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A CLASS OF NON-CONVEX POLYTOPES THAT ADMIT NO ORTHONORMAL BASIS OF EXPONENTIALS

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ABSTRACT. A conjecture of Fuglede states that a bounded measurable set $\Omega \subset \mathbb{R}^d$, of measure 1, can tile $\mathbb{R}^d$ by translations if and only if the Hilbert space $L^2(\Omega)$ has an orthonormal basis consisting of exponentials $e_\lambda(x) = \exp\{2\pi i\langle\lambda, x\rangle\}$. If Ω has the latter property it is called *spectral*. Let Ω be a polytope in $\mathbb{R}^d$ with the following property: there is a direction $\xi \in S^{d-1}$ such that, of all the polytope faces perpendicular to ξ, the total area of the faces pointing in the positive ξ direction is more than the total area of the faces pointing in the negative ξ direction. It is almost obvious that such a polytope Ω cannot tile space by translation. We prove in this paper that such a domain is also not spectral, which agrees with Fuglede's conjecture. As a corollary, we obtain a new proof of the fact that a convex body that is spectral is necessarily symmetric, in the case where the body is a polytope.

Let Ω be a measurable subset of $\mathbb{R}^d$, which we take for convenience to be of measure 1. Let also Λ be a discrete subset of $\mathbb{R}^d$. We write

$$\begin{aligned} e_\lambda(x) &= \exp\{2\pi i\langle\lambda, x\rangle\}, \quad (\lambda, x \in \mathbb{R}^d), \\ E_\Lambda &= \{e_\lambda : \lambda \in \Lambda\} \subset L^2(\Omega). \end{aligned}$$

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g\rangle_\Omega = \int_\Omega f\overline{g}, \quad \text{and} \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

DEFINITION 1. The pair (Ω, Λ) is called a *spectral pair* if E_Λ is an orthonormal basis for $L^2(\Omega)$. A set Ω will be called *spectral* if there is $\Lambda \subset \mathbb{R}^d$ such that (Ω, Λ) is a spectral pair. The set Λ is then called a *spectrum* of Ω.

EXAMPLE. If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$,then $(Q_d, \mathbb{Z}^d)$ is a spectral pair (d-dimensional Fourier series).

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We write $B_R(x) = \{y \in \mathbb{R}^d : |x - y| < R\}$.

Definition 2 (Density). The discrete set $\Lambda \subset \mathbb{R}^d$ has *density* ρ, and we write $\rho = \operatorname{dens} \Lambda$, if we have

$$\rho = \lim_{R \to \infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|},$$

uniformly for all $x \in \mathbb{R}^d$.

We define translational tiling for complex-valued functions below.

Definition 3. Let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable and $\Lambda \subset \mathbb{R}^d$ be a discrete set. We say that f *tiles with* Λ *at level* $w \in \mathbb{C}$, and sometimes write "$f + \Lambda = w\mathbb{R}^d$", if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = w, \quad \text{for almost every (Lebesgue) } x \in \mathbb{R}^d, \tag{1}$$

with the sum above converging absolutely a.e. If $\Omega \subset \mathbb{R}^d$ is measurable, we say that $\Omega + \Lambda$ *is a tiling* when $\mathbf{1}_\Omega + \Lambda = w\mathbb{R}^d$ for some w. If w is not mentioned it is understood to be equal to 1.

Remark 1. If $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, and $f + \Lambda = w\mathbb{R}^d$, then the set Λ has density

$$\operatorname{dens} \Lambda = \frac{w}{\int f}.$$

The following conjecture is still unresolved in all dimensions and in both directions.

Conjecture (Fuglede [F74]). If $\Omega \subset \mathbb{R}^d$ is bounded and has Lebesgue measure 1 then $L^2(\Omega)$ has an orthonormal basis of exponentials if and only if there exists $\Lambda \subset \mathbb{R}^d$ such that $\Omega + \Lambda = \mathbb{R}^d$ is a tiling.

Fuglede's conjecture has been confirmed in several cases.

(1) Fuglede [F74] shows that if Ω tiles with Λ being a lattice then it is spectral with the dual lattice Λ^* being a spectrum. Conversely, if Ω has a lattice Λ as a spectrum then it tiles by the dual lattice Λ^*.
(2) If Ω is a convex non-symmetric domain (bounded, open set) then, as the first author of the present paper has proved [K00], it cannot be spectral. It has long been known that convex domains which tile by translation must be symmetric.
(3) When Ω is a smooth convex domain it is clear that it admits no translational tilings. Iosevich, Katz and Tao [IKT] have shown that it is also not spectral.

