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WEIGHTED INEQUALITIES FOR SOME SPHERICAL MAXIMAL OPERATORS

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Abstract. Given a set $E \subset (0, \infty)$, the spherical maximal operator associated to the parameter set E is defined as the supremum of the spherical means of a function when the radii of the spheres are in E. The aim of the paper is to study boundedness properties of these operators on the spaces $L^p(|x|^\alpha)$. It is shown that the range of values of α for which boundedness holds behaves essentially as follows: (i) for $p > n/(n-1)$ and negative α the range does not depend on E; (ii) when α is positive it depends only on the Minkowski dimension of E; (iii) if $p < n/(n-1)$ and α is negative, sets with the same Minkowski dimension can give different ranges of boundedness.

1. Introduction

Given a function f defined in $\mathbf{R}^n$, a point x in $\mathbf{R}^n$, and $t > 0$, the mean of f over the sphere centered at x of radius t is given by

$$S_t f(x) = \int_{S^{n-1}} f(x - ty) d\sigma(y)$$

where $d\sigma$ is the normalized Lebesgue measure on the unit sphere S^{n-1}. Let E be any subset of $(0, \infty)$; associated to E we define a maximal operator by

$$\mathcal{M}_E f = \sup_{t \in E} |S_t f| .$$

When $E = (0, \infty)$ the operator is the usual spherical maximal operator which is known to be bounded on $L^p(\mathbf{R}^n)$ if and only if $p > n/(n-1)$. This result was first proved by E. Stein [6] for $n \geq 3$ and by J. Bourgain [1] for $n = 2$. There are several alternative proofs for both results. When E is a lacunary set the associated operator is bounded on $L^p(\mathbf{R}^n)$ for all $p > 1$, which was first proved by C. Calderón [2] and R. Coifman and G. Weiss [3]. For general sets E we need the concept of (upper) Minkowski dimension.

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When E is a parameter set contained in $[1,2]$ we define $d(E)$ as its (upper) Minkowski dimension given by

$$d(E)=\limsup_{\delta\to 0}\frac{\log N(E,\delta)}{-\log\delta}\,, \tag{1}$$

where $N(E,\delta)$ is the minimum number of intervals of length δ needed to cover E. Thus $0\le d(E)\le 1$. Among the several equivalent definitions of this number (see [7]) the following is a convenient one for our purposes: Let $E(\delta)=\{x\in\mathbf{R}:d(x,E)<\delta\}$ be the δ-neighbourhood of E. Then

$$d(E)=\inf\left\{a:\lim_{\delta\to 0}\delta^{a-1}|E(\delta)|=0\right\}. \tag{2}$$

From the definitions we have the following inequalities, which will be useful in the calculations: If $d(E)=d$ and $\epsilon>0$, there exist C_ϵ and c_ϵ such that

$$N(E,\delta)\le C_\epsilon\delta^{-(d+\epsilon)}\quad\text{and}\quad |E(\delta)|\le C_\epsilon\delta^{1-(d+\epsilon)}$$

for small δ, and

$$N(E,\delta)\ge c_\epsilon\delta^{-(d-\epsilon)}\quad\text{and}\quad |E(\delta)|\ge c_\epsilon\delta^{1-(d-\epsilon)}\,,$$

for a sequence of values of δ tending to 0.

For a general set E in $(0,\infty)$, we write $E_k=E\cap[2^k,2^{k+1}]$ and define $d(E)$ as in (1) with $N(E,\delta)=\sup_k N(2^{-k}E_k,\delta)$. ($N(2^{-k}E_k,\delta)$ coincides with the minimum number of intervals of length $2^k\delta$ needed to cover E_k.) The equivalent definition (2) can be modified accordingly. We refer to $d(E)$ as the dimension of E and write it simply as d when the set to which it refers has been fixed.

In [8], A. Seeger, S. Wainger and J. Wright proved that $\mathcal{M}_E$ is bounded on $L^p(\mathbf{R}^n)$ for $p>1+d(E)/(n-1)$ and unbounded for $p<1+d(E)/(n-1)$. (A second proof in [4] is closer to the methods we will use in this paper.) Here we are interested on the boundedness properties of $\mathcal{M}_E$ on $L^p(|x|^\alpha)$, that is, we look for inequalities of the type

$$\int_{\mathbf{R}^n}|\mathcal{M}_Ef(x)|^p|x|^\alpha dx\le C\int_{\mathbf{R}^n}|f(x)|^p|x|^\alpha dx, \tag{3}$$

where C can depend on p, α, and E, but not on f. The case of the spherical maximal operator and its lacunary version were studied in [5], where it was proved that (3) holds for $E=(0,\infty)$ if $1-n<\alpha<p(n-1)-n$, which is sharp except perhaps at the left-hand endpoint, and for lacunary E if and only if $1-n\le\alpha<(n-1)(p-1)$.

