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MAPPING THE HOMOLOGY OF A GROUP TO THE K-THEORY OF ITS C^*-ALGEBRA

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Abstract. For a CW-complex X and for $0 \leq j \leq 2$, we construct natural homomorphisms $\beta_j^X : H_j(X;\mathbb{Z}) \longrightarrow K_j(X)$ that are rationally right-inverses of the Chern character. We show that β_j^X is injective for $j = 0$ and $j = 1$. The case $j = 3$ is treated using $\mathbb{Z}[\frac{1}{2}]$-coefficients. The study of these maps is motivated by the connection with the Baum-Connes conjecture on the K-theory of group C^*-algebras.

1. Introduction

For a countable discrete group Γ, there is an assembly map (see [19])

$$\nu_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*(C_r^*\Gamma),$$

called the Novikov assembly map, where $K_*(B\Gamma)$ is the K-homology of the classifying space $B\Gamma$ of Γ, and $K_*(C_r^*\Gamma)$ is the topological K-theory of the reduced C^*-algebra $C_r^*\Gamma$ of Γ. The strong Novikov conjecture is the statement that ν_*^Γ is rationally injective. For Γ torsion-free, the Baum-Connes conjecture asserts that ν_*^Γ is even an isomorphism. More generally, for a countable discrete group Γ (not necessarily torsion-free), the Baum-Connes conjecture predicts that the Baum-Connes assembly map

$$\mu_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C_r^*\Gamma)$$

is an isomorphism, where $K_*^\Gamma(\underline{E\Gamma})$ denotes the Γ-equivariant K-homology with Γ-compact supports of the classifying space for proper actions of Γ (see [3]). In fact, the maps ν_*^Γ and μ_*^Γ are also defined for Γ not necessarily countable (see, for instance, [20] and [26], or [39]). The connection between both assembly maps is embodied by a canonical homomorphism

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$\varphi_*^\Gamma\colon K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$ such that $\nu_*^\Gamma = \mu_*^\Gamma \circ \varphi_*^\Gamma$. Moreover, φ_*^Γ is rationally injective for any group, and even an isomorphism, without tensoring with $\mathbb{Q}$, for torsion-free groups (cf. [39] and [37]).

The Chern character in K-homology yields an isomorphism

$$\mathrm{ch}_* \otimes \mathrm{Id}_{\mathbb{Q}}\colon K_*(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_*(B\Gamma; \mathbb{Q}) = H_*(\Gamma; \mathbb{Q})$$

taking K_0 to H_{ev} and K_1 to H_{odd}. The following basic question arises: "Is there any direct connection between the *integral* homology of Γ and the K-theory of its reduced C^*-algebra ?" As a first step towards answering this question, we prove in Section 2 that the K-homology of a CW-complex of dimension ≤ 3 is isomorphic to its integral homology; we also discuss the naturality properties of the isomorphism in detail. In Section 3, this result is applied to the construction of natural homomorphisms

$$\beta_j^X\colon H_j(X; \mathbb{Z}) \longrightarrow K_j(X) \quad (0 \leq j \leq 2),$$

where X is any pointed connected CW-complex. Several equivalent constructions of these maps are provided, and we prove that they are rationally injective by showing that they are, rationally, cross-sections (i.e., right-inverses) of the Chern character:

$$(\mathrm{ch}_j \otimes \mathrm{Id}_{\mathbb{Q}}) \circ (\beta_j^X \otimes \mathrm{Id}_{\mathbb{Q}}) = \mathrm{Id}_{H_j(X;\mathbb{Q})} \quad (0 \leq j \leq 2).$$

This section also contains a uniqueness result for these maps. To make these constructions as explicit as possible, for the purpose of allowing direct computations, we avoid the use of spectra. In Section 4, we establish the injectivity of β_j^X for $j = 0$ and $j = 1$. If X has its integral homology concentrated in even degrees, except possibly for H_1 and H_3, we prove that β_2^X is also injective. In general, β_2^X is *not* injective, as we will show in Section 5. In Section 6, using the Postnikov tower of the connective K-theory spectrum, we construct a natural transformation

$$\beta_3^X\big[\tfrac{1}{2}\big]\colon H_3(X; \mathbb{Z}\big[\tfrac{1}{2}\big]) \longrightarrow K_1(X) \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]$$

that is rationally a right-inverse of the Chern character in degree 3.

Section 7 contains our main application of these ideas that was already used in [38] and in [37]. Namely, we show that if Γ is torsion-free and satisfies the Baum-Connes conjecture (or if ν_j^Γ is merely injective), then the composition

$$\nu_j^\Gamma \circ \beta_j^{B\Gamma}\colon H_j(\Gamma; \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma)$$

is injective for $j = 0$ and $j = 1$, and also for $j = 2$ if moreover $\dim(B\Gamma) \leq 4$. On the other hand, as established in [22] and in [9], the map $\nu_1^\Gamma \circ \beta_1^{B\Gamma}$ is rationally injective for any group Γ. We prove that this map factorizes through the homomorphism

$$\mathcal{U}_\Gamma\colon H_1(\Gamma/T_\Gamma; \mathbb{Z}) = (\Gamma/T_\Gamma)^{ab} \hookrightarrow K_1(C_r^*\Gamma),\ (\gamma T_\Gamma)^{ab} \longmapsto [\gamma],$$

where T_Γ is the subgroup of Γ generated by the torsion elements. Furthermore, $\mathcal{U}_\Gamma$ is rationally injective, and, if the Baum-Connes assembly map μ_1^Γ is injective, we show that

$$\mathcal{U}_\Gamma\colon (\Gamma/T_\Gamma)^{ab} \hookrightarrow K_1(C_r^*\Gamma).$$

This partially answers a question raised by Bettaieb and Valette in [9].

The assembly map μ_*^Γ is known to be an isomorphism, for example, for amenable groups, free groups, surface groups, Coxeter groups, one-relator groups, braid groups, pure braid groups, knot groups and Gromov hyperbolic groups; it is known to be an injection for discrete subgroups of real Lie groups with finitely many connected components. (See, for example, [51] and [39] and the references therein.)

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2. β_0^X and K-homology of CW-complexes of dimension ≤ 3

We start by fixing our notations and conventions. We then define the map β_0^X and show that it is a split-injection. Next, as a very elementary application of the Atiyah-Hirzebruch spectral sequence, we compute the K-homology of CW-complexes of dimension ≤ 3 in terms of integral homology.

Throughout this paper, we assume all CW-complexes and all maps between them to be pointed. Moreover, for the spectral sequence arguments, we suppose that the 0-skeleton of any connected CW-complex is reduced to the base-point (up to homotopy equivalence, this is no restriction).

We consider complex K-homology, the dual theory of the usual K-theory based on complex vector bundles, associated to the **BU**-spectrum. By Bott periodicity, both theories are 2-periodic and considered as being $\mathbb{Z}/2$-graded. We use the bordism-type description of K-homology, due to Baum and Douglas [4], that we now briefly recall (see also Jakob [27][28]). If X is a connected CW-complex, the K-homology group $K_*(X) = K_0(X) \oplus K_1(X)$ is given by suitable equivalence classes of triples $[M, \xi, f]$, where M is a (not necessarily connected) closed Spin^c-manifold, f is a continuous map from M to X, and ξ is a complex vector bundle over M; the $\mathbb{Z}/2$-grading is given by the reduction modulo 2 of the dimension of M. The equivalence relation imposed on such triples involves bordism and vector bundle modification (we refer to [4] and [27] for details). The group structure is given by taking disjoint unions. We point out that K-homology has automatically compact supports, as opposed to its counterpart defined via Kasparov's KK-theory. The Chern character in K-homology $\mathrm{ch}_*\colon K_*(X) \longrightarrow H_*(X,\mathbb{Q})$ is the natural homomorphism given by $\mathrm{ch}_*([M, \xi, f]) = f_*\big((\mathrm{ch}^*(\xi) \cup \mathrm{Td}(M)) \cap [M]\big)$, where $\mathrm{Td}(M)$ is the Todd class of the Spin^c tangent bundle of M and ch^* is the usual Chern character in K-theory. We denote by $\mathrm{ch}_n\colon K_*(X) \longrightarrow H_n(X;\mathbb{Q})$ the component of ch_* of degree n. The key property is that ch_* is a rational isomorphism. Let us also recall that a given closed connected Spin^c-manifold M admits several such structures: they are parameterized by $H^1(M;\mathbb{Z}/2) \oplus 2 \cdot H^2(M;\mathbb{Z})$ (cf. [34, p. 392]). A complex structure (or a Spin-structure) determines a Spin^c-structure in a canonical way. A closed connected Spin^c-manifold M of dimension m is automatically orientable and also K-orientable. It has consequently two 'fundamental classes', depending on the chosen Spin^c-structure, namely $[M] \in H_m(M;\mathbb{Z})$, the usual orientation class, and $[M]_K = [M, 1_M, \mathrm{Id}_M] \in K_m(M)$, the 'fundamental K-homology class' or 'K-orientation class'. (We denote the trivial 1-dimensional complex vector bundle over a space X simply by 1_X.)

