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# BAER–LIKE DECOMPOSITIONS OF MODULES

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*On the 100th anniversary of the birthday of Reinhold Baer*

ABSTRACT. For certain artinian modules over group rings, we obtain the Baer decomposition, that is, a direct decomposition into two summands such that all chief factors of the first summand are $\mathcal{X}$–central and all chief factors of the second summand are $\mathcal{X}$–eccentric, for some formation $\mathcal{X}$ of finite groups.

## 1. Introduction

Let $\mathcal{X}$ be a class of groups. A factor $C/B$ of a group $G$ is said to be *$\mathcal{X}$–central* (respectively, *$\mathcal{X}$–eccentric*) if $G/C_G(C/B) \in \mathcal{X}$ (respectively, $G/C_G(C/B) \notin \mathcal{X}$). In general, $\mathcal{X}$–central and $\mathcal{X}$–eccentric factors of a group $G$ can appear at arbitrary points in a composition series of $G$. Thus the problem of finding cases in which all $\mathcal{X}$–central factors of a group or of a normal subgroup of this group can be gathered in one place, whereas all $\mathcal{X}$–eccentric factors can be gathered in another place, appears to be very interesting. Motivated by this problem, R. Baer [3] introduced, for a finite group $G$, two important subgroups to rule out the $\mathcal{X}$–centrality and the $\mathcal{X}$–eccentricity with respect to $G$. These subgroups are the *$\mathcal{X}$–hypercenter $HZ_{\mathcal{X}}(G)$ of $G$* and *the $\mathcal{X}$–hypereccenter $HE_{\mathcal{X}}(G)$ of $G$*, and are defined as follows. Every $G$–chief factor of the normal subgroup $HZ_{\mathcal{X}}(G)$ (respectively, $HE_{\mathcal{X}}(G)$) is $\mathcal{X}$–central (respectively, $\mathcal{X}$-eccentric) and $HZ_{\mathcal{X}}(G)$ (respectively, $HE_{\mathcal{X}}(G)$) is a maximal normal subgroup with respect to this property. Clearly $HZ_{\mathcal{X}}(G) \cap HE_{\mathcal{X}}(G) = \langle 1 \rangle$ always holds, but the direct decomposition $G = HZ_{\mathcal{X}}(G) \times HE_{\mathcal{X}}(G)$ usually fails. Baer himself obtained an important result in this direction, which we now quote. A finite group $G$ is called *$\mathcal{X}$–nilpotent* if $HZ_{\mathcal{X}}(G) = G$. Baer's result (see [3]) is the following.

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THEOREM. *Suppose that $\mathcal{X}$ is a local formation of finite groups and $A$ is a normal subgroup of a finite group $G$ such that $Q = G/C_G(A)$ is $\mathcal{X}$–nilpotent. Then $A = (A \cap HZ_{\mathcal{X}}(G)) \times (A \cap HE_{\mathcal{X}}(G)$.*

In passing we mention that under these assumptions the stronger restriction $G \in \mathcal{X}$ holds, since for a local formation $\mathcal{X}$, $\mathcal{X}$–nilpotency of $G$ implies that $G \in \mathcal{X}$ (see, for example, [4, Theorem IV.3.2]).

Later, P. Schmid [15] was able to extend Baer's theorem to finite groups with operators.

The direct decompositions, which are similar to Baer decompositions, have been the subject of much research in the theory of modules over group rings. We briefly mention some of the results. The first is the famous Fitting lemma, which can be formulated as follows.

THEOREM (H. Fitting). *Let $R$ be a ring, $G$ a finite nilpotent group and $A$ an $RG$–module having finite composition series. Then $A = C \oplus E$, where the $RG$–chief factors $U/V$ of $C$ (respectively, of $E$) satisfy $G = C_G(U/V)$ (respectively, $G \neq C_G(U/V)$.*

Other researchers studied the question whether the upper *RG*-hypercenter of a module $A$ is complemented, and considered generalizations to modules that are near modules of finite composition length. This question has found many applications in the study of groups and modules with finiteness conditions and is also connected with the existence of complements of some particular residuals in groups; for more details see the survey paper [6].

In this paper, we prove the existence of Baer decompositions for certain types of artinian modules. We consider infinite groups and artinian modules that are associated in some way with specific formations of groups.

We first develop the concepts necessary to state our results. All of these concepts are closely related to the concept of a $\mathcal{X}$–center, which is due to Baer himself [1]. If $G$ is a group and $x \in G$, we put $x^G = \{g^{-1}xg \mid g \in G\}$; clearly, $C_G(x^G)$ is normal in $G$. Now let $\mathcal{X}$ be a class of groups, and define *the $\mathcal{X}$–center* of $G$ by

$$\mathcal{X}C(G) = \{x \in G \mid G/C_G(x^G) \in \mathcal{X}\}.$$

If $\mathcal{X}$ is a formation of groups, then $\mathcal{X}C(G)$ is a characteristic subgroup of $G$ and $G$ is said to be *an $\mathcal{X}C$–group* if the equality $G = \mathcal{X}C(G)$ holds. If $\mathcal{X} = \mathcal{I}$ is the class of all identity groups, then $\mathcal{X}C(G) = \zeta(G)$ is the ordinary center of $G$, whereas if $\mathcal{X} = \mathcal{F}$ is the class of all finite groups, then $\mathcal{X}C(G) = FC(G)$ is precisely the $FC$–center of $G$ introduced by Baer. From this subgroup, we may construct *the upper $\mathcal{X}$–central series* of $G$ as

$$\langle 1 \rangle = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma,$$

where $C_1 = \mathcal{X}C(G)$, $C_{\alpha+1}/C_\alpha = \mathcal{X}C(G/C_\alpha)$, $\alpha < \gamma$, and $\mathcal{X}C(G/C_\gamma) = \langle 1 \rangle$. The last term $C_\gamma$ of this series is called *the upper $\mathcal{X}$–hypercenter of $G$* and denoted by $HZ_{\mathcal{X}}(G)$. If $G = C_\gamma$, then $G$ is said to be *$\mathcal{X}$–hypercentral*, and if $\gamma$ is finite, $G$ is called *$\mathcal{X}$–nilpotent*. Once again, if $\mathcal{X} = \mathcal{I}$ or $\mathcal{X} = \mathcal{F}$, $HZ_{\mathcal{X}}(G) = \zeta^\infty(G)$ and $FC^\infty(G)$ are called the upper hypercenter and upper $FC$–hypercenter of $G$, respectively.