Due to the fact that in the continuous parameter case $d(E)=1$ and in the lacunary case $d(E)=0$, the previous results suggest that the range of values of α (excepting endpoints) for which (3) holds depends on $d(E)$ for positive α and goes up to $1-n$ for negative α. Notice also that in the unweighted boundedness result the range of values of p depends only on $d(E)$ except perhaps at the endpoint. One of the consequences of the theorems we prove in this paper is that in general it is not true that the dimension of the set is enough to describe the range of boundedness: While the range of values of p and the range of values of $\alpha \geq 0$ depend only on $d(E)$, when $\alpha < 0$ there exist sets E_1 and E_2 with $d(E_1) = d(E_2)$ for which (3) holds for different ranges of values of α (the difference being not only at the endpoint). More precisely, for a given d in $(0, 1)$ there exist a minimal set of values of α for which the boundedness holds for all sets E with $d(E) = d$ and a maximal set of values of α (up to $1-n$) for which (3) is true for particular sets E with $d(E) = d$. In Section 2 we study necessary conditions and in Section 3 we consider weights valid for all sets with the same dimension; in Section 4 we show that operators associated to sets with a regular distribution of points (like Cantor sets, for instance) have always a maximal range of boundedness.

In Section 5 we discuss the restriction to radial functions. Here the difference between the cases $n = 2$ and $n \geq 3$ becomes important. When $n \geq 3$ and α is negative, (3) holds whenever $\alpha > 1 - n$ for all $p > 1 + d/(n-1)$; nevertheless, for $n = 2$ and $1 + d < p < 2$, the lower bound depends on d and is always strictly bigger than -1. (As a consequence, sets with the same dimension can have different ranges of boundedness even for radial functions.)

The results are sharp except for some endpoints when $d = 0, 1/2$, and 1, and also for any $d \in [0, 1]$ when either α is positive, the parameter set satisfies some regularity assumption, or the functions are radial. The remaining cases are partially open.

2. Necessary conditions

THEOREM 1.

(1) *Assume that inequality* (3) *holds for some set E of dimension d. Then the following conditions are necessary:*
   (i) $\max(\alpha, 0) \leq (p-1)(n-1) - d$;
   (ii) $\alpha \geq 1 - n$.

(2) *If the inequality holds for all sets of dimension d and $1 + d/(n-1) < p < 1 + 1/(n-1)$, we have moreover the conditions:*
   (iii) $\alpha \geq -1 + d(1 - p/2)/(1-d)$ *if* $d < 1/2$ *and* $n = 2$;
   (iv) $\alpha \geq 2d + (1-n)p$ *if* $d \leq 1/2$;
   (v) $\alpha \geq -1 + (1 - p/2)/(1-d)$ *if* $d > 1/2$ *and* $n = 2$;
   (vi) $\alpha \geq 1 + (1-n)p$ *if* $d > 1/2$ *and* $n \geq 3$.

Proof. (1) Assume that the set E is contained in $[1, 2]$ and let $E(\delta)$ be as defined in the introduction. Take as f the characteristic function of the ball of radius δ centered at the origin; then the support of $\mathcal{M}_E f$ is in the set $\{y : |y| \in E(\delta)\}$. On $\{y : |y| \in E(\delta/2)\}$, $\mathcal{M}_E f(x)$ is of the order δ^{n-1}. Then

$$\delta^{(n-1)p}|E(\delta/2)| \leq C \int_{\mathbf{R}^n} |\mathcal{M}_E f(x)|^p |x|^\alpha \, dx \leq C\delta^{\alpha+n} . \tag{4}$$

Since the inequality holds for small δ, it follows that $\alpha \le (n-1)(p-1)-d$.

Translating the center of the ball away from the origin, the size of the weight is almost the same on the ball and on the support of $\mathcal{M}_E f$ and the condition $\delta^{(n-1)p}|E(\delta/2)| \le C\delta^n$ implies $0 \le (n-1)(p-1)-d$.

The condition $\alpha \ge 1-n$ appears for each operator S_t. It is enough to consider as f the characteristic function of the annulus $\{y : t-\delta < |y| < t+\delta\}$ and observe that $\mathcal{M}_E f(x)$ is 1 on $\{|x| < \delta\}$.

2. For a given $d \in (0,1)$ there are sets E of dimension d such that $E(\delta)$ contains an interval of length δ^{1-d}, which we take as $[1, 1+\delta^{1-d}]$ (consider, for instance, $E = \{1+k^{-\gamma} : k = 1, 2, \ldots\}$ for which $d = 1/(\gamma+1)$). In this part we assume that the set E has this property.

Take again as f the characteristic function of the annulus $\{y : t-\delta < |y| < t+\delta\}$; when $n = 2$ and $2\delta < |y| < \delta^{1-d}$, $\mathcal{M}_E f$ is of the order $(\delta/|y|)^{1/2}$. This gives the first restriction.