Let us illustrate this by examples. For the n-sphere S^n, one has $\widetilde{K}_j(S^n) \cong \mathbb{Z}$, where $j \in \{0, 1\}$ is the reduction modulo 2 of n. The standard orientation of S^n, as a sub-manifold of $\mathbb{R}^{n+1}$, determines the standard generator $[S^n]_K$ by the equality $\mathrm{ch}_*([S^n]_K) = [S^n] \in H_n(S^n;\mathbb{Q})$. In the bordism-type description of K-homology, $[S^n]_K = [S^n, 1_{S^n}, \mathrm{Id}_{S^n}]$, where, for $n \neq 2$, S^n is equipped with the unique Spin^c-structure inducing the canonical orientation, and $S^2 \approx \mathbb{C}P^1$ is equipped with the Spin^c-structure associated to its canonical complex structure. As usual, we orient Σ_g, 'the' closed connected oriented surface of genus $g \geq 1$, as a sub-manifold of the Euclidean space $\mathbb{R}^3$. The 'fundamental K-homology class' of Σ_g is determined by $\mathrm{ch}_*([\Sigma_g]_K) = [\Sigma_g] \in H_2(\Sigma_g;\mathbb{Z})$. In fact, $[\Sigma_g]_K = [\Sigma_g, 1_{\Sigma_g}, \mathrm{Id}_{\Sigma_g}]$, where Σ_g is equipped with the Spin^c-structure associated to *any* Spin-structure. By the connectedness of the Teichmüller space and by means of the Riemann-Roch-Hirzebruch formula, one can show that for any complex structure on Σ_g, the K-homology class $[\bar{\partial}_g] \in K_0(\Sigma_g)$ associated to the corresponding Dolbeault operator $\bar{\partial}_g$ is the same, and that one has $[\Sigma_g]_K = [\bar{\partial}_g] + (g-1) \cdot [1]$ (see 2.1 (i) below and [46]). In the sequel, we always consider the spheres (and in particular S^1) and the surfaces Σ_g as being equipped with the Spin^c-structure we have just described.

The canonical isomorphism $\mathrm{ch}_n^{\mathbb{Z}}\colon \widetilde{K}_j(S^n) \xrightarrow{\cong} \widetilde{H}_n(S^n;\mathbb{Z})$, $[S^n]_K \longmapsto [S^n]$, clearly extends to wedges of spheres. We would like to define analogous 'integral Chern characters' for arbitrary CW-complexes. For convenience, we will say that a homomorphism $\psi\colon K_j(X) \longrightarrow H_n(X;\mathbb{Z})$ (with $j \equiv n \mod 2$) is *compatible with the Chern character* if the diagram

$$\begin{array}{ccc} K_j(X) & & \\ {\scriptstyle\psi}\downarrow & \searrow^{\mathrm{ch}_n} & \\ H_n(X;\mathbb{Z}) & \longrightarrow & H_n(X;\mathbb{Q}) \end{array}$$

is commutative (we do not require ψ to be defined for all CW-complexes, nor to be natural). We use the same terminology for a map in the reverse direction. We will sometimes write $H_*(X)$ for $H_*(X;\mathbb{Z})$.

PROPOSITION 2.1. *Let X be a connected CW-complex.*

(i) *There is a homotopy-invariant split-injection $\beta_0^X\colon H_0(X;\mathbb{Z})\to K_0(X)$ that is compatible with the Chern character, and whose section will be denoted by $\mathrm{ch}_0^{\mathbb{Z}}$. Both maps are canonical and natural. This yields in particular the canonical and natural splitting*

$$K_0(X)=\mathbb{Z}\cdot[1]\oplus\widetilde{K}_0(X)=\mathbb{Z}\oplus\widetilde{K}_0(X),$$

with the element $[1]$ representing $[\{x_0\},1_{x_0},i_{x_0}]$, where i_{x_0} is the inclusion of the base-point x_0 of X.

(ii) *If X is of dimension ≤ 3, there are isomorphisms*

$$\mathrm{ch}_{\mathrm{ev}}^{\mathbb{Z}}:=\mathrm{ch}_0^{\mathbb{Z}}\oplus\mathrm{ch}_2^{\mathbb{Z}}\colon\ K_0(X)\xrightarrow{\cong}H_0(X;\mathbb{Z})\oplus H_2(X;\mathbb{Z}),$$
$$\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}}:=\mathrm{ch}_1^{\mathbb{Z}}\oplus\mathrm{ch}_3^{\mathbb{Z}}\colon\ K_1(X)\xrightarrow{\cong}H_1(X;\mathbb{Z})\oplus H_3(X;\mathbb{Z}),$$

that are compatible with the Chern character. For $n=0$, 2 and 3, $\mathrm{ch}_n^{\mathbb{Z}}$ is canonical and natural for CW-complexes of dimension ≤ 3.

(iii) *The isomorphisms $\mathrm{ch}_{\mathrm{ev}}^{\mathbb{Z}}$ and $\mathrm{ch}_{\mathrm{odd}}^{\mathbb{Z}}$ of* (ii) are canonical and natural for CW-complexes of dimension ≤ 2. For $X=S^1$, one has $\mathrm{ch}_1^{\mathbb{Z}}([S^1]_K)=[S^1]$, and for $X=\Sigma_g$, a closed oriented surface of genus g, we get an isomorphism $\mathrm{ch}_2^{\mathbb{Z}}\colon\widetilde{K}_0(\Sigma_g)\xrightarrow{\cong}H_2(\Sigma_g;\mathbb{Z})$.

Proof. (i) The map β_0^X is defined as the composition

$$H_0(X;\mathbb{Z})\xrightarrow{\cong}H_0(\{x_0\};\mathbb{Z})\xrightarrow{\cong}K_0(\{x_0\})\longrightarrow K_0(X).$$

The projection $X\twoheadrightarrow\{x_0\}$ yields a splitting of the last map. The rest is trivial.

Before proving (ii) and (iii), let us define a 2-periodic homology theory h_* on the category of finite CW-complexes by setting

$$h_n(X):=\begin{cases}H_{\mathrm{ev}}(X;\mathbb{Q}), & \text{if } n \text{ is even}\\ H_{\mathrm{odd}}(X;\mathbb{Q}), & \text{if } n \text{ is odd}\end{cases}\qquad(n\in\mathbb{Z}).$$

(iii) Since we are working with compact supports, we can assume that X is a finite CW-complex of dimension ≤ 2. The Atiyah-Hirzebruch spectral sequence for reduced K-homology, namely $E^2_{p,q}=\widetilde{H}_p(X;K_q(pt))\Longrightarrow\widetilde{K}_{p+q}(X)$, is trivial, i.e., all the differentials vanish. This yields the desired natural isomorphisms. The compatibility with the Chern character ch_* follows by comparing this spectral sequence with the corresponding one for h_*. Indeed, by naturality, ch_* induces a morphism of spectral sequences that coincides, at the level of the E^2-pages, with the coefficient homomorphism induced by $\mathbb{Z}\hookrightarrow\mathbb{Q}$. The statement about the circle and the surfaces follows readily.

(ii) Similarly, the spectral sequence for K-homology is trivial and yields the natural isomorphism $\mathrm{ch}_{\mathrm{ev}}^{\mathbb{Z}}$, and $0\longrightarrow H_1(X)\xrightarrow{\iota}\widetilde{K}_1(X)\longrightarrow H_3(X)\longrightarrow 0$, a natural short exact sequence that must split (naturally or not) since $H_3(X)$ is a free abelian group. We define $\mathrm{ch}_3^{\mathbb{Z}}$ to be the above surjection, and $\mathrm{ch}_1^{\mathbb{Z}}$ to be any choice of a retraction of ι. The compatibility with the Chern character follows as before. □

For $\dim(X) \leq 3$, we do not know if there is a choice of the retraction $\mathrm{ch}_1^{\mathbb{Z}}$ that is natural and compatible with the Chern character. A completely analogous result for 4-dimensional CW-complexes is stated as Proposition 4.3 below.

It is also possible to prove Proposition 2.1 by carefully comparing the long exact sequences of the pair (X, A), where A is the skeleton of codimension 1 of X, for K-homology, integral homology and rational homology, using the integral Chern characters for spheres (X/A has the homotopy type of a wedge of spheres). This argument is slightly longer than the one presented here, but it has the advantage of avoiding the repeated use of spectral sequences (see also [5]).

3. Definition and first properties of β_1^X and β_2^X

We first define the maps β_1^X and β_2^X. Next, we prove that they are natural homomorphisms and rationally right-inverses of the Chern character. We provide a second construction, show that it is equivalent to the first one, and give a uniqueness result for these maps. Finally, using Spin^c-bordism, we discuss a third equivalent construction.

In the following definition, integral coefficients are understood for homology.

Definition 3.1. Let X be a connected CW-complex, and let i_n denote the inclusion of its n-skeleton $X^{[n]}$. Then the map β_1^X is defined as the composition

$$\beta_1^X\colon\ H_1(X) \xrightarrow[\cong]{(i_2)_*^{-1}} H_1(X^{[2]}) \xrightarrow[\cong]{(\mathrm{ch}_1^{\mathbb{Z}})^{-1}} K_1(X^{[2]}) \xrightarrow{(i_2)_*} K_1(X),$$

where the isomorphism $\mathrm{ch}_1^{\mathbb{Z}}$ is given by Proposition 2.1 (iii). Similarly, the map β_2^X is defined as the composition

$$\beta_2^X\colon\ H_2(X) \xrightarrow[\cong]{(i_3)_*^{-1}} H_2(X^{[3]}) \xrightarrow[\cong]{(\mathrm{ch}_2^{\mathbb{Z}})^{-1}} \widetilde{K}_0(X^{[3]}) \xrightarrow{(i_3)_*} \widetilde{K}_0(X) \hookrightarrow K_0(X),$$

where the isomorphism $\mathrm{ch}_2^{\mathbb{Z}}$ is given by Proposition 2.1 (ii).

The homomorphisms $\mathrm{ch}_1^{\mathbb{Z}}$ and $\mathrm{ch}_2^{\mathbb{Z}}$ being natural for CW-complexes of dimension ≤ 2 and ≤ 3, respectively, those maps are well-defined (i.e., independent of the CW-decomposition of X), natural, and compatible with the Chern character. Together with Proposition 2.1 (i), this establishes the following result.

PROPOSITION 3.2. *For a connected CW-complex X and $0 \leq j \leq 2$, the maps β_j^X are well-defined natural homomorphisms. They are rationally right-inverses of the Chern character, i.e.,*

$$(\mathrm{ch}_j \otimes \mathrm{Id}_{\mathbb{Q}}) \circ (\beta_j^X \otimes \mathrm{Id}_{\mathbb{Q}}) = \mathrm{Id}_{H_j(X;\,\mathbb{Q})}\,.$$

In particular, the maps β_j^X are rationally injective for $0 \leq j \leq 2$. The map β_0^X is split-injective, with splitting $\mathrm{ch}_0^{\mathbb{Z}}$.

