$.*

Proof. The assertion follows from the construction of the design $\mathcal{D}_3$. □

THEOREM 11. *There exists a Hadamard design with parameters $(4\cdot(3\cdot 2^k)^2-1, 2\cdot(3\cdot 2^k)^2-1, (3\cdot 2^k)^2-1)$ having S_8 as an automorphism group which possesses a symmetric incidence matrix with constant diagonal for all $k \in N$.*

Proof. These designs are obtained from Hadamard matrices belonging to Menon designs from Theorem 2. The assertion about the automorphism group follows from Theorem 10. □

THEOREM 12. *For each even positive integer k there exists a strongly regular graph with parameters $(4\cdot(3\cdot 2^k)^2-1, 2\cdot(3\cdot 2^k)^2-2, (3\cdot 2^k)^2-3, (3\cdot 2^k)^2-1)$ having S_8 as an automorphism group.*

Proof. If k is even, the incidence matrices of the Hadamard designs from Theorem 11 have entries 1 everywhere on the diagonal. □

THEOREM 13. *For each positive odd integer k there exists a strongly regular graph with parameters $(4\cdot(3\cdot 2^k)^2-1, 2\cdot(3\cdot 2^k)^2-1, (3\cdot 2^k)^2-1, (3\cdot 2^k)^2-1)$ having S_8 as an automorphism group.*

Proof. If k is odd, the incidence matrices of Hadamard designs from Theorem 11 are symmetric with 0 everywhere on the diagonal. □

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