Let f be the characteristic function of a parallelepiped of sides $\delta \times \delta^{1/2} \times \ldots \times \delta^{1/2}$ centered at $(1, 0, \ldots, 0)$. Then $\mathcal{M}_E f$ is of the order $\delta^{(n-1)/2}$ on $P(\delta) = \{(x_1, \overline{x}) : |x_1| \le \delta^{1-d}, |\overline{x}| < \delta^{1/2}/2\}$. Thus

$$\delta^{p(n-1)/2} \int_{P(\delta)} |x|^\alpha \, dx \le C\delta^{1+(n-1)/2} \,. \tag{5}$$

The computation of the integral depends on d. If $d \le 1/2$,

$$\begin{aligned} \int_{P(\delta)} |x|^\alpha \, dx &= \int_{|x| \le \delta^{1-d}} |x|^\alpha \, dx \\ &\qquad + \int_{|x_1| \le \delta^{1-d}} \int_{\delta^{1-d} < |\overline{x}| < \delta^{1/2}/2} |\overline{x}|^\alpha \, d\overline{x} \, dx_1 \\ &\ge C\big(\delta^{(1-d)(\alpha+n)} + \delta^{1-d}\delta^{(\alpha+n-1)/2}\big) \,. \end{aligned}$$

Together with (5) this implies the condition for $d \le 1/2$ and all n.

If $d > 1/2$ we have

$$\begin{aligned} \int_{P(\delta)} |x|^\alpha \, dx &= \int_{|x| \le \delta^{1/2}} |x|^\alpha \, dx + \int_{\delta^{1/2} < |x_1| \le \delta^{1-d}} \int_{|\overline{x}| < \delta^{1/2}/2} |x_1|^\alpha \, d\overline{x} \, dx_1 \\ &\ge C\big(\delta^{(\alpha+n)/2} + \delta^{(\alpha+1)(1-d)}\delta^{(n-1)/2}\big) \,. \end{aligned}$$

The conditions stated in the theorem for $d > 1/2$ appear using again (5) with this estimate; when $n = 2$ we already had the restriction $\alpha > -1$ so that we use the second summand in the last expression, while for $n \ge 3$ we use the first one. $\square$

It is worth pointing out that a question which remains open for the continuous spherical maximal operator $\mathcal{M}$ (corresponding to $E = (0, \infty)$) is whether the inequality

$$\int_{\mathbf{R}^n} |\mathcal{M}f(x)|^p\, u(x)\, dx \le C_s \int_{\mathbf{R}^n} |f(x)|^p\, (\mathcal{M}u^s(x))^{1/s}\, dx$$

holds for $p > n/(n-1)$ when $s = 1$ (Fefferman-Stein type inequality) or at least for all $s > 1$. Theorem 1 shows that the answer to an analogous conjecture for $\mathcal{M}_E$ is negative for $1 + d/(n-1) < p < 1 + 1/(n-1)$; indeed, $|x|^\alpha$ satisfies $\mathcal{M}_E(|x|^\alpha) \le C|x|^\alpha$ when $1 - n < \alpha \le 0$ and the inequality fails for some values of α in this range.

3. Sufficient conditions

In this section we prove the following theorem.

THEOREM 2. *Let E be a subset of $(0, \infty)$ and d its dimension. Let $\mathcal{M}_E$ be the above defined maximal operator. Then $\mathcal{M}_E$ is bounded on $L^p(|x|^\alpha)$ if one of the following conditions holds:*

(1) $p > 1 + 1/(n-1)$ *and* $1 - n < \alpha < (n-1)(p-1) - d$;
(2) $d \in [0, 1)$, $1+d/(n-1) < p \le 1+1/(n-1)$ *and* $1-n+\frac{d}{1-d}[n-(n-1)p] < \alpha < (n-1)(p-1) - d$.

The proof will be split into several parts.

Proof for positive values of α. The method is standard and applies to any positive operator. The operator $f \mapsto w^{-1}\mathcal{M}_E(wf)$ is bounded on L^∞ if $\mathcal{M}_E w \le Cw$ a.e. and on $L^{p_0}(w^{p_0})$ if $\mathcal{M}_E$ is bounded on L^{p_0}; by interpolation, it is bounded on $L^p(w^{p_0})$ if both conditions are fulfilled and $p > p_0$. Then $\mathcal{M}_E$ is bounded on $L^p(w^{p_0 - p})$ whenever it is bounded on L^{p_0} and w is such that $\mathcal{M}_E w \le Cw$ a.e. Particularizing to power weights $w(x) = |x|^\beta$, this condition is satisfied if $1 - n < \beta \le 0$; moreover, we can choose any p_0 greater than $1 + d/(n-1)$. Thus we deduce the boundedness of $\mathcal{M}_E$ on $L^p(|x|^\alpha)$ if $0 \le \alpha < (n-1)(p-1) - d$. □

The result for $p > 1 + 1/(n-1)$ and negative α was proved in [5] for the spherical maximal operator which is bigger than any $\mathcal{M}_E$. We only need to consider $d \in [0, 1)$.