The following proposition is a useful complement to this result. (The part concerning CW-complexes of dimension 4 refers to Proposition 4.3.)

PROPOSITION 3.3. *For a connected CW-complex X of dimension ≤ 3, one has*

$$\mathrm{ch}_j^{\mathbb{Z}} \circ \beta_j^X = \mathrm{Id}_{H_j(X;\,\mathbb{Z})} \quad (0 \leq j \leq 2),$$

so that β_j^X is split-injective. (For $j = 1$, this is independent of the choice of the retraction $\mathrm{ch}_1^{\mathbb{Z}}$.) If X is of dimension 4, the same holds for $j = 0$ and 2. (For $j = 2$, this is independent of the choice of the retraction $\mathrm{ch}_2^{\mathbb{Z}}$.)

Proof. Assume that $\dim(X) \leq 3$. For $j = 0$ and 2 the result is obvious. For $j = 1$, we have to be more careful, because of the choice of the retraction $\mathrm{ch}_1^{\mathbb{Z}}$ in the exact sequence $0 \longrightarrow H_1(X) \stackrel{\iota}{\longrightarrow} K_1(X) \longrightarrow H_3(X) \longrightarrow 0$, as in the proof of Proposition 2.1 (ii). By the naturality of the latter sequence for CW-complexes of dimension ≤ 3, the diagram

$$\begin{array}{ccc} H_1(X^{[2]}) & \xrightarrow[\cong]{\iota^{X^{[2]}} = (\mathrm{ch}_1^{\mathbb{Z}})^{-1}} & K_1(X^{[2]}) \\ {\scriptstyle (i_2)_*}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle (i_2)_*} \\ H_1(X) & \xrightarrow{\iota^X} & K_1(X) \end{array}$$

commutes. Since by definition $\beta_1^X = (i_2)_* \circ (\mathrm{ch}_1^{\mathbb{Z}})^{-1} \circ (i_2)_*^{-1} = \iota^X$, we see that for any choice of the retraction $\mathrm{ch}_1^{\mathbb{Z}}$ of ι^X, we have $\mathrm{ch}_1^{\mathbb{Z}} \circ \beta_1^X = \mathrm{Id}_{H_1(X;\,\mathbb{Z})}$.

The case where X is of dimension 4 is completely similar. □

The following result gives another construction of β_1^X, referring to the fundamental group $\pi_1(X)$.

PROPOSITION 3.4. *For any connected CW-complex X, the map*

$$\widetilde{\alpha}_1^X : \pi_1(X) \longrightarrow K_1(X),\ [f] \longmapsto [S^1,\, 1_{S^1},\, f] = f_*([S^1]_K)$$

is a natural homomorphism. Factorizing through the Hurewicz homomorphism h_1^X in degree 1, it defines a natural map $\alpha_1^X : H_1(X;\mathbb{Z}) \longrightarrow K_1(X)$ that *coincides with* β_1^X*, so that we have the following commutative diagram:*

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\widetilde{\alpha}_1^X} & K_1(X) \\ {\scriptstyle h_1^X}\big\downarrow & \nearrow {\scriptstyle \beta_1^X} & \\ H_1(X;\mathbb{Z}) & & \end{array}$$

Proof. Since the bordism-type description of K-homology is homotopy-invariant (by the 'bordism relation'; see [28]), $\widetilde{\alpha}_1^X$ is well-defined. Let us show that it is a homomorphism. Let $[f]$, $[g] \in \pi_1(X)$. The product $[f] \cdot [g]$ in $\pi_1(X)$ is given by the class $[f \cdot g]$ of the composition

$$f \cdot g \colon S^1 \twoheadrightarrow S^1/S^0 \xrightarrow{\cong} S^1 \vee S^1 \xrightarrow{f \vee g} X.$$

On the other hand, $[S^1,\, 1_{S^1},\, f] + [S^1,\, 1_{S^1},\, g] = [S^1 \amalg S^1,\, 1_{S^1 \amalg S^1},\, f \amalg g]$ holds. It is easy to show that there is a continuous map $h \colon M \longrightarrow X$, where M is a pair of pants, i.e., a compact, connected and orientable surface with boundary $\partial M = S^1 \amalg S^1 \amalg S^1$, such that the restrictions of h to the three components of ∂M are f, g and $f \cdot g$, respectively. This shows that

$$\begin{aligned} \widetilde{\alpha}_1^X([f \cdot g]) &= [S^1,\, 1_{S^1},\, f \cdot g] = [S^1 \amalg S^1,\, 1_{S^1 \amalg S^1},\, f \amalg g] \\ &= \widetilde{\alpha}_1^X([f]) + \widetilde{\alpha}_1^X([g]). \end{aligned}$$

The naturality of $\widetilde{\alpha}_1^X$ and of α_1^X is clear. To prove that α_1^X and β_1^X coincide, first observe that any homology class $x \in H_1(X)$ is '*Steenrod-representable*'. This simply means that there exists a pointed continuous map $f \colon S^1 \longrightarrow X$ such that $f_* \colon H_1(S^1) \longrightarrow H_1(X)$ takes $[S^1]$ to x; this is a direct consequence of the surjectivity of the Hurewicz map $h_1^X \colon \pi_1(X) \longrightarrow H_1(X)$. By the naturality of α_1^X and of β_1^X, it is enough to check that $\alpha_1^{S^1}([S^1]) = \beta_1^{S^1}([S^1])$. The class $[S^1]$ is 'Steenrod-represented' by the identity map on S^1; therefore $\alpha_1^{S^1}([S^1]) = [S^1]_K$. The equality $\beta_1^{S^1}([S^1]) = [S^1]_K$ is a consequence of Proposition 2.1 (iii). □

Notice that $\widetilde{\alpha}_1^X \colon \pi_1(X) \longrightarrow K_1(X)$ is nothing but the Hurewicz homomorphism in K-homology.

For the second description of β_2^X, we discuss 'Steenrod-representability' further.

Theorem 3.5. *Consider a connected CW-complex X. Then any homology class $x \in H_2(X;\mathbb{Z})$ is 'Steenrod-representable'; in other words, there exists a surface Σ_g of genus $g \geq 1$ and a pointed continuous map $f \colon \Sigma_g \longrightarrow X$ (with both g and f depending on x) such that $f_* \colon H_2(\Sigma_g;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$ takes the fundamental class $[\Sigma_g]$ to x.*

For $x \in H_2(X;\mathbb{Z})$, we write $x = [\Sigma_g,\, f]$ to express the fact that x is 'Steenrod-represented' by $f \colon \Sigma_g \longrightarrow X$.

This theorem is well-known; see [18] and [50]. (The condition on the genus, namely $g \geq 1$, is no restriction, since the 2-sphere S^2 is a suitable quotient of the 2-torus $\mathbb{T}^2$, and the quotient map takes $[\mathbb{T}^2]$ to $[S^2]$.) We point out that in [35, Lemma 2.2.4]), there is an 'elementary' proof of Theorem 3.5 for $X = B\Gamma$, the classifying space of a discrete group Γ. We also mention Zimmermann's paper [52] in connection with this result.

We now turn to the second description of β_2^X.

Proposition 3.6. *For any connected CW-complex X, the map β_2^X is given by*

$$\beta_2^X : H_2(X;\mathbb{Z}) \longrightarrow K_0(X),\ [\Sigma_g,\, f] \longmapsto [\Sigma_g,\, 1_{\Sigma_g},\, f];$$

in other words, $f_*([\Sigma_g]) \longmapsto f_*([\Sigma_g]_K)$.

Proof. The result follows from the following computation:

$$\begin{aligned}\beta_2^X([\Sigma_g,\, f]) &= \beta_2^X \circ f_*([\Sigma_g,\, \mathrm{Id}_{\Sigma_g}]) = \beta_2^X \circ f_*([\Sigma_g]) = f_* \circ \beta_2^{\Sigma_g}([\Sigma_g]) \\ &= f_*([\Sigma_g]_K) = f_*([\Sigma_g,\, 1_{\Sigma_g},\, \mathrm{Id}_{\Sigma_g}]) = [\Sigma_g,\, 1_{\Sigma_g},\, f]. \qquad \square\end{aligned}$$

The next proposition follows from 'Steenrod-representability' and the fact that the point, the circle and the surfaces Σ_g are torsion-free. (For convenience, we set $[pt] := 1 \in H_0(pt;\mathbb{Z})$ and $[pt]_K := [1] \in K_0(pt)$.)

Proposition 3.7. *Fix $0 \leq j \leq 2$; the map β_j^X is the unique natural transformation for connected CW-complexes satisfying either of the following properties:*

(i) *$\beta_j^M([M]) = [M]_K$, for $M = pt$, S^1, and Σ_g for all $g \geq 1$, respectively.*
(ii) *It is rationally a right-inverse of the Chern character.*

Finally, we give a description of the maps β_j^X based on the Spin^c-bordism $\Omega_*^{\mathrm{Spin}^c}$. The first few coefficient groups are $\Omega_0^{\mathrm{Spin}^c} \cong \mathbb{Z}$, $\Omega_1^{\mathrm{Spin}^c} = 0$, $\Omega_2^{\mathrm{Spin}^c} \cong \mathbb{Z}$ and $\Omega_3^{\mathrm{Spin}^c} = 0$ (see, for example, Gilkey [24]). For a connected CW-complex X, there is a natural map $\rho_*^X : \Omega_*^{\mathrm{Spin}^c}(X) \longrightarrow H_*(X;\mathbb{Z})$, $[M,\, f] \longmapsto f_*([M])$, and the Atiyah-Hirzebruch spectral sequence yields natural isomorphisms $\rho_j^X : \Omega_j^{\mathrm{Spin}^c}(X) \xrightarrow{\cong} H_j(X;\mathbb{Z})$ for $j = 0, 1$, and $p_* \oplus \rho_2^X : \Omega_2^{\mathrm{Spin}^c}(X) \xrightarrow{\cong} \mathbb{Z} \oplus H_2(X;\mathbb{Z})$, where $p\colon X \twoheadrightarrow pt$ (compare with 7.2 on page 17 of [18]). We write $(\rho_2^X)^{-1}$ for the restriction of $(p_* \oplus \rho_2^X)^{-1}$ to the direct summand $H_2(X;\mathbb{Z})$. On the other hand, there is a graded natural map $\kappa_*^X : \Omega_*^{\mathrm{Spin}^c}(X) \longrightarrow K_*(X)$, $[M,\, f] \longmapsto f_*([M]_K)$. The following result follows readily from Propositions 3.4 and 3.6.