Let $R$ be a ring, $G$ a group, $\mathcal{X}$ a class of groups and $A$ an $RG$–module. Recall that if $B \leq C$ are $RG$–submodules of $A$, the factor $C/B$ is said to be *$\mathcal{X}$–central* (respectively, *$\mathcal{X}$–eccentric*) if $G/C_G(C/B) \in \mathcal{X}$ (respectively, $G/C_G(C/B) \notin \mathcal{X}$). We then define

$$\mathcal{X}C_{RG}(A) = \{a \in A \mid G/C_G(aRG) \in \mathcal{X}\}.$$

If $\mathcal{X}$ is a formation of groups, then $\mathcal{X}C_{RG}(A)$ is an $RG$–submodule of $A$ called the *$\mathcal{X}$–center of* $A$ (more precisely, *the $\mathcal{X}$–$RG$–center of* $A$). Proceeding as in the case of groups, we construct the upper $\mathcal{X}$–central series of the module $A$ as

$$\langle 0\rangle = A_0 \leq A_1 \leq \cdots A_\alpha \leq A_{\alpha+1} \leq \cdots \leq A_\gamma,$$

where $A_1 = \mathcal{X}C_{RG}(A)$, $A_{\alpha+1}/A_\alpha = \mathcal{X}C_{RG}(A/A_\alpha)$ for all ordinals $\alpha < \gamma$, and $\mathcal{X}C_{RG}(A/A_\gamma) = \langle 0\rangle$. The last term $A_\gamma = HZ_{\mathcal{X}-RG}(A)$ of this series is called *the upper $\mathcal{X}$–hypercenter of* $A$ (or *the $\mathcal{X}$–$RG$–hypercenter*) and the other terms $A_\alpha$ are called *the $\mathcal{X}$–hypercenters of* $A$. If $A = A_\gamma$, then $A$ is said to be *$\mathcal{X}$–hypercentral* (*$\mathcal{X}$–nilpotent*, if $\gamma$ is finite). If $\mathcal{X} = \mathcal{I}$ and $\mathcal{X} = \mathcal{F}$, we have the $RG$–center $\zeta_{RG}(A)$ of $A$, the upper $RG$–hypercenter $\zeta_{RG}^\infty(A)$ of $A$, the $FC$–center $FC_{RG}(A)$ of $A$ and the upper $FC$–hypercenter $FC_{RG}^\infty(A)$ of $A$.

On the other hand, an $RG$–submodule $C$ of $A$ is said to be *$\mathcal{X}$–$RG$–hypereccentric* if it has an ascending series

$$\langle 0\rangle = C_0 \leq C_1 \leq \cdots C_\gamma \leq C_{\alpha+1} \leq \cdots C_\gamma = C$$

of $RG$–submodules of $A$ such that each factor $C_{\alpha+1}/C_\alpha$ is an $\mathcal{X}$–eccentric simple $RG$–module for every $\alpha < \gamma$.

We say that *the $RG$–module $A$ has the Baer decomposition for the formation $\mathcal{X}$* (the Baer *$\mathcal{X}$–decomposition* or, more precisely, the Baer *$\mathcal{X}$–$RG$–decomposition*) if the following equality holds:

$$A = HZ_{\mathcal{X}-RG}(A) \bigoplus HE_{\mathcal{X}-RG}(A),$$

where $HE_{\mathcal{X}-RG}(A)$ is the unique maximal $\mathcal{X}$–$RG$–hypereccentric $RG$–submodule of $A$. In fact, let $B$ be an $\mathcal{X}$–$RG$-hypereccentric $RG$–submodule of $A$ and $E = HE_{\mathcal{X}-RG}(A)$. If $(B+E)/E$ is non-zero, it contains a non-zero simple $RG$–submodule $U/E$. Since $(B+E)/E \cong B/(B\cap E)$, $U/E$ is $RG$–isomorphic to some simple $RG$–factor of $B$ and it follows that $G/C_G(U/E) \notin \mathcal{X}$. On the other hand, $(B+E)/E \leq A/E \cong HZ_{\mathcal{X}-RG}(A)$, that is, $G/C_G(U/E) \in \mathcal{X}$. This contradiction shows that $B \leq E$. Hence $HE_{\mathcal{X}-RG}(A)$ contains every $\mathcal{X}$–$RG$–hypereccentric $RG$–submodule and, in particular, is unique.

If $\mathcal{X} = \mathcal{I}$, the decomposition is simply called *the $\mathcal{Z}$–decomposition*, whereas if $\mathcal{X} = \mathcal{F}$, we call it *the $\mathcal{F}$–decomposition*.

In module theory modules having a finite composition series correspond closely to finite groups. For these modules we can obtain the following result, which is analogous to Baer's theorem.

Theorem A. *Let $\mathcal{X}$ be a formation of finite groups, $G$ an $\mathcal{X}$–hypercentral group, $D$ a Dedekind domain and $A$ a $DG$–module. If $A$ has a finite $DG$–composition series, then $A$ has the Baer $\mathcal{X}$–decomposition.*

Since $G$ is $FC$–hypercentral by [19, Corollary 1 of Theorem 2], $A = B \oplus C$ has the $\mathcal{F}$–decomposition, where $B = HZ_{\mathcal{F}-DG}(A)$ and $C = HE_{\mathcal{F}-DG}(A)$. (In fact, in [19], only the case $D = \mathbb{Z}$ was the treated, but the proof remains valid for an arbitrary Dedekind domain $D$.) In particular, $C \leq HE_{\mathcal{X}-DG}(A)$. Since $B$ has a finite composition series every factor of which is $\mathcal{F}$–central, $G/C_G(B)$ is finite. By [12], $B$ has the Baer $\mathcal{X}$–decomposition and therefore $A$ also has the Baer $\mathcal{X}$–decomposition.

Note that in this result $\mathcal{X}$ is an arbitrary formation of finite groups.

It is natural to raise the question of the existence of the Baer $\mathcal{X}$–decomposition for artinian modules. If $\mathcal{X} = \mathcal{I}$, this question was solved by D. I. Zaitsev [18]. Another important formation is the formation $\mathcal{X} = \mathcal{F}$. For artinian $DG$–modules, the case $\mathcal{X} = \mathcal{F}$ was considered by Zaitsev [20] for hyperfinite groups and by Duan [5] for groups having an ascending series of normal subgroups with finite or infinite cyclic factors (again, Zaitsev and Duan only considered the case $D = \mathbb{Z}$). The most recent result in this direction was obtained by Kurdachenko, Petrenko and Subbotin [10]. The existence of the $\mathcal{F}$–decomposition is an important special case and led to the solution of the above question for many important formations $\mathcal{X}$.

We now restrict ourselves to two special cases, the case $\mathcal{F} \leq \mathcal{X}$, and the case when $\mathcal{X}$ is a proper formation of finite groups.

A formation $\mathcal{X}$ is said to be *overfinite* if it satisfies the following conditions:

(i) If $G \in \mathcal{X}$ and $H$ is a normal subgroup of $G$ of finite index, then $H \in \mathcal{X}$.
(ii) If $G$ is a group, $H$ is a normal subgroup of finite index of $G$ and $H \in \mathcal{X}$, then $G \in \mathcal{X}$.
(iii) $\mathcal{I} \leq \mathcal{X}$.

Clearly, an overfinite formation always contains $\mathcal{F}$. The most important examples of these formations are polycyclic groups, Chernikov groups, soluble minimax groups, soluble groups of finite special rank and soluble groups of finite abelian section rank. For locally soluble $FC$–hypercentral groups, the existence of the Baer decomposition for an overfinite formation $\mathcal{X}$ in an artinian $DG$–module $A$ was also proved by Kurdachenko, Petrenko and Subbotin [11].

Since every overfinite formation $\mathcal{X}$ contains $\mathcal{F}$, every $FC$–hypercentral group is likewise $\mathcal{X}$–hypercentral.