Proof for negative α and $E \subset [1, 2]$. Write

$$\begin{aligned}
&\int_{\mathbf{R}^n} \sup_{t\in E} |S_t f(x)|^p\, |x|^\alpha\, dx \\
&\quad\le \int_{|x|\le 1/2} \sup_{t\in E} |S_t f(x)|^p\, |x|^\alpha\, dx + \sum_{m=0}^{\infty} 2^{m\alpha} \int_{2^{m-1}<|x|<2^m} \sup_{t\in E} |S_t f(x)|^p\, dx\,.
\end{aligned}$$

For $m \geq 3$ and $2^{m-1} < |x| < 2^m$, $S_t f(x) = S_t(f\chi_{2^{m-2}<|\cdot|<2^{m+1}})(x)$, and using the unweighted boundedness of the operator,

$$\int_{|x|>4} \sup_{t\in E} |S_t f(x)|^p \, |x|^\alpha \, dx \leq C \int_{|x|>2} |f(x)|^p \, |x|^\alpha \, dx \, .$$

For $1/2 < |x| < 4$, $|x|^\alpha < 2^{-\alpha}$ and $S_t f(x) = S_t(f\chi_{|\cdot|<6})(x)$; then

$$\int_{1/2<|x|<4} \sup_{t\in E} |S_t f(x)|^p \, |x|^\alpha \, dx \\ \leq C \int_{|x|<6} |f(x)|^p \, dx \leq C \int_{|x|<6} |f(x)|^p \, |x|^\alpha \, dx \, .$$

We are left with the integral over $|x| < 1/2$. In this range, $S_t f(x) = S_t(f\chi_{1/2<|\cdot|<3})(x)$, so that we only need to prove

$$\int_{|x|<1/2} \sup_{t\in E} |S_t f(x)|^p \, |x|^\alpha \, dx \leq C \int |f(x)|^p \, dx \tag{6}$$

for functions f supported on $1/2 \leq |x| \leq 3$, which will be assumed in the rest of the proof for $E \subset [1, 2]$.

Choose a $\mathcal{C}^\infty$ function Φ, compactly supported, with integral 1, and null moments up to the order M with $M \geq (n-1)/2$, that is,

$$\int_{\mathbf{R}^n} x^\beta \Phi(x) \, dx = 0, \text{ for } |\beta| \leq M \, .$$

Define $\Phi_i(x) = 2^{in}\Phi(2^i x)$; this is an approximation of the identity, and for smooth f we have

$$\lim_{i\to\infty} \Phi_i * f(x) = f(x)$$

pointwise. Then it is enough to prove (6) for $S_t(\Phi_i * f)$ instead of $S_t f$ with bounds independent of i and to use Fatou's lemma to obtain the desired conclusion. Define now $\Psi_j = \Phi_j - \Phi_{j-1}$, so that $\Phi_i = \Phi + \sum_{j=1}^{i} \Psi_j$. Since the functions $S_t(\Phi)$ for $t \in [1, 2]$ have a common compact support and are uniformly bounded by $\|\Phi\|_\infty$, we have $|S_t(\Phi * f)(x)| \leq CMf(x)$, where M denotes the Hardy-Littlewood maximal operator and C is independent of t. Then

$$|S_t(\Phi_i * f)(x)| \leq CMf(x) + \sum_{j=1}^{i} |S_t(\Psi_j * f)(x)| \, .$$

We do not need to consider the first term because the Hardy-Littlewood maximal operator already satisfies the required estimates; consequently, we define $S_t^j f(x) = S_t(\Psi_j * f)(x)$ and work with these operators. An important observation is the following: The kernel K_t^j of S_t^j is the function $S_t(\Psi_j)(x)$ which is supported in the annulus $t - 2^{-j+2} < |x| < t + 2^{-j+2}$ and is bounded by a constant (in t) times 2^j. The following is the crucial lemma.

Lemma 3.

(1) *For* $j>0$, $p\geq 1$, $t,t'\in[1,2]$, $1-n<\alpha\leq 0$ *and* $\delta>2^{-j}$, *we have*

$$\int_{|x|<\delta}|S_t^j f(x)-S_{t'}^j f(x)|^p\,|x|^\alpha\,dx \leq C\delta^{(\alpha+n-1)}\,\min\{2^j|t-t'|,1\}\int|f(x)|^p\,|x|^\alpha\,dx\ .$$

(2) *For* $j>0$, $1\leq p\leq 2$, $t,t'\in[1,2]$ *and* $\delta>2^{-j}$, *we have*

$$\left(\int_{|x|<\delta}|S_t^j f(x)-S_{t'}^j f(x)|^p dx\right)^{1/p} \leq C_p\delta^{(n-1)((2/p)-1)}2^{-j(n-1)(1-1/p)}\,\min\{2^j|t-t'|,1\}||f||_p\ .$$

Proof of the lemma. For the first part it is enough to prove the result for $p=1$ and to interpolate with the (trivial) uniform bound for L^∞. Assume first that we have $S_t^j f$ instead of the difference inside the integrals. Then

$$\int_{|x|<\delta}|S_t^j f(x)|\,|x|^\alpha\,dx\leq\int|f(x)|(|K_t^j|*|\cdot|^\alpha\chi_{B(0,\delta)})(x)\,dx$$

and the estimate follows from the above observation about the localization and size of K_t^j. To get the estimate with the difference $S_t^j f-S_{t'}^j f$ we consider its kernel and see that

$$|K_t^j(x)-K_{t'}^j(x)|\leq\int|\Psi_j(x-ty)-\Psi_j(x-t'y)|\,d\sigma(y)\leq C2^{2j}|t-t'|$$

by applying the mean value theorem. Moreover, when $|t-t'|<2^{-j}$, the left-hand side is supported on an annulus of width 2^{6-j}. The estimate follows as before.