(4) There has also been significant progress in dimension 1 (the conjecture is still open there as well) by Łaba [La], [Lb]. For example, the conjecture has been proved in dimension 1 if the domain Ω is the union of two intervals.

In this paper we describe a wide class of, generally non-convex, polytopes for which Fuglede's conjecture holds.

THEOREM 1. *Suppose Ω is a polytope in $\mathbb{R}^d$ with the following property: there is a direction $\xi \in S^{d-1}$ such that*

$$\sum_i \sigma^*(\Omega_i) \neq 0.$$

Here the finite sum is extended over all faces Ω_i of Ω which are orthogonal to ξ and $\sigma^*(\Omega_i) = \pm\sigma(\Omega_i)$, where $\sigma(\Omega_i)$ is the surface measure of Ω_i and the $\pm$ sign depends upon whether the outward unit normal vector to Ω_i is in the same or opposite direction with ξ.

Then Ω is not spectral.

Such polytopes cannot tile space by translation for the following, intuitively clear, reason. In any conceivable such tiling the set of positive-looking faces perpendicular to ξ must be countered by an equal area of negatively-looking ξ-faces, which is impossible because there is more (say) area of the former than the latter.

The following corollary is a special case of the result in [K00], which says that all spectral convex domains are symmetric.

COROLLARY 1. *If Ω is a spectral convex polytope then it is necessarily symmetric.*

Proof. If Ω is spectral, then by Theorem 1 the area measure of Ω is symmetric. (See [S] for the definition of the area measure.) This implies that Ω is itself symmetric, as the area measure determines a convex body up to translation [S, Th. 4.3.1]. Therefore Ω and $-\Omega$, which have the same surface measure, are translates of each other. □

It has been observed in recent work on this problem (see, e.g., [K00]) that a domain (of volume 1) is spectral with spectrum Λ if and only if $|\widehat{\chi_\Omega}|^2 + \Lambda$ is a tiling of Euclidean space at level 1. By Remark 1 this implies that Λ has density 1.

By the orthogonality of e_λ and e_μ for any two different λ and μ in Λ, it follows that

$$\widehat{\chi_\Omega}(\lambda - \mu) = 0. \tag{2}$$

It is only this property, and the fact that any spectrum of Ω must have density 1, that are used in the proof.

Proof of Theorem 1. The quantities P, Q, N, ℓ and K, which are introduced in the proof below, will depend only on the domain Ω. (The letter K will denote several different constants.)

Suppose that Λ is a spectrum of Ω. Define the Fourier transform of χ_Ω as

$$\widehat{\chi_\Omega}(\eta) = \int_\Omega e^{-2\pi i\langle x,\eta\rangle}\, dx.$$

By an easy application of the divergence theorem we get

$$\widehat{\chi_\Omega}(\eta) = -\frac{1}{i|\eta|}\int_{\partial\Omega} e^{-2\pi i\langle x,\eta\rangle}\left\langle \frac{\eta}{|\eta|}, \nu(x)\right\rangle d\sigma(x), \quad \eta \neq 0,$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$.

From the last formula we easily see that for some $K \geq 1$

$$|\nabla\widehat{\chi_\Omega}(\eta)| \leq \frac{K}{|\eta|}, \quad |\eta| \geq 1. \tag{3}$$

Without loss of generality we assume that $\xi = (0, \ldots, 0, 1)$. Hence

$$\widehat{\chi_\Omega}(t\xi) = -\frac{1}{it}\int_{\partial\Omega} e^{-2\pi i t x_d}\nu_d(x)\, d\sigma(x).$$

Now it is easy to see that each face of the polytope other than the faces Ω_i contributes $O(t^{-2})$ to $\widehat{\chi_\Omega}(t\xi)$ as $t \to \infty$. Therefore

$$\left|\widehat{\chi_\Omega}(t\xi) + \frac{1}{it}\sum_i e^{-2\pi i\lambda_i t}\sigma^*(\Omega_i)\right| \leq \frac{K}{t^2}, \quad t \geq 1, \tag{4}$$

where λ_i is the value of x_d for $x = (x_1, \ldots, x_d) \in \Omega_i$.