A formation $\mathcal{X}$ of finite groups is said to be *infinitely hereditary for a class of groups* $\mathcal{Y}$ if it satisfies the following condition:

(IH) Whenever a $\mathcal{Y}$–group $G$ belongs to the class $\mathbf{R}\mathcal{X}$, then every finite factor group of $G$ belongs to $\mathcal{X}$.

Many important formations of finite groups are infinitely hereditary for the class of $FC$–hypercentral groups, for example:

(1) $\mathcal{A} \cap \mathcal{F}$, finite abelian groups,
(2) $\mathcal{N}_c \cap \mathcal{F}$, finite nilpotent groups of class at most $c$,
(3) $\mathcal{S}_d \cap \mathcal{F}$, finite soluble groups of derived length at most $d$,
(4) $\mathcal{S} \cap \mathcal{F}$, finite soluble groups,
(5) $\mathcal{B}(n) \cap \mathcal{F}$, finite groups of exponent dividing $n$.

Moreover, these five examples, and the examples

(6) $\mathcal{N} \cap \mathcal{F}$, finite nilpotent groups,
(7) $\mathcal{U} \cap \mathcal{F}$, finite supersoluble groups,

are infinitely hereditary for the classes of $FC$–groups and hyperfinite groups.

In the present paper we study the question of the existence of the Baer decomposition in artinian modules for infinitely hereditary formations of finite groups. Our main results are as follows:

Theorem B. *Let $D$ be a Dedekind domain, $\mathcal{X}$ a formation of finite groups, $G$ an infinite $\mathcal{X}$–hypercentral group, $A$ an artinian $DG$–module. If $\mathcal{X}$ is infinitely hereditary for the class of $FC$–hypercentral groups, then $A$ has the Baer decomposition for the formation $\mathcal{X}$.*

Corollary B1. *Let $D$ be a Dedekind domain, $\mathcal{X}$ a formation of finite groups, $G$ an infinite $\mathcal{X}$–hypercentral group, $A$ an artinian $DG$–module. Then $A$ has the Baer decomposition for the formations $\mathcal{A}\cap\mathcal{F}$, $\mathcal{N}_c\cap\mathcal{F}$, $\mathcal{S}_d\cap\mathcal{F}$, $\mathcal{S}\cap\mathcal{F}$ and $\mathcal{B}(n) \cap \mathcal{F}$.*

Corollary B2. *Let $D$ be a Dedekind domain, $\mathcal{X}$ a formation of finite groups, $G$ an infinite $\mathcal{X}$–hypercentral group, $A$ an artinian $DG$–module. If $G$ is an $FC$–group, then $A$ has the Baer decomposition for the formations $\mathcal{A}\cap\mathcal{F}$, $\mathcal{N}_c \cap \mathcal{F}$, $\mathcal{S}_d \cap \mathcal{F}$, $\mathcal{S} \cap \mathcal{F}$, $\mathcal{B}(n) \cap \mathcal{F}$, $\mathcal{N} \cap \mathcal{F}$ and $\mathcal{U} \cap \mathcal{F}$.*

Corollary B3. *Let $D$ be a Dedekind domain, $\mathcal{X}$ a formation of finite groups, $G$ an infinite $\mathcal{X}$–hypercentral group, $A$ an artinian $DG$–module. If $G$ is a hyperfinite group, then $A$ has the Baer decomposition for the formations $\mathcal{A} \cap \mathcal{F}$, $\mathcal{N}_c \cap \mathcal{F}$, $\mathcal{S}_d \cap \mathcal{F}$, $\mathcal{S} \cap \mathcal{F}$, $\mathcal{B}(n) \cap \mathcal{F}$, $\mathcal{N} \cap \mathcal{F}$ and $\mathcal{U} \cap \mathcal{F}$.*

Corollary B4. *Let $D$ be a Dedekind domain, $\mathcal{X}$ a formation of finite groups, $G$ an infinite $\mathcal{X}$–hypercentral group, $A$ an artinian $DG$–module. If $G$*

*is a Chernikov group, then $A$ has the Baer decomposition for the formation $\mathcal{X}$.*

THEOREM C. *Let $D$ be a Dedekind domain, $\mathcal{X}$ a formation of finite groups, $G$ an infinite $\mathcal{X}$–hypercentral group, $A$ an artinian $DG$–module. If $G$ is finitely generated, then $A$ has the Baer decomposition for the formation $\mathcal{X}$.*

The other classical generalization of modules with finite composition series are the noetherian modules. Here the situation is quite different: we cannot expect to find direct decompositions for a noetherian module, as the following example shows. Let $A = \langle u\rangle \times \langle v\rangle$ be a free abelian group of rank 2. We construct the split extension $G$ of $A$ by a finite cyclic group $\langle g\rangle$ of order 3, where the action is given by: $u^g = v$ and $v^g = u^{-1}v^{-1}$. Then every non-identity $G$–invariant subgroup of $A$ has finite index. In particular, the $\mathbb{Z}\langle g\rangle$–module $A$ is noetherian. However, $A$ is directly indecomposable and has central and non-central $G$–chief factors.

In spite of this, D. J. S. Robinson [14] was able to obtain the best result known, which gives a weak form of the $\mathbb{Z}$–$RG$–decomposition:

THEOREM (Robinson). *If $R$ is a commutative ring, $G$ a nilpotent group, $W$ the augmentation ideal of the group ring $RG$ and $A$ a noetherian $RG$-module, then the lower $RG$–central series $\{A_\alpha \mid A_\alpha = AW^\alpha\}$ terminates at the first infinite ordinal $\omega$ and there is some positive integer $n$ such that $AW^n \cap \zeta^\infty_{RG}(A) = \langle 0\rangle$.*

## 2. Some preliminary results

We list some elementary properties of $\mathcal{X}$–hypercentral groups.

LEMMA 2.1. *Let $\mathcal{X}$ be a formation of groups and $G$ a finitely generated $\mathcal{X}C$–group. Then $G$ is a central–by–$\mathcal{X}$–group.*

*Proof.* Indeed, let $G = \langle g_1, \dots, g_s\rangle$. Then each $G/C_G(\langle g_i\rangle^G) \in \mathcal{X}$, $1 \le i \le s$. Since $Z = C_G(\langle g_1\rangle^G) \cap \dots \cap C_G(\langle g_s\rangle^G) \le \zeta(G)$ and $\mathcal{X}$ is a formation, $G/Z \in \mathcal{X}$ and therefore $G/\zeta(G) \in \mathcal{X}$. □

LEMMA 2.2. *Let $\mathcal{X}$ be a formation of groups, $G$ a group, $P = \mathcal{X}C(G)$, $Q = HZ_{\mathcal{X}}(G)$, $H$ a $G$–invariant subgroup of $Q$. If $H \ne \langle 1\rangle$, then $H \cap P \ne \langle 1\rangle$.*