The estimate of the second part for $p=1$ is contained in the first one; we only need to prove the corresponding one for $p=2$ and interpolate. We use now Plancherel's theorem so that

$$\int_{\mathbf{R}^n}|S_t^j f(x)-S_{t'}^j f(x)|^2\,dx\leq C\int_{\mathbf{R}^n}|\hat\sigma(t\xi)-\hat\sigma(t'\xi)|^2|\hat\Psi_j(\xi)|^2|\hat f(\xi)|^2\,d\xi\,.$$

On the one hand we have $|\hat\sigma(\xi)|\leq C\min(1,|\xi|^{(1-n)/2})$, and $|\hat\sigma(t\xi)-\hat\sigma(t'\xi)|\leq C|t-t'||\xi|\min(1,|\xi|^{(1-n)/2})$ (using the mean value theorem and the fact that the derivative of $\hat\sigma$ has its same decay). On the other hand, we built our Φ function so that

$$|\hat\Psi_0(\xi)|\leq C|\xi|^{(n-1)/2}\text{ for }|\xi|\leq 1$$

(using the assumption on the null moments), and $\hat\Psi_0$ is rapidly decreasing at infinity because it belongs to the Schwartz class. From these size estimates the desired inequality for $p=2$ follows. □

We continue with the proof of the theorem. Without loss of generality we shall work with finite sets. Indeed, we can consider our general set as a limit of finite sets. If the condition we get for L^p-boundedness is independent of the finite sets involved, a limiting argument will complete the proof.

Let E be a finite set and assume that we have a decreasing family of sets contained in E, $E = E_N \supset E_{N-1} \supset \cdots \supset E_{m_0}$, and a function τ such that for each $t \in E_m \setminus E_{m-1}$, $\tau(t)$ is in E_{m-1}. Our approach (presented also in [4] and used previously by S. Wainger) is based on the following observation:

$$\sup_{t\in E_m} |S_t^j f(x)| \leq \left(\sum_{t\in E_m\setminus E_{m-1}} |S_t^j f(x) - S_{\tau(t)}^j f(x)|^p \right)^{1/p} + \sup_{t\in E_{m-1}} |S_t^j f(x)| \,.$$

Raising both sides to the p-th power and adding the inequalities from $m = m_0 + 1$ to $m = N$ we get

$$\begin{aligned} (7) \qquad & \sup_{t\in E_N} |S_t^j f(x)|^p \\ & \leq \sum_{m=m_0+1}^{N} \sum_{t\in E_m\setminus E_{m-1}} |S_t^j f(x) - S_{\tau(t)}^j f(x)|^p + \sup_{t\in E_{m_0}} |S_t^j f(x)|^p \,. \end{aligned}$$

The sets $E_m, m < N$, are built in such a way that the length of the smallest interval of $[1, 2] \setminus E_m$ is at least 2^{-m}. Given a finite set E contained in $[1, 2]$, let N be the largest integer for which the length of all the component intervals of $[1, 2] \setminus E$ is bigger than 2^{-N}, and put $E_N = E$. To pass from E_m to E_{m-1} we look at the intervals of $[1, 2] \setminus E_m$ whose length is smaller than 2^{-m+1}. If there are no such intervals, define $E_{m-1} = E_m$; when intervals of this type exist, two cases can occur: (i) an interval of length smaller than 2^{-m+1} is adjacent to two intervals of length larger than 2^{-m+1}; (ii) several intervals of length smaller than 2^{-m+1} are next to each other. In the first case, remove one endpoint of the small interval; in the second case, label consecutively the endpoints of the intervals and remove those with even label. The remaining points form E_{m-1}. (This defines E_m up to $m = 1$; in (7) we stop at E_{m_0} for convenience.)

For each $t \in E_m \setminus E_{m-1}$ define $\tau(t)$ as the point (or one of the points) in E_{m-1} closest to t. It is clear that the smallest number of intervals of length 2^{-m} needed to cover E is essentially $\#E_m$ (where $\#$ denotes the cardinality of the set). Note also that by construction $\#(E_m \setminus E_{m-1}) \leq \#E_{m-1}$, so that $\#E_m \leq 2\#E_{m-1}$.