Proposition 3.8. *For a connected CW-complex X and for $0 \leq j \leq 2$, one has*

$$\beta_j^X = \kappa_*^X \circ (\rho_j^X)^{-1} : H_j(X;\mathbb{Z}) \longrightarrow K_j(X),\ f_*([M]) \longmapsto f_*([M]_K).$$

One could replace $\Omega_*^{\mathrm{Spin}^c}(X)$ by the complex bordism $\Omega_*^U(X)$, since they coincide in degree ≤ 3 (cf. [24]). The fact that $\Omega_2^{\mathrm{Spin}^c} \neq 0$ illustrates the difficulties one faces in trying to define a natural map β_3^X by the same method: there is a natural exact sequence

$$0 \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow \Omega_3^{\mathrm{Spin}^c}(X) \xrightarrow{\rho_3^X} H_3(X;\mathbb{Z}) \longrightarrow 0,$$

but, in general, it does *not* split (see Lemma 6.1 and the proof of Theorem 2.1).

4. Injectivity of β_1^X

We prove the injectivity of β_1^X for any connected CW-complex X. Roughly speaking, the idea is to first prove injectivity for $X = B\mathbb{Z}$ and for $X = B(\mathbb{Z}/n)$. This poses no difficulty, since their reduced integral homology is concentrated in odd degree. As an application, we describe the K-homology of CW-complexes of dimension ≤ 4 in terms of the integral homology.

Theorem 4.1. *The map $\beta_1^X\colon H_1(X;\mathbb{Z}) \longrightarrow K_1(X)$ is injective for any connected CW-complex X.*

In the proof of Theorem 4.1, we need the following proposition that gives a fourth description of the maps β_j^X for $j=1$ and $j=2$. We have to recall some facts before stating it. We have defined a 2-periodic integral homology theory h_* in the proof of Proposition 2.1. Since we are working with compact supports, it is defined for all connected CW-complexes. Consider the Atiyah-Hirzebruch spectral sequence with $E^2_{p,\,q} = \widetilde{H}_p(X;\, h_q(pt)) \implies \widetilde{K}_{p+q}(X)$. Convergence means that $E^\infty_{p,\,q} = J_{p,\,q}/J_{p-1,\,q+1}$, where $\{J_{p,\,q}\}$ is the filtration

$$0 = J_{0,\,n} \subseteq J_{1,\,n-1} \subseteq \ldots \subseteq J_{p,\,n-p} \subseteq \ldots \subseteq J_{n,\,0} \subseteq \ldots \subseteq \bigcup_{p+q=n} J_{p,\,q} = \widetilde{K}_n(X)$$

defined by $J_{p,\,q} := \mathrm{Im}(\widetilde{K}_{p+q}(X^{[p]}) \longrightarrow \widetilde{K}_{p+q}(X))$. Notice that for $1 \leq j \leq 3$, there is a natural epimorphism $H_j(X;\mathbb{Z}) = E^2_{j,\,0} \twoheadrightarrow E^\infty_{j,\,0} = J_{j,\,0}/J_{j-1,\,1}$, with $J_{0,\,1} = 0$ (since $X^{[0]} = pt$, as a standing assumption) and $J_{1,\,1} = 0$ because, by Proposition 2.1 (iii), $\widetilde{K}_0(X^{[1]}) = 0$. We therefore get a natural map

$$\delta_j^X : H_j(X;\mathbb{Z}) = E^2_{j,\,0} \twoheadrightarrow E^\infty_{j,\,0} = J_{j,\,0} = \mathrm{Im}\left(\widetilde{K}_j(X^{[j]}) \longrightarrow \widetilde{K}_j(X)\right) \hookrightarrow \widetilde{K}_j(X),$$

for $j=1$ and $j=2$.

Proposition 4.2. *Let X be a connected CW-complex. For $j=1$ and $j=2$, the map δ_j^X coincides with β_j^X. In particular, β_j^X is injective if and only if no non-zero differential reaches $E^{\geq 2}_{j,\,0}$ in the spectral sequence. Consequently, if the reduced integral homology of X is concentrated in odd (resp. even) degree, except possibly for H_2 (resp. H_1 and H_3), then β_1^X (resp. $\beta_0^X \oplus \beta_2^X$) is injective.*

Proof. By naturality, it is enough to check the equality between both maps in the case of S^1 for $j = 1$, and in the case of the surfaces Σ_g for $j = 2$. This is obvious. The injectivity of the map is clear. □

This proposition also illustrates the difficulty in trying to define a map β_3^X: The group $J_{2,1}$ is in general non-zero, since it is isomorphic to $H_1(X;\mathbb{Z})$, by virtue of Proposition 2.1 (iii).

We now turn to the main proof in this section.

Proof of Theorem 4.1. Let us first assume that $H := H_1(X;\mathbb{Z})$ is of finite type (as, for example, is the case when X is finite). So, H is a finite direct sum of cyclic groups. Let G be a direct summand of H, isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}/n$ for some $n \geq 2$. The projection $p\colon H \twoheadrightarrow G$ together with the map $f\colon X \longrightarrow B\pi_1(X)$ classifying the universal covering of X define a map

$$F\colon X \xrightarrow{f} B\pi_1(X) \xrightarrow{B\pi} BH \xrightarrow{Bp} BG,$$

where $\pi\colon \pi_1(X) \twoheadrightarrow \pi_1(X)^{ab} = H$. Clearly, $H_1(F)$ is merely the projection p. Since the reduced integral homology of G (i.e., of $\mathbb{Z}$ or of $\mathbb{Z}/n$) is concentrated in odd degree, by Proposition 4.2, the map β_1^{BG} is injective. By the naturality of β_1, one has a commutative diagram

$$\begin{array}{ccc} H_1(X;\mathbb{Z}) & \xrightarrow{\beta_1^X} & K_1(X) \\ {\scriptstyle H_1(F)=p}\downarrow & & \downarrow{\scriptstyle K_1(F)} \\ H_1(BG;\mathbb{Z}) & \xhookrightarrow{\beta_1^{BG}} & K_1(BG) \end{array}$$

So, $K_1(F) \circ \beta_1^X$ is injective on the direct summand G of H and is the zero map on the supplementary direct summand. This establishes the injectivity of β_1^X for X of finite type. Obviously, $H_1(X;\mathbb{Z})$ and $K_1(X)$ coincide with the direct limits over the connected finite sub-complexes of X, of the corresponding groups. Since the direct limit over a directed set is an exact functor, it preserves injectivity (see [43, Theorem 2.18]). This shows that β_1^X is injective for any X, as was to be shown. □

The following result can be proved in the same way as Proposition 2.1, using Theorem 4.1 and Proposition 4.2.

Proposition 4.3. *For a connected CW-complex X of dimension ≤ 4, there are natural short exact sequences*

$$0 \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow K_1(X) \xrightarrow{\mathrm{ch}_3^{\mathbb{Z}}} H_3(X;\mathbb{Z}) \longrightarrow 0\ ,$$

$$0 \longrightarrow H_2(X;\mathbb{Z}) \longrightarrow \widetilde{K}_0(X) \xrightarrow{\mathrm{ch}_4^{\mathbb{Z}}} H_4(X;\mathbb{Z}) \longrightarrow 0.$$

The latter sequence splits and yields an (abstract) isomorphism

$$\mathrm{ch}_0^{\mathbb{Z}} \oplus \mathrm{ch}_2^{\mathbb{Z}} \oplus \mathrm{ch}_4^{\mathbb{Z}}\colon K_0(X) \xrightarrow{\cong} H_0(X;\mathbb{Z}) \oplus H_2(X;\mathbb{Z}) \oplus H_4(X;\mathbb{Z}).$$

In particular, if X is simply-connected, then $\mathrm{ch}_3^{\mathbb{Z}}\colon K_1(X) \xrightarrow{\cong} H_3(X;\mathbb{Z})$ *is a canonical and natural isomorphism. The second short exact sequence above is also valid and natural in case X is of dimension* 5.

Remark 4.4. In general, for a non-simply-connected CW-complex X of dimension 4, one cannot expect an isomorphism $K_1(X) \cong H_1(X;\mathbb{Z}) \oplus H_3(X;\mathbb{Z})$. For example, the projective space $\mathbb{R}P^4$ satisfies $H_1(\mathbb{R}P^4;\mathbb{Z}) \cong \pi_1(\mathbb{R}P^4)^{ab} \cong \mathbb{Z}/2$ and $H_3(\mathbb{R}P^4;\mathbb{Z}) \cong \mathbb{Z}/2$, whereas $K_1(\mathbb{R}P^4) \cong \mathbb{Z}/4$.

5. (Non-)Injectivity properties of β_2^X

We prove an injectivity result for β_2^X (after inverting some primes) when the *co*homology group $H^3(X;\mathbb{Z})$ is "under control". We also show that β_2^X is in general not injective. For this purpose, we first establish a 'Künneth-type' result for β_2^X.

The first injectivity result for β_2^X is contained in Proposition 4.2. Before stating the second one, we state and prove the following lemma from standard K-theory (for convenience, we set $(-1)! := 1$).

Lemma 5.1.