*Proof.* Let $\langle 1\rangle = C_0 \le C_1 \le \dots \le C_\alpha \le C_{\alpha+1} \le \dots C_\gamma$ be the upper $\mathcal{X}$–central series of $G$. Let $\beta$ be the least ordinal such that $H \cap C_\beta \ne \langle 1\rangle$. Clearly $\beta$ is not a limit ordinal, so that $H \cap C_{\beta-1} = \langle 1\rangle$. If $1 \ne y \in H \cap C_\beta$ and $Y = \langle y\rangle^G$, since $yC_{\beta-1} \in C_\beta/C_{\beta-1}$, we have $G/C_G(YC_{\beta-1}/C_{\beta-1}) \in \mathcal{X}$. Let $g \in C_G(YC_{\beta-1}/C_{\beta-1})$. Then $[g, Y] \le C_{\beta-1}$. On the other hand, $H$ is a normal subgroup, so that $[g, Y] \le H$, that is, $[g, Y] \le H \cap C_{\beta-1} = \langle 1\rangle$. This means that $C_G(YC_{\beta-1}/C_{\beta-1}) \leq C_G(Y)$ and, in particular, $G/C_G(Y) \in \mathcal{X}$, because $\mathcal{X}$ is a formation. Hence $Y \leq \mathcal{X}C(G)$. □

Corollary 2.3. *Let $\mathcal{X}$ be a formation of groups, $G$ a $\mathcal{X}$–hypercentral group. If $H$ is a non-identity normal subgroup of $G$, then $H \cap \mathcal{X}C(G) \neq \langle 1\rangle$.*

Lemma 2.4. *Let $\mathcal{X}$ be a formation of finite groups, $G$ a finitely generated $\mathcal{X}$–nilpotent group. Then $G$ is a nilpotent–by–$\mathcal{X}$–group. In particular, $G$ is polycyclic–by–finite.*

*Proof.* Let $\langle 1\rangle = F_0 \leq F_1 \leq \cdots \leq F_n = G$ be the upper $\mathcal{X}$–central series of $G$. It suffices to show that $G/C_G(F_{i+1}/F_i) \in \mathcal{X}$, for every $0 \leq i \leq n-1$. We proceed by induction on $n$. If $n = 1$, then $G$ is an $\mathcal{X}C$–group and it suffices to apply Lemma 2.1.

Now let $n > 1$ and assume that we have already proved $G/C_G(F_{i+1}/F_i) \in \mathcal{X}$, for $n > i \geq 1$. Let $H = C_G(F_2/F_1) \cap \cdots \cap C_G(F_n/F_{n-1})$. Then $G/H \in \mathcal{X}$ and $H/F_1$ is nilpotent (see [7, Theorem 1.C.1]). In particular, $G/F_1$ is finitely presented (see [13, Corollary 1.43]) and it follows that $F_1 = \langle g_1\rangle^G \cdots \langle g_s\rangle^G$, where $g_1, \ldots, g_s \in \mathcal{X}C(G)$ ([13, Corollary 1.43]). Put $U = C_G(\langle g_1\rangle^G) \cap \cdots \cap C_G(\langle g_s\rangle^G)$. Then $G/U \in \mathcal{X}$ and $U = C_G(F_1)$. If $C = H \cap U$, $G/C \in \mathcal{X}$ and $C$ is nilpotent (see [7, Theorem 1.C.1]). □

Lemma 2.5. *Let $\mathcal{X}$ be a formation of finite groups, $G$ a finitely generated $\mathcal{X}$–hypercentral group. Then $G$ is a nilpotent–by–$\mathcal{X}$–group.*

*Proof.* Let $\langle 1\rangle = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = G$ be the upper $\mathcal{X}$–central series of $G$. Let $\Sigma = \{\alpha \mid G/C_\alpha \text{ is nilpotent-by-}\mathcal{X}\}$. Since $G = \bigcup_{\alpha\leq\gamma} C_\alpha$, we have $\Sigma \neq \emptyset$. Let $\beta$ be the least element of $\Sigma$. If $\beta = 0$, then $G$ is a nilpotent–by–$\mathcal{X}$–group and we are done.

Suppose that $\beta > 0$. First consider the case when $\beta$ is not a limit ordinal. Since $C_\beta/C_{\beta-1}$ is the $\mathcal{X}C$–center of $G/C_{\beta-1}$, $G/C_{\beta-1}$ is $\mathcal{X}$–nilpotent. By Lemma 2.4, $G/C_{\beta-1}$ is a nilpotent–by–$\mathcal{X}$–group. But this contradicts the choice of $\beta$. Hence $\beta$ has to be a limit ordinal. Since $G/C_\beta$ is a nilpotent–by–$\mathcal{X}$–group, and, in particular, nilpotent–by–finite, it is finitely presented. By [13, Corollary 1.43], $C_\beta = \langle g_1\rangle^G \cdots \langle g_s\rangle^G$, where $g_1, \ldots, g_s \in C_\beta$. It follows that there is an ordinal $\delta < \beta$ such that $g_1, \ldots, g_s \in C_\delta$, which yields $C_\beta = C_\delta$, a contradiction. □

Corollary 2.6. *Let $\mathcal{X}$ be a formation of finite groups such that $\mathcal{X} = \mathbf{S}\mathcal{X}$, $G$ an $\mathcal{X}$–hypercentral group, $H$ a finitely generated subgroup of $H$. Then $H$ is a nilpotent–by–$\mathcal{X}$–group. In particular, $G$ is locally (polycyclic–by–finite).*

Lemma 2.7. *Let $\mathcal{X}$ be a formation of groups such that $\mathcal{X} = \mathbf{S}_n\mathcal{X}$, $G$ a group, $g_1, \ldots, g_s \in \mathcal{X}C(G)$, $H = \langle g_1\rangle^G \cdots \langle g_s\rangle^G$. Then $H$ is a central–by–$\mathcal{X}$–group.*

*Proof.* Indeed, since $C_G(\langle g_1\rangle^G) \cap \cdots \cap C_G(\langle g_s\rangle^G) \cap H \leq \zeta(H)$,

$$H/\left(\left(\bigcap_{i=1}^{s} C_G(\langle g_i\rangle^G)\right) \cap H\right) \leq \bigtimes_{i=1}^{s} H/(H \cap C_G(\langle g_i\rangle^G))$$

and

$$H/(H \cap C_G(\langle g_i\rangle^G)) \cong HC_G(\langle g_i\rangle^G)/C_G(\langle g_i\rangle^G) \leq G/C_G(\langle g_i\rangle^G),$$

we obtain that $H/\zeta(H) \in \mathcal{X}$. □

COROLLARY 2.8. *Let $\mathcal{X}$ be a formation of locally finite groups such that $\mathcal{X} = \mathbf{S}_n\mathcal{X}$, and let $G$ be an $\mathcal{X}C$–group. Then:*

1. *The derived subgroup $[G, G]$ is locally finite.*
2. *If $G$ is torsion-free, then $G$ is abelian.*
3. *If $H$ is a normal torsion-free subgroup, then $H \leq \zeta(G)$.*
4. *If $g \in G$, $L = \langle g\rangle^G$, then either $L$ is locally finite or $L$ contains a $G$–invariant locally finite subgroup $T$ such that $L/T$ is an infinite cyclic group.*

*Proof.* (1) Let $g_1, \ldots, g_s \in G$, $H = \langle g_1\rangle^G \cdots \langle g_s\rangle^G$. By Lemma 2.7, $H$ is a central–by–$\mathcal{X}$–group. By [13, Corollary to Theorem 4.12], $[H, H]$ is locally finite. It follows that $[G, G]$ is also locally finite.

(2) and (3) are obvious.