Now define

$$f(t) = \sum_i \sigma^*(\Omega_i)e^{-2\pi i\lambda_i t}, \quad t \in \mathbb{R}.$$

f is a finite trigonometric sum and has the following properties:

(i) f is an almost-periodic function.
(ii) $f(0) \neq 0$ by assumption. Without loss of generality assume $f(0) = 1$.
(iii) $|f'(t)| \leq K$, for every $t \in \mathbb{R}$.

By (i), for every $\epsilon > 0$ there exists an $\ell > 0$ such that every interval of $\mathbb{R}$ of length ℓ contains a translation number τ of f belonging to ϵ:

$$\sup_t |f(t+\tau) - f(t)| \leq \epsilon \tag{5}$$

(see [B32]).

Fix $\epsilon > 0$ to be determined later ($\epsilon = 1/6$ will do) and the corresponding ℓ. Fix the partition of $\mathbb{R}$ in consecutive intervals of length ℓ, one of them being

$[0, \ell]$. Divide each of these ℓ-intervals into N consecutive equal intervals of length ℓ/N, where

$$N > \frac{6K\ell\sqrt{d-1}}{\epsilon}.$$

In each ℓ-interval there is at least one (ℓ/N)-interval containing a number τ satisfying (5). For example, in $[0, \ell]$ we may take $\tau = 0$ and the corresponding (ℓ/N)-interval to be $[0, \ell/N]$.

Define the set L to be the union of all these (ℓ/N)-intervals in $\mathbb{R}$. Then $L\xi$ is a copy of L on the x_d-axis. Construct M by translating copies of the cube $[0, \ell/N]^d$ along the x_d-axis so that they have their x_d-edges on the ℓ/N-intervals of $L\xi$.

The point now is that there can be no two elements λ of Λ in the same translate of M, at distance $D > 2K/\epsilon$ from each other. Suppose, on the contrary, that

$$\lambda_1, \lambda_2 \in \Lambda, \quad |\lambda_1 - \lambda_2| \geq D, \quad \lambda_1, \lambda_2 \in M + \eta.$$

Then $\lambda_1 = t_1\xi + \eta + \eta_1$, $\lambda_2 = t_2\xi + \eta + \eta_2$, for some $t_1, t_2 \in L$, $\eta_1, \eta_2 \in \mathbb{R}^d$ with

$$|\eta_1|, |\eta_2| < \frac{\ell}{N}\sqrt{d-1} < \frac{\epsilon}{6K}.$$

Hence, $\lambda_1 - \lambda_2 = (t_1 - t_2)\xi + \eta_1 - \eta_2$, and an application of the mean value theorem together with (2) and (3) gives

$$|\widehat{\chi_\Omega}((t_1 - t_2)\xi)| \leq \frac{3K}{|t_1 - t_2|}|\eta_1 - \eta_2|.$$

From (4) we get

$$|f(t_1 - t_2)| \leq 3K|\eta_1 - \eta_2| + \frac{K}{|t_1 - t_2|} < 2\epsilon.$$

Now, since $t_1, t_2 \in L$, there exist τ_1, τ_2 satisfying (5) so that

$$|\tau_1 - t_1|, |\tau_2 - t_2| < \frac{\ell}{N}$$

and hence (by (iii))

$$|f(\tau_1 - \tau_2) - f(\tau_1 - t_2)|, |f(\tau_1 - t_2) - f(t_1 - t_2)| < K\frac{\ell}{N} < \epsilon.$$

Therefore

$$\begin{aligned} 2\epsilon &> |f(t_1 - t_2)| \\ &\geq |f(0)| - |f(0) - f(-\tau_2)| - |f(-\tau_2) - f(\tau_1 - \tau_2)| \\ &\quad - |f(\tau_1 - \tau_2) - f(\tau_1 - t_2)| - |f(\tau_1 - t_2) - f(t_1 - t_2)| \\ &\geq 1 - \epsilon - \epsilon - \epsilon - \epsilon. \end{aligned}$$

It suffices to take $\epsilon = 1/6$ for a contradiction.

Therefore, as the distance between any two λ's is bounded below by the modulus of the zero of $\widehat{\chi_\Omega}$ that is nearest to the origin, there exists a natural number P so that every translate of M contains at most P elements of Λ. Hence there exists a natural number Q (we may take $Q = 2NP$) so that every translate of

$$\mathbb{R}\xi + [0, \ell/N]^d$$

contains at most Q elements of Λ.

It follows that Λ cannot have positive density, a contradiction as any spectrum of Ω (which has volume 1) must have density equal to 1. □

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