To find a bound for the $L^p(|x|^\alpha)$ norm of $\sup_{t\in E} |S_t^j f(x)|$ we decompose the integral as

$$\int_{|x|<1/2} \sup_{t\in E} |S_t^j f(x)|^p |x|^\alpha \, dx = \int_{|x|<2^{-j}} \sup_{t\in E} |S_t^j f(x)|^p |x|^\alpha \, dx + \sum_{k=3}^{j} 2^{-k\alpha} \int_{2^{-k}\le |x|<2^{-k+1}} \sup_{t\in E} |S_t^j f(x)|^p dx \, .$$

Define $E(l) = E \cap [1+2^{-j}(l-1), 1+2^{-j}l)$ for $l = 1, \ldots, 2^j$. Then

$$\int_{|x|<2^{-j}} \sup_{t\in E} |S_t^j f(x)|^p |x|^\alpha \, dx = \sum_{l=1}^{2^j} \int_{|x|<2^{-j}} \sup_{t\in E(l)} |S_t^j f_l(x)|^p |x|^\alpha \, dx \, ,$$

where $f_l = f \cdot \chi_{A(l)}$ and $A(l) = \{x : 1+2^{-j}(l-3) \le x \le 1+2^{-j}(l+3)\}$. This is due to the fact that only this part of f gives a nonzero contribution when $t \in E(l)$ and $|x| < 2^{-j}$.

With $m_0 = j$ multiply both sides of the inequality (7) by $|x|^\alpha$, integrate over $|x| < 2^{-j}$ and use the first inequality of the previous lemma (with $\delta = 2^{-j}$ and $|t - \tau(t)| < 2^{-m}$ for $t \in E(l)_m \setminus E(l)_{m-1}$) to get

$$\begin{aligned} (8) \qquad & \sum_{l=1}^{2^j} \int_{|x|<2^{-j}} \sup_{t\in E(l)} |S_t^j f_l(x)|^p |x|^\alpha \, dx \\ \le & C \sum_{l=1}^{2^j} [\sum_{m\ge j} N(E(l), 2^{-m}) 2^{m-j} + N(E(l), 2^{-j})] 2^{-j(\alpha+n-1)} \int |f_l(x)|^p \, dx \, . \end{aligned}$$

Since $N(E(l), 2^{-m}) \le \min(C2^{m(d+\epsilon)}, 2^{m-j})$, we introduce this bound in the previous sum and get

$$\int_{|x|<2^{-j}} \sup_{t\in E} |S_t^j f(x)|^p |x|^\alpha dx \le Cj 2^{-j(\alpha+n-1)} \|f\|_p^p \, .$$

For $k < j$ decompose E into smaller sets as before using now intervals of length 2^{-k} to get the sets $E(l)$ with $l = 1, \ldots, 2^k$. (Although the decomposition depends on k we will not show explicitly this dependence in the notation.) We have

$$\int_{|x|<2^{-k}} \sup_{t\in E} |S_t^j f(x)|^p \, dx = \sum_{l=1}^{2^k} \int_{|x|<2^{-k}} \sup_{t\in E(l)} |S_t^j f_l(x)|^p \, dx \, .$$

From the second inequality of the previous lemma we have

$$
\begin{aligned}
(9)\qquad & \sum_{l=1}^{2^k} \int_{|x|<2^{-j}} \sup_{t\in E(l)} \left| S_t^j f_l(x)\right|^p \, dx \\
&\le C \sum_{l=1}^{2^j} \left[\sum_{m\ge j} N(E(l),2^{-m}) 2^{(j-m)p} + N(E(l),2^{-j}) \right] \cdot \\
&\qquad\qquad \cdot\ 2^{-k(n-1)(2-p)}\, 2^{-j(n-1)(p-1)} \int |f_l(x)|^p \, dx \,.
\end{aligned}
$$

Since $N(E(l),2^{-m}) \le \min(C2^{m(d+\epsilon)}, 2^{m-k})$, we must use the first bound for $m > \max(j, k/(1-d-\epsilon))$, and the second one for $j < m < k/(1-d-\epsilon)$. (This situation appears only for those k satisfying $k > j(1-d-\epsilon)$.) Introducing these bounds we get

$$
\begin{aligned}
&\sum_{k=-2}^{j} \int_{2^{-k}\le|x|<2^{-k+1}} \sup_{t\in E} |S_t^j f(x)|^p |x|^\alpha \, dx \\
&\qquad \le C 2^{-j(n-1)(p-1)} (2^{j(d+\epsilon)}\ B_1 + 2^j\ B_2) ||f||_p^p \,,
\end{aligned}
$$

with

$$
B_1 = \sum_{k=-2}^{j(1-d-\epsilon)} 2^{-k(\alpha+(n-1)(2-p))} \quad \text{and} \quad B_2 = \sum_{k=j(1-d-\epsilon)}^{j} 2^{-k(\alpha+(n-1)(2-p)+1)} \,.
$$

It is enough to consider the case $\alpha < 1-n+d$. Then the exponent in B_1 is positive and that in B_2 is negative and we conclude that

$$
\int_{|x|<1/2} \sup_{t\in E} |S_t^j f(x)|^p \, dx \le C 2^{-j[(\alpha+n-1)(1-d-\epsilon)+(d+\epsilon)(n-(n-1)p)]} \int |f|^p \,.
$$

The sum in j of the L^p-norms is finite if $\alpha > 1-n+\frac{d}{1-d}[n-(n-1)p]$ as stated in the theorem. $\square$

Proof for general E and negative α. For a general set $E \subset (0,\infty)$ define $E_k = E \cap [2^k, 2^{k+1}]$ and write $\mathcal{M}_E f = \sup_k \mathcal{M}_{E_k} f$.