(4) By (1), the subgroup $[L, L]$ is locally finite. All elements of finite order in $L/[L, L]$ form a characteristic subgroup $T/[L, L]$, so that $T$ is a $G$–invariant subgroup of $L$ and $L/T$ is torsion-free abelian or $T = L$. In the first case we have $L/T \leq \zeta(G/T)$ by (3). But $L/T = \langle gT\rangle^{G/T}$, so $L/T = \langle gT\rangle$. □

COROLLARY 2.9. *Let $G$ be a group, $g_1, \ldots, g_s \in FC(G)$, $H = \langle g_1\rangle^G \cdots \langle g_s\rangle^G$. Then:*

1. *$H$ is a finitely generated subgroup.*
2. *$H$ is central-by-finite.*
3. *Either $H$ is finite or $H$ has a $G$–invariant finite subgroup $T$ such that $H/T$ is a finitely generated free abelian group.*

*Proof.* Since each $g_i \in FC(G)$, $G/C_G(\langle g_i\rangle^G)$ is finite; in particular, $g_i$ has only finitely many conjugates in $G$, $1 \leq i \leq s$. Since $H = \langle g_i^x \mid x \in G,\ 1 \leq i \leq s\rangle$, $H$ is a finitely generated subgroup. By Lemma 2.7, the subgroup $H$ is central–by–finite. By [13, Corollary to Theorem 4.12], the derived subgroup $[H, H]$ is finite. Statement (3) follows from Corollary 2.8. □

## 3. On the existence of the relational $Z$–decomposition

In this section, we consider the question of the existence of the $Z$–$RH$–decomposition in artinian modules over a group ring $RG$ for a hypercentral normal subgroup H (which we simply call *the relational* $Z$*–decomposition*). The results of this section play a very important role in the proof of Theorem B.

Lemma 3.1. *Let $G$ be a group, $Y$ and $H$ normal subgroups of $G$ such that $H \leq Y$ and $G/Y$ is finite, $R$ a ring, $A$ an $RG$–module. Suppose that $B/C$ is a chief $RG$–factor of $A$ such that $\zeta_{RH}(B/C) = \langle 0\rangle$. If $U/V$ is a chief $RY$–factor such that $C \leq V \leq U \leq B$, then $\zeta_{RH}(U/V) = \langle 0\rangle$.*

*Proof.* The $RG$–submodule $B$ contains an $RY$–submodule $E \geq C$ such that $E/C$ is a simple $RY$–module and $B/C = \oplus_{i=1}^{n}(E/C)g_i$ for some elements $g_1, \dots, g_n$ ([17, Lemma]). If we assume that $C_H(E/C) = H$, then from $C_H((E/C)g_i) = g_i^{-1}C_H(E/C)g_i = g_i^{-1}Hg_i = H$ we deduce that $H = C_H(B/C)$, a contradiction. Since $(E/C)g_i$ is a simple $RY$–module and $H$ is a normal subgroup of $Y$, $\zeta_{RH}((E/C)g_i)$ is an $RY$–submodule and hence $\zeta_{RH}((E/C)g_i) = \langle 0\rangle$ for every $i$, $1 \leq i \leq n$. Since the $RY$–module $B/C$ is semisimple, a chief $RY$–factor $U/V$ is $RY$–isomorphic with some $(E/C)g_i$. It follows that $\zeta_{RH}(U/V) = \zeta_{RH}((E/C)g_i) = \langle 0\rangle$. □

Lemma 3.2. *Let $G$ be a group, $H$ a normal subgroup of $G$, $\mathcal{X}$ a formation of groups, $R$ a ring, $A$ an $RG$–module. Suppose that $A$ contains an $RH$–submodule $B$ satisfying the following conditions:*

1. *Every non-zero $RH$–factor of $B$ is $\mathcal{X}$–eccentric.*
2. *$HZ_{\mathcal{X}-RG}(A/B) = A/B$.*

*Then $B$ is an $RG$–submodule of $A$.*

The proof of this lemma is obvious.

Lemma 3.3. *Let $G$ be an $FC$–hypercentral group, $H$ a normal hypercentral subgroup of $G$, $R$ a ring, $A$ an artinian $RG$–module. Suppose that $A$ contains an $RH$–submodule $B$ satisfying the following conditions:*

1. *If $U/V$ is a non-zero $RH$–factor of $B$, then $C_H(U/V) \neq H$.*
2. *$C_H(A/B) = H$.*

*Then $B$ is an $RG$–submodule of $A$ and there is an $RG$–submodule $M$ such that $A = B \oplus M$.*

*Proof.* By Lemma 3.2, $B$ is an $RG$–submodule. Put

$$\Sigma = \{C \mid C \text{ is an } RG\text{–submodule such that } A = B + C\}.$$

Clearly $\Sigma \neq \emptyset$ since $A \in \Sigma$. Since $A$ is an artinian $RG$–module, $\Sigma$ has a minimal element $M$. We may assume that $C_G(M) = \langle 1\rangle$. Suppose that $M_1 = M \cap B \neq \langle 0\rangle$. By Corollary 2.3, $\zeta(H) \cap FC(G) \neq \langle 1\rangle$. Let $1 \neq x \in \zeta(H) \cap FC(G)$, $X = \langle x\rangle^G$. Then the subgroup $Y = C_G(X)$ has finite index in $G$ and $H \leq Y$. In particular, $X \leq \zeta(Y)$. Put

$$\Sigma_1 = \{S \mid S \text{ is an } RY\text{–submodule such that } M = M_1 + S\}.$$

Clearly, $\Sigma_1 \neq \emptyset$. By a theorem by Wilson [16], $A$ is an artinian $RY$–module, so that $\Sigma_1$ has a minimal element $U$. Put $U_1 = M_1 \cap U$. If $z \in X$, then $z \in \zeta(H)$, so that the mapping $\phi_z : u \mapsto u(z-1)$, $u \in U$, is an $RY$–endomorphism. Therefore $U(z-1)/U_1(z-1)$ is an $RY$–epimorphic image of

$$\begin{aligned} U/U_1 = U/(M_1 \cap U) &\cong_{RY} (U + M_1)/M_1 = M/M_1 \\ = M/(M \cap B) &\cong_{RY} (M + B)/B = A/B. \end{aligned}$$

It follows that $H = C_H(U(z-1)/U_1(z-1))$. On the other hand, $A(z-1) \leq B$ by condition (2). Since $U(z-1)$ and $U_1(z-1)$ are $RY$–submodules, they are also $RH$–submodules. So if we assume that $U(z-1) \neq U_1(z-1)$, then, by condition (1), $H \neq C_H(U(z-1)/U_1(z-1))$. This contradiction shows that $U(z-1) = U_1(z-1)$. In this case $U = U_1 + C_U(z)$ and hence $M = M_1 + U = M_1 + U_1 + C_U(z) = M_1 + C_U(z)$, which implies that $C_U(z) \in \Sigma_1$. By the choice of $U$ we have $U = C_U(z)$. Since this is true for every $z \in X$, we obtain $U = C_U(X)$, and, in particular, $U \leq C_M(X)$. Note that $C_M(X)$ is an $RG$–submodule since $X$ is normal in $G$. Thus $A = B + M = B + M_1 + U = B + C_M(X)$, and by the choice of $M$ we conclude that $M = C_M(X)$. Hence $X \leq C_G(M) = \langle 1 \rangle$, a contradiction. Therefore $M \cap B = \langle 0 \rangle$ and hence $A = B \oplus M$. □

LEMMA 3.4. *Let $G$ be an $FC$–hypercentral group, $H$ a normal hypercentral subgroup of $G$, $D$ a Dedekind domain, $A$ an artinian $DG$–module. Suppose that $A$ contains a $DG$–submodule $B$ satisfying the following conditions:*

(1) $B \leq \zeta^{\infty}_{DH}(A)$.
(2) *$A/B$ is a simple $DG$–module.*
(3) $C_H(A/B) \neq H$.