(i) *For a finite CW-complex X and for $n \geq 0$, there are canonical and natural homomorphisms* $\mathrm{ch}^{2n}_{\mathbb{Z}[\frac{1}{n!}]}$ *and* $\mathrm{ch}^{2n+1}_{\mathbb{Z}[\frac{1}{(n+1)!}]}$ *fitting in the following commutative triangles:*

$$\begin{array}{ccc} & & H^{2n}(X;\mathbb{Z}[\tfrac{1}{n!}]) \\ & \overset{\mathrm{ch}^{2n}_{\mathbb{Z}[\frac{1}{n!}]}}{\nearrow} & \downarrow \\ K^0(X) & & \\ & \underset{\mathrm{ch}^{2n}}{\searrow} & \\ & & H^{2n}(X;\mathbb{Q}) \end{array} \qquad \begin{array}{ccc} & & H^{2n+1}(X;\mathbb{Z}[\tfrac{1}{(n+1)!}]) \\ & \overset{\mathrm{ch}^{2n+1}_{\mathbb{Z}[\frac{1}{(n+1)!}]}}{\nearrow} & \downarrow \\ K^1(X) & & \\ & \underset{\mathrm{ch}^{2n+1}}{\searrow} & \\ & & H^{2n+1}(X;\mathbb{Q}) \end{array}$$

(ii) *If X is finite and of dimension $\leq 2n+1$, then the map* $\mathrm{ch}^{\mathrm{ev}}_{\mathbb{Z}[\frac{1}{n!}]}$ *defined as*

$$\bigoplus_{j=0}^{n} \mathrm{ch}^{2j}_{\mathbb{Z}[\frac{1}{j!}]} \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{n!}]}\colon K^0(X)\otimes\mathbb{Z}\big[\tfrac{1}{n!}\big] \longrightarrow H^{\mathrm{ev}}(X;\mathbb{Z}\big[\tfrac{1}{n!}\big])$$

is an isomorphism.

(iii) *If X is finite and of dimension $\leq 2n$, then the map* $\mathrm{ch}^{\mathrm{odd}}_{\mathbb{Z}[\frac{1}{n!}]}$ *defined as*

$$\bigoplus_{j=0}^{n-1} \mathrm{ch}^{2j+1}_{\mathbb{Z}[\frac{1}{(j+1)!}]} \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{n!}]}\colon K^1(X)\otimes\mathbb{Z}\big[\tfrac{1}{n!}\big] \longrightarrow H^{\mathrm{odd}}(X;\mathbb{Z}\big[\tfrac{1}{n!}\big])$$

is an isomorphism.

Proof. (i) We first deal with the even case. For $n = 0$, the result is trivial, so we suppose $n \geq 1$. The Chern character ch^{2n} is expressed in terms of the (rational) universal Chern classes as $\mathrm{ch}^{2n} = (-1)^{n-1}\frac{1}{(n-1)!}c_n + P_n(c_1, \ldots, c_{n-1})$ with $P_n \in \mathbb{Z}\big[\frac{1}{n!}\big][x_1, \ldots, x_{n-1}]$ a suitable polynomial *without constant term* (see, for example, Karoubi [29, pp. 255 and 282]). This proves the existence and naturality of $\mathrm{ch}^{2n}_{\mathbb{Z}[\frac{1}{n!}]}$. The proof that this map is a homomorphism is the same as for ch^{2n} (cf. *loc. cit.*). The 'compatibility' with ch^{2n} is clear. By the very definition of $\mathrm{ch}^{\mathrm{odd}}$, the odd case is obtained by replacing X by its suspension ΣX and invoking the suspension isomorphism in K-theory and in cohomology.

(ii) For simplicity, we denote the ring $\mathbb{Z}\big[\frac{1}{(n-1)!}\big]$ by Λ. From Peterson's computations [41] of the k-invariants of the infinite Grassmannians $BU(k)$ it follows that $(BU_\Lambda)^{(2n+1)}$, the $(2n+1)$st Postnikov approximation of BU localized at Λ, is homotopically equivalent to the product (in the category of CW-complexes) $\prod_{j=1}^{n} K(\Lambda, 2j)$. The equivalence is realized by the product $c_1^\Lambda \times \ldots \times c_n^\Lambda$ of the Chern classes $c_j^\Lambda \in H^{2j}(BU; \Lambda)$ (compare with Chapter VII in [25]). It follows that for $\dim(X) \leq 2n+1$, the map $c_1^\Lambda \oplus \ldots \oplus c_n^\Lambda \colon \widetilde{K}^0(X) \otimes \Lambda \longrightarrow \bigoplus_{j=1}^{n} H^{2j}(X; \Lambda)$ is bijective (but in general *not* a homomorphism). Now, the expression ch^{2j} given in the proof of (i) above shows inductively – with j ranging from 1 to n – that $\mathrm{ch}^{\mathrm{ev}}_{\mathbb{Z}[\frac{1}{n!}]}$ is injective (recalling that P_j is without constant term). Surjectivity is proved in the same way.

(iii) Apply (ii) to ΣX. □

Let us also introduce a notation. For a finitely generated abelian group A, we denote by $\mathrm{Tors}(A)$ its torsion subgroup, i.e., the subgroup of elements of finite order in A; this subgroup is a finite direct summand of A.

PROPOSITION 5.2. *Let X be a connected finite CW-complex. Assume that there exists $N \geq 1$ (depending on X) such that the composition*

$$K^1(X) \otimes \mathbb{Z}\big[\tfrac{1}{2N}\big] \xrightarrow{\mathrm{ch}^3_{\mathbb{Z}[\frac{1}{2}]} \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]}} H^3(X; \mathbb{Z}\big[\tfrac{1}{2N}\big]) \twoheadrightarrow \mathrm{Tors}\big(H^3(X; \mathbb{Z}\big[\tfrac{1}{2N}\big])\big)$$

is surjective. (This assumption is, for example, fulfilled for $N = n!/2$ when X is of dimension $\leq 2n$ for some $n \geq 2$.) Then the map $(\beta_0^X \oplus \beta_2^X) \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]}$ is injective.

Proof. By the Universal Coefficient Theorems in homology (see Spanier [48, Theorem 5.5.12, p. 248]) and in K-homology (see Brown [13]), we have the commutative diagram with (non-naturally) split-exact rows

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(H^3(X);\mathbb{Z}[\tfrac{1}{2N}]) & \longrightarrow & H_2(X;\mathbb{Z}[\tfrac{1}{2N}]) & \longrightarrow & \mathrm{Hom}(H^2(X);\mathbb{Z}[\tfrac{1}{2N}]) & \to 0 \\
& \downarrow \mathrm{Ext}(\mathrm{ch}^3_{\mathbb{Z}[\frac{1}{2}]};\mathbb{Z}[\tfrac{1}{2N}]) & & \downarrow \beta_2^X \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]} & & \downarrow \mathrm{Hom}(c_1;\mathbb{Z}[\tfrac{1}{2N}]) & \\
0 \to & \mathrm{Ext}(K^1(X);\mathbb{Z}[\tfrac{1}{2N}]) & \longrightarrow & K_0(X)\otimes\mathbb{Z}[\tfrac{1}{2N}] & \longrightarrow & \mathrm{Hom}(K^0(X);\mathbb{Z}[\tfrac{1}{2N}]) & \to 0
\end{array}$$

We have identified $\mathrm{Ext}(H^3(X)\otimes\mathbb{Z}[\frac{1}{2}];\,\mathbb{Z}[\frac{1}{2N}])$ with $\mathrm{Ext}(H^3(X);\mathbb{Z}[\frac{1}{2N}])$ in the obvious way, and similarly for K^1 in place of H^3. For the application of Brown's result note that a finite CW-complex is compact metrizable (see, for instance, [23, Proposition 1.5.17]). By assumption, the left vertical map is injective, and so is the right one, since the first Chern class map c_1 is surjective. Now, the injectivity result follows from a diagram chase. □

We now turn to the 'Künneth-type' result for β_2^X.

In the bordism-type description of K-homology, for two connected CW-complexes X and Y, the external cross product is given by

$$\begin{aligned}
\times\colon K_i(X)\times K_j(Y) &\longrightarrow K_{i+j}(X\times Y)\\
([M,\,\xi,\,f],\,[N,\,\eta,\,g]) &\longmapsto [M\times N,\,\xi\boxtimes\eta,\,f\times g],
\end{aligned}$$

where the product $M\times N$ is endowed with the product Spin^c-structure (see [27, p. 18]). Here, $\mathrm{Vect}(M)\times\mathrm{Vect}(N)\longrightarrow\mathrm{Vect}(M\times N)$, $(\xi,\,\eta)\longmapsto\xi\boxtimes\eta$ denotes the external tensor product of vector bundles.

Let us explain our notation concerning the gradings: $[K_*(X)\otimes K_*(Y)]_1$ denotes $(K_0(X)\otimes K_1(Y))\oplus(K_1(X)\otimes K_0(Y))$; similarly, $[\mathrm{Tor}(K_*(X);\,K_*(Y))]_1$ stands for $\mathrm{Tor}(K_0(X);\,K_1(Y))\oplus\mathrm{Tor}(K_1(X);\,K_0(Y))$, and so on. Now, we recall that the Künneth Theorem in K-homology (see Bousfield [11]) states that for two connected CW-complexes X and Y, and for $n\in\mathbb{Z}$, there is a natural short exact sequence

$$0\longrightarrow[K_*(X)\otimes K_*(Y)]_n\xrightarrow{\ \times\ }K_n(X\times Y)\longrightarrow[\mathrm{Tor}(K_*(X);\,K_*(Y))]_{n-1}\longrightarrow 0.$$

(We do not know if the sequence splits.) For the next lemma, using the decomposition $K_0=\mathbb{Z}\oplus\widetilde{K}_0$ of Proposition 2.1 (i), $[K_*(X)\otimes K_*(Y)]_0$ identifies with

$$\Big(\big(K_1(X)\otimes K_1(Y)\big)\,\oplus\,\widetilde{K}_0(X)\,\oplus\,\widetilde{K}_0(Y)\Big)\;\oplus\;\Big(\mathbb{Z}\,\oplus\,\big(\widetilde{K}_0(X)\otimes\widetilde{K}_0(Y)\big)\Big).$$

Since β_2 maps H_2 in $\widetilde{K}_0$, the following statement makes full sense.