Lemma 4. *Assume that $\mathcal{M}_E$ is bounded on $L^p(\mathbf{R}^n)$ and that*

$$
\int |\mathcal{M}_{E_k} f(x)|^p \, |x|^\alpha \, dx \le C \int |f(x)|^p \, |x|^\alpha \, dx
$$

for some $\alpha < 0$ and C independent of k. Then $\mathcal{M}_E$ is bounded on $L^p(|x|^\alpha)$.

Proof of the lemma. Let $A_j = \{x : 2^j < |x| \le 2^{j+1}\}$ and $B_j = \{x : |x| \le 2^j\}$. Decompose f as $f = f_j + f^j$, where $f_j = f\chi_{B_{j+1}}$.

$$\begin{aligned}\int |\mathcal{M}_E f(x)|^p\,|x|^\alpha\,dx &\le \sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{A_j} |\mathcal{M}_E f(x)|^p\,dx \\ &\le 2^p \left[\sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{A_j} |\mathcal{M}_E f_j(x)|^p\,dx + \sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{A_j} |\mathcal{M}_E f^j(x)|^p\,dx \right].\end{aligned}$$

Applying the boundedness of $\mathcal{M}_E$ on L^p to the first summand we get the bound

$$\sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{B_{j+1}} |f(x)|^p\,dx$$

and this sum is bounded by the $L^p(|x|^\alpha)$ norm of f (because $\alpha < 0$). For the second summand we proceed as follows:

$$\sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{A_j} |\mathcal{M}_E f^j(x)|^p\,dx \le \sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{A_j} \sum_{k=-\infty}^{\infty} |\mathcal{M}_{E_k} f^j(x)|^p\,dx.$$

For $x \in A_j$, $\mathcal{M}_{E_k} f^j(x) = 0$ if $k < j$, and for fixed $k > j$, $\mathcal{M}_{E_k} f^j = \mathcal{M}_{E_k}(f\chi_{A_k})$. Then the last sum is bounded by

$$\begin{aligned}C \sum_{j=-\infty}^{\infty} 2^{j\alpha} \int_{A_j} \sum_{k=j}^{\infty} &|\mathcal{M}_{E_k}(f\chi_{A_k})(x)|^p\,dx \\ &\le C \sum_{k=-\infty}^{\infty} \int_{B_k} |\mathcal{M}_{E_k}(f\chi_{A_k})(x)|^p\,|x|^\alpha\,dx \\ &\le C \sum_{k=-\infty}^{\infty} \int_{A_k} |f(x)|^p\,|x|^\alpha\,dx\,. \qquad \square\end{aligned}$$

Since we are under the hypotheses of the lemma, the proof of Theorem 2 is complete. □

Observe that we have proved the boundedness for $\alpha > 1 - n + d$ for all $p > 1 + d/(n-1)$ and that the result we get in the range $1 + d/(n-1) < p < 1 + 1/(n-1)$ is the interpolation between this and the range $\alpha > 1 - n$, valid for $p > 1 + 1/(n-1)$. When $d = 1/2$ the necessary and sufficient conditions are the same except at the endpoint $\alpha = 1 - n$; when $d < 1/2$ this is only true in the limit (when p tends to either $1 + d/(n-1)$ or $1 + 1/(n-1)$).

4. Regular sets of parameters

Compare the ternary Cantor set in $[1, 2]$ and the set $E_\gamma = \{1 + n^{-\gamma}, n = 1, 2, \dots\}$ with the same dimension. When both sets are covered with intervals of length δ, the amount of intervals needed for the covering is essentially the same in both cases, but their distributions are rather different. They are quite regularly distributed in the case of the Cantor set, while half of them are packed together in the second case. A different way of looking at this property is to remark that the δ-neighbourhood of the Cantor set is formed by equal intervals with big "holes" and the δ-neighbourhood of the second set contains a very big interval.

We say that a parameter set E contained in $[1, 2]$ is *regular* if for any interval I in $[1, 2]$, the number of intervals of length δ needed to cover $E \cap I$ is bounded by $C_\epsilon(|I|/\delta)^{d+\epsilon}$ for all $\epsilon > 0$. A general set E will be regular if $2^{-k}(E \cap [2^k, 2^{k+1}])$ is regular for all k.

Theorem 5. *Let E be a regular set of parameters and d its dimension. Then $\mathcal{M}_E$ is bounded on $L^p(|x|^\alpha)$ if $p > 1 + d/(n-1)$ and $1 - n < \alpha < (n-1)(p-1) - d$.*

To obtain this result, repeat the proof of the previous theorem with the following changes: In inequality (8) use the bound $N(E(l), 2^{-m}) \leq C2^{(m-j)(d+\epsilon)}$, and in (9) use $N(E(l), 2^{-m}) \leq C2^{(m-k)(d+\epsilon)}$.

5. Radial functions

The action of the spherical maximal operators on radial functions has been carefully studied in [9] for the unweighted case. The examples used in Section 2 to describe necessary conditions are radial only in the first part of the theorem when $n \geq 3$; for $n = 2$, the characteristic function of an annulus gives an extra restriction which appeared in the second part of Theorem 1. We show that these conditions are also sufficient except maybe at the endpoints.