*Then there exists a $DG$–submodule $M$ such that $A = B \oplus M$.*

*Proof.* Let $a \in A \backslash B$, $A_1 = aDG$. It suffices to show that $A_1 = (A_1 \cap B) \oplus M$ for some $DG$–submodule $M$, for then $A = A_1 + B = ((A_1 \cap B) \oplus M) + B = B \oplus M$, as required. In other words, we may assume that $A$ can be generated by any element $a \in A \setminus B$.

Put

$$\Sigma = \{C \mid C \text{ is a } DG\text{–submodule such that } A = B + C\}.$$

Since $A \in \Sigma$, $\Sigma \neq \emptyset$. Since $A$ is an artinian $DG$–module, $\Sigma$ has a minimal element $M$. We may assume that $C_G(M) = \langle 1 \rangle$. Suppose that $M_1 = M \cap B \neq \langle 0 \rangle$. By Corollary 2.3, $\zeta(H) \cap FC(G) \neq \langle 1 \rangle$. Let $1 \neq x \in \zeta(H) \cap FC(G)$,

$X = \langle x\rangle^G$. Then the subgroup $Y = C_G(X)$ has finite index in $G$ and we have $H \le Y$ and, in particular, $X \le \zeta(Y)$. Since

$$M/M_1 = M/(M \cap B) \cong (M + B)/B = A/B$$

is a simple $DG$/-module, $M$ contains a $DY$–submodule $U \ge M_1$ such that $U/M_1$ is a simple $DY$–module and, moreover, $M/M_1 = \oplus_{i=1}^{n}(U/M_1)g_i$, for some $g_1, \dots, g_n$ ( [17, Lemma]). By Lemma 3.1, $C_H(U/M_1) \ne H$. Put

$$\Sigma_1 = \{S \mid S \text{ is a } DY\text{–submodule such that } U = M_1 + S\}.$$

Obviously, $\Sigma_1 \ne \emptyset$. By a theorem of Wilson [16], $A$ is an artinian $RY$–module and hence $\Sigma_1$ has a minimal element $Q$. Put $Q_1 = M_1 \cap Q = M \cap Q$. Let $R/S$ be a chief $DY$–factor of $M_1$. Since $H$ is a normal subgroup of $G$, $\zeta_{DH}(R/S)$ is a $DY$–submodule of $R/S$. The inclusion $M_1 \le \zeta^{\infty}_{DH}(A)$ implies that $\zeta_{DH}(R/S) \ne \langle 0\rangle$. It follows that $C_H(R/S) = H$. From

$$Q/Q_1 = Q/(M_1 \cap Q) \cong_{DY} (Q + M_1)/M_1 = U/M_1$$

we deduce that $Q/Q_1$ is a simple $DY$–module. Then either $\zeta_{DH}(Q/Q_1) = Q/Q_1$ or $\zeta_{DH}(Q/Q_1) = \langle 0\rangle$. If we assume that $H \le C_Y(Q/Q_1)$, then also $H \le C_Y(U/M_1)$. But we have already proved that this is impossible. This contradiction shows that $C_Y(Q/Q_1)$ does not contain $H$.

If $Q_1 = \langle 0\rangle$, then we consider the $DG$–submodule $QDG$. In this case $Q$ is a simple $DY$–submodule, so $QDG$ is a semisimple $DY$–submodule, that is, $QDG = Qw_1 \oplus \cdots \oplus Qw_m$ for some elements $w_1, \dots, w_m \in G$. If $QDG \cap B \ne \langle 0\rangle$, then it contains a simple $DY$–submodule $V$ and so there is an index $j$, $1 \le j \le m$, such that $V \cong_{DY} Qw_j$ and, in particular, $C_Y(V) = C_Y(Qw_j)$. However, $C_Y(Qw_j) = w_j^{-1}C_Y(Q)w_j$. Since $Q \cong_{DY} U/M_1$ and $C_H(U/M_1) \ne H$, we obtain $C_H(Q) \ne H$ and $H = w_j^{-1}Hw_j \ne w_j^{-1}C_H(Q)w_j = C_H(Qw_j) = C_H(V)$. Since $V$ is a $DY$–submodule and $H$ is normal in $Y$, $C_V(H)$ is a $DY$–submodule. It follows that $C_V(H) = \langle 0\rangle$. On the other hand, $B \le \zeta^{\infty}_{DH}(A)$ and, in particular, $C_V(H) \ne \langle 0\rangle$. This contradiction shows that $QDG \cap B = \langle 0\rangle$. Since $A/B$ is a simple $DG$–module, $QDG + B = A$. But the choice of $M$ implies that $QDG = M$ and we assumed that $M \cap B \ne \langle 0\rangle$, so we arrive at a contradiction.

Thus we may assume that $Q_1 \ne \langle 0\rangle$. Note that $Q$ can be generated by any element $a \in Q \setminus M_1$. If $z \in X$, then $z \in \zeta(Y)$. By [10, Lemma 14], $Q(z-1) \ne Q$. Since $Q$ is a minimal element of $\Sigma_1$, $Q(z-1) \notin \Sigma_1$. This means that $Q(z-1) + M_1 \ne U$. It follows that $Q(z-1) \le M_1$ because the $DY$–module $U/M_1$ is simple. Since $z \in \zeta(Y)$, the mapping $\phi_z : u \mapsto u(z-1)$, $u \in U$, is a $DY$–endomorphism. Thus the factor $Q(z-1)/Q_1(z-1)$ is a $DY$–epimorphic image of $Q/Q_1$. Since $Q/Q_1$ is a simple $DY$–module, we have either $Q(z-1)/Q_1(z-1) \cong_{DY} Q/Q_1$ or $Q(z-1) = Q_1(z-1)$. We have already proved that $C_Y(Q/Q_1)$ does not contain $H$ and so, in the first case, $C_Y(Q(z-1)/Q_1(z-1))$ does not contain $H$. On the other hand,

$Q(z-1) \leq M_1$ and we have already shown that the centralizer of every chief $DY$–factor of $M_1$ contains $H$. So the first case is impossible and we therefore have $Q(z-1) = Q_1(z-1)$. In this case $Q = Q_1 + C_Q(z)$. Since $z \in \zeta(Y)$, $C_Q(z)$ is a $DY$–submodule of $Q$ and thus

$$U = M_1 + Q = M_1 + Q_1 + C_Q(z) = M_1 + C_Q(z),$$

which yields $C_Q(z) \in \Sigma_1$. By the choice of $Q$ we have $Q = C_Q(z)$. Since this is true for every $z \in X$, we have $Q = C_Q(X)$ and, in particular, $Q \leq C_M(X)$. Since $U \neq M_1$, we deduce that $M_1$ cannot contain $C_M(X)$. Since $M/M_1$ is a simple $DG$–module and $X$ is normal in $G$, $M_1 + C_M(X) = M$. Therefore $A = B + M = B + M_1 + C_M(X) = B + C_M(X)$ and, by the choice of $M$, we deduce that $M = C_M(X)$. Hence $X \leq C_G(M) = \langle 1 \rangle$, a contradiction which shows that $M \cap B = \langle 0 \rangle$. Therefore $A = B \oplus M$, as required. □