LEMMA 5.3. *Let X and Y be two connected CW-complexes. Then the map $\beta_2^{X\times Y}$ fits into the commutative diagram*

$$\begin{array}{ccc}
\big(H_1(X)\otimes H_1(Y)\big)\oplus H_2(X)\oplus H_2(Y) & \xrightarrow{(\beta_1^X\otimes\beta_1^Y)\oplus\beta_2^X\oplus\beta_2^Y} & [K_*(X)\otimes K_*(Y)]_0 \\
\times\downarrow\cong & & \times\downarrow \\
H_2(X\times Y) & \xrightarrow{\beta_2^{X\times Y}} & K_0(X\times Y)
\end{array}$$

In particular, $\beta_2^{X\times Y}$ is injective if and only if the three maps $\beta_1^X\otimes\beta_1^Y$, β_2^X and β_2^Y are injective.

Proof. The vertical maps are given by the corresponding Künneth Theorem. We split the proof of the commutativity into two parts. First, by the naturality and additivity of β_2, the diagram

$$\begin{array}{ccc}
H_2(X)\oplus H_2(Y) & \xrightarrow{\beta_2^X\oplus\beta_2^Y} & \widetilde{K}_0(X)\oplus\widetilde{K}_0(Y) \\
(j_X)_*\oplus(j_Y)_*\downarrow & & \downarrow(j_X)_*\oplus(j_Y)_* \\
H_2(X\times Y)\oplus H_2(X\times Y) & \xrightarrow{\beta_2^{X\times Y}\oplus\beta_2^{X\times Y}} & \widetilde{K}_0(X\times Y)\oplus\widetilde{K}_0(X\times Y) \\
+\downarrow & & \downarrow+ \\
H_2(X\times Y) & \xrightarrow{\beta_2^{X\times Y}} & K_0(X\times Y)
\end{array}$$

commutes, where j_X (resp. j_Y) denotes the inclusion (with respect to the corresponding base-point) of X (resp. Y) in $X\times Y$. Both vertical compositions coincide with the corresponding restrictions of the vertical maps in the statement. This is well-known for the homology, and for the K-homology it follows from the following explicit computation:

$$\begin{aligned}
\big(K_j(X)\otimes 1\big)\oplus\big(1\otimes K_j(Y)\big) &\overset{\times}{\hookrightarrow} K_j(X\times Y),\\
([M,\,\xi,\,f],\,[N,\,\eta,\,g]) &\longmapsto [M,\,\xi,\,f\times c_{y_0}]+[N,\,\eta,\,c_{x_0}\times g]\\
&\qquad = [M\amalg N,\,\xi\amalg\eta,\,(f\times c_{y_0})\amalg(c_{x_0}\times g)],
\end{aligned}$$

where c_{x_0} and c_{y_0} are the constant maps taking the base-point of X and Y, respectively, as value, and $j=0$ or $j=1$.

For the second part, let $x=[S^1,\,f]\in H_1(X)$ and $y=[S^1,\,g]\in H_1(Y)$ be 'Steenrod-represented' homology classes. By the naturality of the external cross product in homology, the left-hand square of the following diagram commutes:

$$\begin{array}{ccccc}
H_1(S^1)\otimes H_1(S^1) & \xrightarrow{f_*\otimes g_*} & H_1(X)\otimes H_1(Y) & \xrightarrow{\beta_1^X\otimes\beta_1^Y} & K_1(X)\otimes K_1(Y) \\
\times\downarrow & & \times\downarrow & & \times\downarrow \\
H_2(\mathbb{T}^2) & \xrightarrow{(f\times g)_*} & H_2(X\times Y) & \xrightarrow{\beta_2^{X\times Y}} & K_0(X\times Y)
\end{array}$$

We claim that the right-hand square also commutes. For one composition

$$\beta_1^X([S^1,\, f]) \times \beta_1^Y([S^1,\, g]) = [S^1,\, 1_{S^1},\, f] \times [S^1,\, 1_{S^1},\, g] = [\mathbb{T}^2,\, 1_{\mathbb{T}^2},\, f \times g]$$

holds. Since $[S^1] \times [S^1] = [\mathbb{T}^2]$, for the second composition, we first compute

$$[S^1,\, f] \times [S^1,\, g] = f_*([S^1]) \times g_*([S^1]) = (f \times g)_*([\mathbb{T}^2]) = [\mathbb{T}^2,\, f \times g],$$

from which we deduce that $\beta_2^{X \times Y}([S^1,\, f] \times [S^1,\, g]) = [\mathbb{T}^2,\, 1_{\mathbb{T}^2},\, f \times g]$, as claimed.

Altogether, this proves that the diagram of the lemma commutes. The statement about injectivity is clear. □

The 'Künneth-type' Lemma 5.3 allows us to construct an example of a finite CW-complex for which β_2^X is not injective: Namely, we claim that the map $\beta_2^{\mathbb{R}P^2 \times \mathbb{R}P^4}$ is *not* injective. For $\mathbb{R}P^2$, one has $\beta_1^{\mathbb{R}P^2} \colon \mathbb{Z}/2 \cong H_1(\mathbb{R}P^2;\, \mathbb{Z}) \xrightarrow{\cong} K_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ and for $\mathbb{R}P^4$, $\beta_1^{\mathbb{R}P^4} \colon \mathbb{Z}/2 \cong H_1(\mathbb{R}P^4;\, \mathbb{Z}) \hookrightarrow K_1(\mathbb{R}P^4) \cong \mathbb{Z}/4$, and we see that $\beta_1^{\mathbb{R}P^2} \otimes \beta_1^{\mathbb{R}P^4}$ is the zero map $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$. The non-injectivity of $\beta_2^{\mathbb{R}P^2 \times \mathbb{R}P^4}$ follows from Lemma 5.3, as claimed. Since for applications we are mainly interested in classifying spaces of discrete groups, it is worth mentioning that the same argument shows that the map $\beta_2^{\mathbb{R}P^\infty \times \mathbb{R}P^\infty}$ is *not* injective, and, of course, $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ is nothing but $B(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$.

6. The map $\beta_3^X\left[\frac{1}{2}\right]$

In this section, we construct the homomorphism $\beta_3^X\left[\frac{1}{2}\right]$ of the Introduction using the Postnikov tower of the connective K-theory spectrum.

Let **bu** denote the connective complex K-theory spectrum. We write $\mathbf{bu}_*(-)$ for the associated homology theory, namely connective K-homology, on the category of CW-complexes (and similarly for other spectra). Recall that there is a (connected covering) map $\kappa \colon \mathbf{bu} \longrightarrow \mathbf{BU}$ inducing the following relation for the coefficients:

$$\mathbf{bu}_n(pt) \overset{\kappa_*}{\cong} \begin{cases} 0, & \text{if } n < 0 \\ K_n(pt) \cong \mathbb{Z}, & \text{if } n \geq 0 \text{ is even} \\ K_n(pt) = 0, & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

By Adams-Priddy [1] (see [44, Theorem VI.2.6]), the Postnikov tower of **bu** is given by a diagram

$$\begin{array}{ccccccc} & & \Sigma^4\mathbf{H}\mathbb{Z} & & \Sigma^2\mathbf{H}\mathbb{Z} & & \\ & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \\ \cdots & \xrightarrow{p_3} & \mathbf{k^2} & \xrightarrow{p_2} & \mathbf{k^1} & \xrightarrow{p_1} & \mathbf{k^0} = \mathbf{H}\mathbb{Z} \\ & & \downarrow{\scriptstyle \sigma_3} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} \\ & & \Sigma^7\mathbf{H}\mathbb{Z} & & \Sigma^5\mathbf{H}\mathbb{Z} & & \Sigma^3\mathbf{H}\mathbb{Z} \end{array}$$

with $\mathbf{k^r}$ as the Postnikov $2r$-stage of $\mathbf{bu}$, i.e., $\mathbf{k^r} = \mathbf{bu}^{(2r)}$. In particular, there is a map $\tau_{2r}\colon \mathbf{bu} \longrightarrow \mathbf{k^r}$ that is an isomorphism on π_j for $j \leq 2r$, and $\pi_j(\mathbf{k^r}) = 0$ for $j > 2r$. Note that the relation $p_r \circ \tau_{2r} \simeq \tau_{2r-2}$ holds, and for any CW-complex X, $(\tau_{2r})_*\colon \mathbf{bu}_j(X) \xrightarrow{\cong} \mathbf{k}^{\mathbf{r}}_j(X)$ is an isomorphism for $j \leq 2r$ (by an easy comparison of the Atiyah-Hirzebruch spectral sequences). Therefore, the cofiber sequence

$$\Sigma^4\mathbf{H}\mathbb{Z} \xrightarrow{j_2} \mathbf{k^2} \xrightarrow{p_2} \mathbf{k^1} \xrightarrow{\sigma_2} \Sigma^5\mathbf{H}\mathbb{Z}$$

yields the natural isomorphism $(\tau_2)_* = (p_2)_* \circ (\tau_4)_*\colon \mathbf{bu}_j(X) \xrightarrow{\cong} \mathbf{k}^{\mathbf{1}}_j(X)$ for $j \leq 3$.