Theorem 6. *Inequality (3) holds for radial functions if $p > 1 + d/(n-1)$ and $1 - n < \alpha < (n-1)(p-1) - d$ when either $n \geq 3$ or $n = 2$ and $p \geq 2$. When $n = 2$ and $1 + d < p < 2$, it holds if $-1 + d(1 - p/2)/(1 - d) < \alpha < (n-1)(p-1) - d$.*

Proof. We can assume $E \subset [1, 2]$ and use Lemma 4 as before to extend the result to $E \subset (0, \infty)$. According to the preparation of the proof of Theorem 2 for negative α, we only need to check inequality (6) for radial functions supported in $1/2 < |x| < 3$. Write $|f(x)| = f_0(|x|)$. Then the $L^p(|x|^\alpha)$-norm of f is essentially the same as $\int f_0(s)^p \, ds$. Using [9, Lemma 3.1] for $n \geq 3$ and [9, Lemma 5.1] for $n = 2$, the maximal operator can be bounded by a sum of one-dimensional operators acting on f_0. Moreover, when $|x| < 1/4$ and $t \in [1, 2]$, some of these vanish and we are left with

$$\mathcal{M}_E f(x) \leq C \sup_{t \in E} \frac{1}{|x|} \int_{t-|x|}^{t+|x|} f_0(s) \, ds$$

if $n \geq 3$, and

$$\mathcal{M}_E f(x) \leq C \sup_{t\in E} \frac{1}{|x|^{1/2}} \left[\int_{t-|x|}^{t} f_0(s)(s-t+|x|)^{-1/2}\, ds + \int_{t}^{t+|x|} f_0(s)(|x|+t-s)^{-1/2}\, ds \right] \tag{10}$$

if $n = 2$.

In the first case, using Hölder's inequality,

$$|\mathcal{M}_E f(x)|^p \leq C \frac{1}{|x|} \int_{1/2}^{3} f_0(s)^p\, ds\,,$$

and the integral of $|\mathcal{M}_E f(x)|^p |x|^\alpha$ is finite if $\alpha > 1-n$.

For $n = 2$ and $p < 2$ we work with the first term of the right-hand side of (10) because the other one is similar. Write

$$\begin{aligned}\mathcal{M}_E^1 f(x) &= \sup_{t\in E} \frac{1}{|x|^{1/2}} \int_{t-|x|}^{t} f_0(s)(s-t+|x|)^{-1/2}\, ds \\ &\leq \sum_{m=0}^{\infty} \frac{2^{m/2}}{|x|} \sup_{t\in E} \int_{t-|x|+2^{-m-1}|x|}^{t-|x|+2^{-m}|x|} f_0(s)\, ds\,.\end{aligned}$$

Then

$$\begin{aligned}&\left(\int_{|x|<1/2} |\mathcal{M}_E^1 f(x)|^p |x|^\alpha dx \right)^{1/p} \\ &\leq \sum_{m=0}^{\infty} 2^{m/2} \left(\sum_{k=1}^{\infty} 2^{-k(\alpha+1-p)} \int_{2^{-k-1}}^{2^{-k}} \left(\sup_{t\in E} \int_{t-r+2^{-m-1}r}^{t-r+2^{-m}r} f_0(s)\, ds \right)^p dr \right)^{1/p}.\end{aligned} \tag{11}$$

The L^∞ norm of the operator

$$f_0 \mapsto \sup_{t\in E} \int_{t-r+2^{-m-1}r}^{t-r+2^{-m}r} f_0(s)\, ds$$

is bounded by $2^{-(m+k)}$. To bound its L^1 norm we have

$$\begin{aligned}&\int_{2^{-k-1}}^{2^{-k}} \sup_{t\in E} \int_{t-r+2^{-m-1}r}^{t-r+2^{-m}r} f_0(s)\, ds\, dr \\ &\leq \int_{2^{-k-1}}^{2^{-k}} \sup_{t\in E} \int_{t-r}^{t-r+2^{-(m+k)}} f_0(s)\, ds\, dr \\ &\leq \int_{2^{-k-1}}^{2^{-k}} \int_{E(2^{-(m+k)})} f_0(s+r)\, ds \\ &\leq C \min(2^{-k}, 2^{-(m+k)(1-d+\epsilon)})\, \|f_0\|_1\,.\end{aligned}$$

($E(\delta)$ has been defined in the introduction and its size is given by (2).) Interpolating and substituting into (11) we deduce that

$$\left(\int_{|x|<1/2} |\mathcal{M}_E^1 f(x)|^p |x|^\alpha dx\right)^{1/p} \leq C \sum_{m=0}^{\infty} 2^{m/2} \cdot$$
$$\cdot \left(\sum_{k=1}^{\infty} 2^{-k(\alpha+1-p)} 2^{-(m+k)(p-1)} \min(2^{-k}, 2^{-(m+k)(1-d+\epsilon)})\right)^{1/p} \|f_0\|_p .$$

The sum is finite when α satisfies the condition stated in the theorem. □

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