Theorem 3.5. *Let $G$ be an $FC$–hypercentral group, $H$ a normal hypercentral subgroup of $G$, $D$ a Dedekind domain, $A$ an artinian $DG$–module. Then $A$ has the $Z$–$DH$–decomposition.*

*Proof.* Suppose that the result is false. If $\Sigma$ is the family of all $DG$–submodules $B$ of $A$ that do not have the $Z$–$DH$–decomposition, then clearly $\Sigma \neq \emptyset$. Since $A$ is an artinian $DG$–module, $\Sigma$ has a minimal element $C$. By [11, Corollary 3 to Lemma 2], $C$ has a unique maximal $DG$–submodule $M$ having the $Z$–$DH$–decomposition. By the choice of $C$, $M$ must contain every proper $DG$–submodule. In particular, $M$ is a maximal $DG$–submodule of $C$.

Let $M = M_1 \oplus M_2$, where $M_1 = \zeta^{\infty}_{DH}(M)$ is the corresponding hypercenter and $M_2 = \zeta^{*}_{DH}(M)$ the hypereccenter. Since $H$ is a normal subgroup of $G$, $\zeta_{DH}(C/M)$ is a $DG$–submodule of $C/M$. Then either $\zeta_{DH}(C/M) = C/M$ or $\zeta_{DH}(C/M) = \langle 0 \rangle$. Suppose first that $\zeta_{DH}(C/M) = C/M$ and consider the factor module $C/M_1$. By Lemma 3.3, there exists a $DG$–submodule $M_3/M_1$ such that $C/M_1 = M/M_1 \oplus M_3/M_1$. In this case $M_3 = \zeta^{\infty}_{DH}(C)$ and so $C = M_3 \oplus M_2$. Let $\zeta_{DH}(C/M) = \langle 0 \rangle$. By Lemma 3.4, there exists a $DG$–submodule $M_4/M_2$ such that $C/M_2 = M/M_2 \oplus M_4/M_2$ and we arrive at $M_4 = \zeta^{*}_{DH}(C)$. Again $C = M_1 \oplus M_4$. □

If $D = \mathbb{Z}$ and $G = H$, we obtain [18, Theorem 1’].

## 4. Proof of main results

*Proof of Theorem B.* Since $G$ is infinite, $G \notin \mathcal{X}$. Suppose that $A$ does not have the Baer $\mathcal{X}$-$DG$-decomposition. Thus if $\Sigma$ is the family of all $DG$–submodules $B$ of $A$ that do not have the $\mathcal{X}$-$DG$-decomposition, then clearly $\Sigma \neq \emptyset$. Since $A$ is an artinian $DG$–module, $\Sigma$ has a minimal element $C$. By [11, Corollary 4], $C$ contains a unique maximal $DG$–submodule $M$ having the $\mathcal{X}$–$DG$–decomposition. By the choice of $C$ we see that $M$ contains every proper $DG$–submodule of $C$ and so $M$ is a maximal $DG$–submodule of $C$.

Let $M = M_1 \oplus M_2$, where $M_1 = HZ_{\mathcal{X}-DG}(M)$ and $M_2 = HE_{\mathcal{X}-DG}(M)$. Suppose first that $G/C_G(C/M) \notin \mathcal{X}$ and consider the factor module $C/M_2$. In other words, we may assume that $M = HZ_{\mathcal{X}-DG}(M)$. By [10, Theorem 3], $C = HZ_{\mathcal{F}-DG}(C) \oplus HE_{\mathcal{F}-DG}(C)$. Since $C$ is indecomposable and $M \leq HZ_{\mathcal{F}-DG}(C)$, $C = HZ_{\mathcal{F}-DG}(C)$. In particular, the factor module $C/M$ is finite. Let $S$ be the $\mathcal{X}$–residual of $G$. If $S = \langle 1 \rangle$, then $G \in \mathbf{R}\mathcal{X}$. Since $\mathcal{X}$ is infinitely hereditary for the class of $FC$–hypercentral groups, the finite factor group $G/C_G(C/M)$ belongs to $\mathcal{X}$. This contradiction shows that this case cannot occur and so $S \neq \langle 1 \rangle$.

The group $G$ has an ascending series of normal subgroups

$$\langle 1 \rangle = G_0 \leq G_1 \leq \cdots \leq G_\alpha \leq G_{\alpha+1} \leq \cdots G_\gamma = G$$

such that $G/C_G(G_{\alpha+1}/G_\alpha) \in \mathcal{X}$ for every $\alpha < \gamma$.

It follows that $\bigcap_{\alpha<\gamma} C_G(G_{\alpha+1}/G_\alpha) \geq S$. By [9, Theorem 1], the Baer radical of $G$ contains $\bigcap_{\alpha<\gamma} C_G(G_{\alpha+1}/G_\alpha)$. Note that the Baer radical of the group $G$ is locally nilpotent ([1], [8]) and a locally nilpotent $FC$–hypercentral group is hypercentral, so that, in particular, $S$ is hypercentral. Since $G/S \in \mathbf{R}\mathcal{X}$, $C_G(C/M) \ngeq S$. Further, $M$ has an ascending series of $DG$–submodules

$$\langle 0 \rangle = U_0 \leq U_1 \leq \cdots U_\alpha \leq U_{\alpha+1} \leq \cdots U_\gamma = M$$

such that $G/C_G(U_{\alpha+1}/U_\alpha) \in \mathcal{X}$ for every $\alpha < \gamma$.

It follows that $\bigcap_{\alpha<\gamma} C_G(U_{\alpha+1}/U_\alpha) \geq S$ and thus $M$ is $DS$–hypercentral. We have already remarked that $C_S(C/M) \neq S$. Therefore $M = \zeta^{\infty}_{DH}(C)$. By Theorem 3.5, $C = M \oplus E$, where $E = HE_{\mathcal{I}-DG}(M)$. Since $E \cong_{DG} C/M$, we get $E = HE_{\mathcal{X}-DG}(A)$, a contradiction.

Suppose now that $G/C_G(C/M) \in \mathcal{X}$ and consider the factor module $C/M_1$. In other words, we may assume that $M = HE_{\mathcal{X}-DG}(M)$. By [10, Theorem 3], $C = HZ_{\mathcal{F}-DG}(C) \oplus HE_{\mathcal{F}-DG}(C)$. Since $C$ is indecomposable and $G/C_G(C/M)$ is finite, $C = HZ_{\mathcal{F}-DG}(C)$ again. The $DG$–submodule $M$ has an ascending series of $DG$–submodules

$$\langle 0 \rangle = U_0 \leq U_1 \leq \cdots \leq U_\alpha \leq U_{\alpha+1} \leq \cdots U_\gamma = M$$

such that $U_{\alpha+1}/U_\alpha$ is a $DG$–chief factor and $G/C_G(U_{\alpha+1}/U_\alpha) \notin \mathcal{X}$, if $\alpha < \gamma$.