Lemma 6.1. *After localizing at* $\mathbb{Z}\big[\frac{1}{2}\big]$, *the spectrum* $\mathbf{k^1}$ *splits as*

$$\mathbf{k^1}\big[\tfrac{1}{2}\big] \simeq \mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big] \vee \Sigma^2\mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big]$$

(but $\mathbf{k^1}$ *itself does not split).*

Proof. The cofiber sequence $\Sigma^2\mathbf{H}\mathbb{Z} \xrightarrow{j_1} \mathbf{k^1} \xrightarrow{p_1} \mathbf{H}\mathbb{Z} \xrightarrow{\sigma_1} \Sigma^3\mathbf{H}\mathbb{Z}$ yields, after localizing at $\mathbb{Z}\big[\frac{1}{2}\big]$, the cofiber sequence

$$\Sigma^2\mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big] \xrightarrow{j_1[\frac{1}{2}]} \mathbf{k^1}\big[\tfrac{1}{2}\big] \xrightarrow{p_1[\frac{1}{2}]} \mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big] \xrightarrow{\sigma_1[\frac{1}{2}]} \Sigma^3\mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big]$$

(see [44, Proposition II.5.3]). By Adams-Priddy [1] (see also [44, Theorem VI.2.6]), the map σ_1 is the composition

$$\sigma_1 = \delta \circ Sq^2 \circ \mathbf{H}\rho\colon \mathbf{H}\mathbb{Z} \xrightarrow{\mathbf{H}\rho} \mathbf{H}\mathbb{Z}/2 \xrightarrow{Sq^2} \Sigma^2\mathbf{H}\mathbb{Z}/2 \xrightarrow{\delta} \Sigma^3\mathbf{H}\mathbb{Z},$$

where ρ is the reduction modulo 2, and δ is the integral Bockstein morphism. Since $\mathbf{H}\mathbb{Z}/2\big[\frac{1}{2}\big] \simeq *$ (all the homotopy groups being trivial by [44, Theorem II.5.4]), the map $\sigma_1\big[\frac{1}{2}\big]$ is null-homotopic. It follows that $\mathbf{k^1}\big[\frac{1}{2}\big] \simeq \mathbf{H}\mathbb{Z}\big[\frac{1}{2}\big] \vee \Sigma^2\mathbf{H}\mathbb{Z}\big[\frac{1}{2}\big]$ (by [44, Proposition II.1.17]), as was to be shown. The non-splitting statement follows, since σ_1 is a non-trivial cohomology operation. □

Consequently, there exists a section $s\colon \mathbf{H}\mathbb{Z}\big[\frac{1}{2}\big] \longrightarrow \mathbf{k^1}\big[\frac{1}{2}\big]$ of $p_1\big[\frac{1}{2}\big]$; we choose one. By [44, Theorem II.5.4], for a CW-complex X, there is a natural isomorphism $\mathbf{k}^{\mathbf{1}}_3\big[\frac{1}{2}\big](X) \cong \mathbf{k}^{\mathbf{1}}_3(X) \otimes \mathbb{Z}\big[\frac{1}{2}\big]$. This allows us to define the map $\beta^X_3\big[\frac{1}{2}\big]$ as follows.

Definition 6.2. For a CW-complex X, the map $\beta^X_3\big[\frac{1}{2}\big]$ is the composition

$$\begin{array}{ccc}
H_3(X;\mathbb{Z}[\frac{1}{2}]) \overset{s_*}{\hookrightarrow} \mathbf{k}^{\mathbf{1}}_3[\frac{1}{2}](X) \cong \mathbf{k}^{\mathbf{1}}_3(X)\otimes\mathbb{Z}[\frac{1}{2}] & \xrightarrow[\cong]{(\tau_2)_*^{-1}\otimes\mathbb{Z}[\frac{1}{2}]} & \mathbf{bu}_3(X)\otimes\mathbb{Z}[\frac{1}{2}] \\
{\scriptstyle \beta^X_3[\frac{1}{2}]}\,\Big\downarrow & & \Big\downarrow\,{\scriptstyle \kappa_*\otimes\mathbb{Z}[\frac{1}{2}]} \\
K_1(X)\otimes\mathbb{Z}[\frac{1}{2}] & \xleftarrow[\cong]{\mathcal{B}\otimes\mathbb{Z}[\frac{1}{2}]} & K_3(X)\otimes\mathbb{Z}[\frac{1}{2}]
\end{array}$$

where $\mathcal{B}$ is the Bott periodicity isomorphism.

The main properties of this construction are collected in the next theorem.

THEOREM 6.3. *Let X be a connected CW-complex. Then:*

(i) *$\beta_3^X\bigl[\frac{1}{2}\bigr]$ does not depend on the choice of the section s of $p_1\bigl[\frac{1}{2}\bigr]$ (more precisely, s is unique up to homotopy).*

(ii) *$\beta_3^X\bigl[\frac{1}{2}\bigr]$ is a natural homomorphism and is rationally a right-inverse of the Chern character, that is, $(\mathrm{ch}_3 \otimes \mathrm{Id}_{\mathbb{Q}}) \circ (\beta_3^X\bigl[\frac{1}{2}\bigr] \otimes \mathrm{Id}_{\mathbb{Q}}) = \mathrm{Id}_{H_3(X;\,\mathbb{Q})}$ holds.*

(iii) *Assume X is finite and that there exists $N \geq 1$ such that the Chern maps $\mathrm{ch}^3_{\mathbb{Z}[\frac{1}{2}]} \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]}$ and $\mathrm{ch}^4_{\mathbb{Z}[\frac{1}{2}]} \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]}$ are surjective onto the torsion like in Proposition 5.2 (as, for example, for $N = n!/2$ if X is of dimension $\leq 2n$ with $n \geq 2$); then $(\beta_1^X \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]}) \oplus (\beta_3^X\bigl[\frac{1}{2}\bigr] \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{2N}]})$ is injective.*

(iv) *For $0 \leq j \leq 2$, the map $\beta_j^X\bigl[\frac{1}{2}\bigr]$ obtained by adapting definition 6.2 in the obvious way coincides with β_j^X after tensoring with $\mathbb{Z}\bigl[\frac{1}{2}\bigr]$.*

Proof. (i) By [44, Proposition II.1.16], there is a canonical isomorphism

$$\Bigl[\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr],\, \mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr] \vee \Sigma^2\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\Bigr] \cong \Bigl[\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr],\, \mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\Bigr] \oplus \Bigl[\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr],\, \Sigma^2\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\Bigr].$$

A choice of a section s provides an equivalence $f\colon \mathbf{k^1}\bigl[\frac{1}{2}\bigr] \xrightarrow{\simeq} \mathbf{H}\mathbb{Z}\bigl[\frac{1}{2}\bigr] \vee \Sigma^2\mathbf{H}\mathbb{Z}\bigl[\frac{1}{2}\bigr]$, and any other section s' is, up to homotopy, uniquely determined by the composition $f \circ s' \in \Bigl[\mathbf{H}\mathbb{Z}\bigl[\frac{1}{2}\bigr],\, \mathbf{H}\mathbb{Z}\bigl[\frac{1}{2}\bigr] \vee \Sigma^2\mathbf{H}\mathbb{Z}\bigl[\frac{1}{2}\bigr]\Bigr]$, which is an element taken to $(\mathrm{Id}_{\mathbf{H}\mathbb{Z}[\frac{1}{2}]},\, 0)$ via the above isomorphism, where 0 denotes the constant map. It follows that the homotopy classes of sections of $p_1\bigl[\frac{1}{2}\bigr]$ are parameterized by the group

$$\begin{aligned}\Bigl[\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr],\, \Sigma^2\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\Bigr] &= \bigl(\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\bigr)^2\bigl(\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\bigr) \cong (\mathbf{H}\mathbb{Z})^2\bigl(\mathbf{H}\mathbb{Z}\bigl[\tfrac{1}{2}\bigr]\bigr) \otimes \mathbb{Z}\bigl[\tfrac{1}{2}\bigr] \\ &\cong (\mathbf{H}\mathbb{Z})^2(\mathbf{H}\mathbb{Z}) \otimes \mathbb{Z}\bigl[\tfrac{1}{2}\bigr] = 0.\end{aligned}$$

Here, the first equality is the definition of the cohomology associated to the spectrum $\mathbf{H}\mathbb{Z}\bigl[\frac{1}{2}\bigr]$, the second follows from [44, Theorem II.5.4], the third from [44, Corollary II.5.5], and the final equality is due to the standard fact that

$$(\mathbf{H}\mathbb{Z})^2(\mathbf{H}\mathbb{Z}) \cong \operatorname*{colim}_n H^{n+2}(K(\mathbb{Z};\, n);\, \mathbb{Z}) \cong H^5(K(\mathbb{Z};\, 3);\, \mathbb{Z}) = 0$$

(adapt the argument in Switzer [49, pp. 446–447] to $\mathbb{Z}$ in place of $\mathbb{Z}/2$ for the indicated isomorphisms, and see Serre's and Cartan's computations in [45] and [14] for the equality).

(ii) The additivity and the naturality of $\beta_3^X\bigl[\frac{1}{2}\bigr]$ are clear. We turn to the Chern character. The composition $\mathbf{bu}_n(X) \xrightarrow{\kappa_*} K_n(X) \xrightarrow{\mathrm{ch}_n} H_n(X;\, \mathbb{Q})$ is given by the map of spectra $\mathbf{H}\varrho \circ \tau_0 \colon \mathbf{bu} \longrightarrow \mathbf{H}\mathbb{Z} \longrightarrow \mathbf{H}\mathbb{Q}$, where ϱ is the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$. The commutative diagram (up to homotopy)

$$\begin{array}{ccc} \mathbf{bu}\big[\tfrac{1}{2}\big] & \xrightarrow{\tau_2[\frac{1}{2}]} & \mathbf{k^1}\big[\tfrac{1}{2}\big] \\ {\scriptstyle \tau_0[\frac{1}{2}]}\downarrow & & {\scriptstyle p_1[\frac{1}{2}]}\downarrow \;\; \uparrow{\scriptstyle s} \\ \mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big] & = & \mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big] \end{array}$$

says that $\big((\tau_2)_*^{-1} \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \circ s_* \colon H_3(X;\, \mathbb{Z}\big[\tfrac{1}{2}\big]) \longrightarrow \mathbf{bu}_3(X) \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]$ is a splitting of $\tau_0\big[\tfrac{1}{2}\big]_*$. It follows that the composition $\mathrm{ch}_3 \circ \beta_3^X\big[\tfrac{1}{2}\big]$ corresponds to the obvious map of spectra $\mathbf{H}\mathbb{Z}\big[\tfrac{1}{2}\big] \longrightarrow \mathbf{H}\mathbb{Q}$, so it is the identity after tensoring with $\mathbb{Q}$.

(iii) This follows from Lemma 5.1, by adapting the proof of Proposition 5.2.