It follows that $C_G(U_{\alpha+1}/U_\alpha) \ngeq S$ and therefore $\zeta_{DS}(U_{\alpha+1}/U_\alpha) = \langle 0 \rangle$ for every $\alpha < \gamma$, which means that $M = \zeta^{*}_{DH}(C)$. By Theorem 3.5, $C = M \oplus E$, where $E = \zeta^{\infty}_{DH}(C)$. Since $E \cong_{DG} C/M$, we obtain $E = HZ_{\mathcal{X}-DG}(A)$, which again is a contradiction. □

*Proof of Corollary B1.* It suffices to show that the formation of all finite soluble groups is infinitely hereditary for the class of $FC$–hypercentral groups. To do this, it is enough to show that if an $FC$–hypercentral group $G$ belongs to the class $\mathbf{R}(\mathcal{S} \cap \mathcal{F})$, then it is locally soluble, for $G$ is then hyperabelian and its finite factor groups are therefore soluble. Let $K$ be a finitely generated subgroup of $G$. By Corollary 2.6, $K$ is nilpotent–by–finite. In particular, $K$

satisfies **Max** and so $K$ has a maximal normal soluble subgroup $S$. Assume that $K \neq S$. Then $K$ contains a subnormal subgroup $L$ such that $L \geq S$ and $L/S$ is finite non-abelian simple. Since $L \in \mathbf{R}(\mathcal{S} \cap \mathcal{F})$, $L$ has a family $\{L\lambda \mid \lambda \in \Lambda\}$ of normal subgroups such that each $L/L_\lambda$ is finite soluble and $\bigcap_{\lambda \in \Lambda} L_\lambda = \langle 1 \rangle$. If we assume that some $L_\lambda S \neq L$, then $L/L_\lambda S$ must be a finite simple non-abelian group. This contradiction shows that $L_\lambda S = L$, for each $\lambda \in \Lambda$. Therefore $L/L_\lambda = L_\lambda S/L_\lambda \cong S/(S \cap L_\lambda)$. It follows that each $L/L_\lambda$ is a finite soluble group of derived length at most $d$, where $d$ is the derived length of the soluble radical $S$. By Remak's theorem $L \leq \prod_{\lambda \in \Lambda} L/L_\lambda$ and so $L$ is a soluble subgroup of derived length at most $d$. In particular, $L = S$, a contradiction which shows $K = S$. Hence $K$ is soluble. □

*Proof of Corollary B4.* Let $G^{\mathcal{X}}$ be the $\mathcal{X}$–residual of $G$. Since $G/G^{\mathcal{X}}$ is residually finite, this factor group becomes finite. This means that any formation $\mathcal{X}$ of finite groups is infinitely hereditary for the class of Chernikov groups. The result then follows from Theorem B. □

*Proof of Theorem C.* Since $G$ is infinite, $G \notin \mathcal{X}$. Suppose that $A$ does not have the Baer $\mathcal{X}$-$DG$-decomposition. If $\Sigma$ is the family of all $DG$–submodules $B$ of $A$ that do not have the $\mathcal{X}$-$DG$-decomposition, then clearly $\Sigma \neq \emptyset$. Since $A$ is an artinian $DG$–module, $\Sigma$ has a minimal element $C$. By [11, Corollary 4], $C$ contains a unique maximal $DG$–submodule $M$ having the Baer $\mathcal{X}$–$DG$–decomposition. By the choice of $C$, we deduce that $M$ contains every proper $DG$–submodule of $C$ and, in particular, $M$ is a maximal $DG$–submodule of $C$. Let $c \in C \setminus M$. Thus $M$ does not contain $cDG$. This means that C= $cDG$. Since $G$ is finitely generated, by Lemma 2.4, $G$ is nilpotent–by–finite. Since a Dedekind domain is noetherian, the group ring $DG$ is noetherian (see [13, Corollary of Lemma 5.35]). It follows that the cyclic $DG$–submodule $C$ is noetherian and, since it is also artinian, $C$ has a finite composition series. Then Theorem A gives us a contradiction. □

## References

[1] R. Baer, *Finiteness properties of groups*, Duke Math. J. **15** (1948), 1021–1022.

[2] ———, *Erreichbare und engelsche Gruppenelemente*, Abh. Math. Sem. Univ. Hamburg **27** (1964), 44–74.

[3] ———, *Durch Formationen bestimmte Zerlegungen von Normalteilern endlicher Gruppen*, J. Algebra **20** (1972), 38–56.

[4] K. Doerk and T. Hawkes, *Finite soluble groups*, de Gruyter, Berlin, 1992.

[5] Z. Y. Duan, *Extensions of abelian–by–hyper (cyclic or finite) groups*, Comm. Algebra **20** (1992), 2305–2321.

[6] L. S. Kazarin and L. A. Kurdachenko, *The finiteness conditions and the factorizations in infinite groups*, Russian Math. Surveys **47** (1992), 81–126.

[7] O. H. Kegel and B. A. F. Wehrfritz, *Locally finite groups*, North-Holland, Amsterdam, 1973.

[8] Sh. S. Kemhadze, *Quasinilpotent groups*, Dokl. Akad. Nauk SSSR **155** (1964), 1003–1005.

[9] ______, *On stable groups of automorphisms*, Dokl. Akad. Nauk SSSR **158** (1964), 510–512.

[10] L. A. Kurdachenko, B. V. Petrenko, and I. Ya. Subbotin, *On generalized hypercenters in artinian modules*, Comm. Algebra **25** (1997), 1023–1046.

[11] ______, *Direct decompositions in artinian modules over FC–hypercentral groups*, Mat. Contemp. **14** (1998), 89–99.

[12] B. V. Petrenko, *A generalization of Fitting's theorem defined by formations of finite groups*, The classes of groups with restrictions for subgroups, Math. Inst. Kiev, 1997, pp. 73–78.

[13] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer-Verlag, Berlin, 1972.

[14] ______, *Hypercentral ideals, noetherian modules and a theorem of Stroud*, J. Algebra **32** (1974), 234–239.

[15] P. Schmid, *Lokale Formationen endlicher Gruppen*, Math. Z. **197** (1974), 31–48.

[16] J. S. Wilson, *Some properties of groups inherited by normal subgroups*, Math. Z. **114** (1970), 19–21.

[17] D. I. Zaitsev, *On the existence of direct complements in groups with operators*, Investigations in Group Theory, Math. Inst. Kiev, Kiev, 1976, pp. 26–44.

[18] ______, *Hypercyclic extensions of abelian groups*, Groups defined by properties of systems of subgroups, Math. Inst. Kiev, Kiev, 1979, pp. 16–37.

[19] ______, *On direct decompositions of abelian groups with operators*, Ukrain. Math. J. **40** (1988), 303–309.

[20] ______, *The hyperfinite extensions of abelian groups*, Investigations of groups with restrictions on subgroups, Math. Inst. Kiev, Kiev, 1988, pp. 17–26.

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