(iv) By the Atiyah-Hirzebruch spectral sequence, we have natural isomorphisms $H_i(X) \cong \mathbf{bu}_i(X)$ for $i = 0$ and $i = 1$, and $H_0(X) \oplus H_2(X) \cong \mathbf{bu}_2(X)$, and the composition $\big((\tau_2)_*^{-1} \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \circ s_*$ is merely the inclusion (tensored with $\mathbb{Z}\big[\tfrac{1}{2}\big]$) of $H_j(X)$ in $\mathbf{bu}_j(X)$ for $0 \leq j \leq 2$. It follows from Proposition 4.2 that, under these identifications (modulo Bott periodicity), $\kappa_* \colon \mathbf{bu}_j(X) \longrightarrow K_j(X)$ is precisely β_0^X in degree 0, β_1^X in degree 1, and $\beta_0^X \oplus \beta_2^X$ in degree 2. The result follows. □

7. Application to the K-theory of group C^*-algebras

In this section, we apply the preceding results on the maps β_j^X to the K-theory of group C^*-algebras, in connection with the strong Novikov and the Baum-Connes conjectures. We illustrate the situation by examples. We keep the same notations as in the Introduction, in particular, for the Novikov assembly map ν_*^Γ and the map $\varphi_*^\Gamma \colon K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$ (and we do not assume that Γ is countable).

In the following statement, we use Bott periodicity implicitly, and we write $\beta_j^X\big[\tfrac{1}{k}\big]$ for $\beta_j^X \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{k}]}$, and similarly for $\nu_j^\Gamma \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{k}]}$ and for $\beta_3^X\big[\tfrac{1}{2}\big] \otimes \mathrm{Id}_{\mathbb{Z}[\frac{1}{k}]}$.

Theorem 7.1. *Let Γ be a discrete torsion-free group such that the Novikov assembly map ν_*^Γ is injective. Consider the compositions*

$$\nu_j^\Gamma \circ \beta_j^{B\Gamma} \colon\ H_j(\Gamma;\, \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma) \qquad (0 \leq j \leq 2),$$

and $\nu_1\big[\tfrac{1}{2}\big] \circ \beta_3^X\big[\tfrac{1}{2}\big] \colon H_3(\Gamma;\, \mathbb{Z}\big[\tfrac{1}{2}\big]) \longrightarrow K_1(C_r^*\Gamma) \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]$. Then the following holds:

(i) *$\nu_j^\Gamma \circ \beta_j^{B\Gamma}$ is injective for $j = 0$ and $j = 1$.*

(ii) *$\nu_0^\Gamma \circ (\beta_0^{B\Gamma} \oplus \beta_2^{B\Gamma})$ is injective if Γ has its integral homology concentrated in even degree, except possibly for H_1 and H_3, as, for example, if there is a model for $B\Gamma$ of dimension ≤ 4.*

(iii) *If $B\Gamma$ is a finite CW-complex of dimension $\leq 2n$ for some ≥ 2, then the maps $\nu_0\left[\frac{1}{n!}\right] \circ (\beta_0^{B\Gamma}\left[\frac{1}{n!}\right] \oplus \beta_2^{B\Gamma}\left[\frac{1}{n!}\right])$ and $\nu_1\left[\frac{1}{n!}\right] \circ (\beta_1^{B\Gamma}\left[\frac{1}{n!}\right] \oplus \beta_3^{B\Gamma}\left[\frac{1}{n!}\right])$ are injective.*

Proof. This follows immediately from Proposition 2.1 (i), Theorem 4.1, Propositions 4.2 and 5.2, and from Theorem 6.3 (iii). □

Note that part (iii) of Theorem 7.1 can be improved, using the full statements of Proposition 5.2 and Theorem 6.3 (iii).

In [36] and in [8] it is shown that the compositions $\nu_j^\Gamma \circ \beta_j^{B\Gamma}$ are given by

$$\nu_0^\Gamma \circ \beta_0^{B\Gamma} \colon \mathbb{Z} = H_0(\Gamma; \mathbb{Z}) \longrightarrow K_0(C_r^*\Gamma),\ 1 \longmapsto [1],$$

where [1] is the K-theory class of the unit, and

$$\nu_1^\Gamma \circ \beta_1^{B\Gamma} \colon \Gamma^{ab} = H_1(\Gamma; \mathbb{Z}) \longrightarrow K_1(C_r^*\Gamma),\ \gamma^{ab} \longmapsto [\gamma],$$

where γ^{ab} is the class in Γ^{ab} of an element $\gamma \in \Gamma$ and $[\gamma]$ is the corresponding K-theory class. (For $j = 0$, this is plain, and for $j = 1$, it suffices to check it for the easy case where $\Gamma = \mathbb{Z}$.) The case $j = 2$ is also discussed in [36], [8] and [38], in terms of the Hopf formula and of Steinberg symbols in algebraic K-theory.

As shown by V. Lafforgue [32][33], the Baum-Connes assembly map μ_*^Γ factorizes through the K-theory of $\ell^1\Gamma$, the Banach algebra of ℓ^1-summable complex valued functions on Γ, which injects in $C_r^*\Gamma$. So there is a map, sometimes called the Bost assembly map,

$$\hat{\mu}_*^\Gamma \colon K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(\ell^1\Gamma).$$

This map is natural with respect to Γ, whereas for μ_*^Γ this is still an open problem (that would follow from the Baum-Connes conjecture). It is conjectured that $\hat{\mu}_*^\Gamma$ is an isomorphism for any countable discrete group Γ; this is the Bost conjecture (see Skandalis [47]). Recently, Berrick, Chatterji and Mislin [6] have proven the striking result that a discrete group Γ, all of whose countable subgroups verify the Bost conjecture, satisfies the Bass conjecture on the values taken by the Hattori-Stallings trace on $K_0(\mathbb{C}\Gamma)$ (and even a more general version of it). This implies, for instance, that the Bass conjecture holds for all amenable groups. In our situation, by a standard argument (cf. [10, Proposition 3.4.1]), $\nu_1^\Gamma \circ \beta_1^{B\Gamma}$ lifts to a map

$$\hat{\beta}_1^{B\Gamma} \colon H_1(\Gamma; \mathbb{Z}) \longrightarrow K_1(\ell^1\Gamma),\ \gamma^{ab} \longmapsto [\gamma].$$

THEOREM 7.2. *Let Γ be a discrete group, and T_Γ the subgroup generated by the torsion elements. Then the homomorphisms*

$$\varphi_1^\Gamma \circ \beta_1^{B\Gamma} \colon H_1(\Gamma; \mathbb{Z}) \longrightarrow K_1^\Gamma(\underline{E\Gamma}) \quad \text{and} \quad \hat{\beta}_1^{B\Gamma} \colon H_1(\Gamma; \mathbb{Z}) \longrightarrow K_1(\ell^1\Gamma)$$

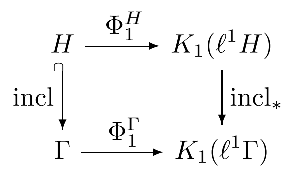
factor through the canonical map $H_1(\Gamma; \mathbb{Z}) \twoheadrightarrow H_1(\Gamma/T_\Gamma; \mathbb{Z})$. The corresponding factorization $H_1(\Gamma/T_\Gamma; \mathbb{Z}) \longrightarrow K_1^\Gamma(\underline{E\Gamma})$ is injective. In particular, if the

Baum-Connes assembly map μ_1^Γ is injective, then the homomorphism

$$\mathcal{U}_\Gamma\colon (\Gamma/T_\Gamma)^{ab} = H_1(\Gamma/T_\Gamma;\mathbb{Z}) \longrightarrow K_1(C_r^*\Gamma),\ (\gamma T_\Gamma)^{ab} \longmapsto [\gamma]$$

is injective. Furthermore, $\mathcal{U}_\Gamma$ is rationally injective for an arbitrary group Γ.

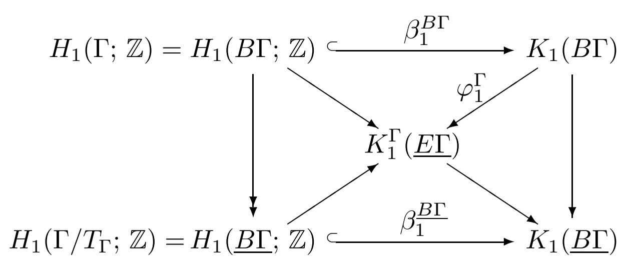
Proof. Consider the map $\Phi_1^\Gamma\colon \Gamma \longrightarrow K_1(\ell^1\Gamma)$ taking γ to $[\gamma]$. This map is a natural homomorphism. If $h \in \Gamma$ is a torsion element, let $H := \langle h\rangle$ be the finite subgroup of Γ generated by h. Then one has a commutative diagram



As $K_1(\ell^1 H)$ vanishes, this shows that Φ_1^Γ is zero on the generators of T_Γ, and hence on T_Γ. This allows us to define a map $\Gamma/T_\Gamma \longrightarrow K_1(\ell^1\Gamma)$ taking γT_Γ to $[\gamma]$. Since $K_1(\ell^1\Gamma)$ is abelian, this determines the map $H_1(\Gamma/T_\Gamma;\mathbb{Z}) \longrightarrow K_1(\ell^1\Gamma)$, $(\gamma T_\Gamma)^{ab} \longmapsto [\gamma]$ we are looking for, since $H_1(\Gamma;\mathbb{Z}) \twoheadrightarrow H_1(\Gamma/T_\Gamma;\mathbb{Z})$ precisely takes γ^{ab} to $(\gamma T_\Gamma)^{ab}$.

Since $K_1^H(\underline{EH}) = 0$ for a finite group H, the same process shows that $\varphi_1^\Gamma \circ \beta_1^{B\Gamma}$ factorizes through $H_1(\Gamma/T_\Gamma;\mathbb{Z})$.

Now, we prove that $H_1(\Gamma/T_\Gamma;\mathbb{Z}) \longrightarrow K_1^\Gamma(\underline{E\Gamma})$ is injective. There is an 'induction map' $K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(\underline{B\Gamma})$, where $\underline{B\Gamma} := \Gamma\backslash\underline{E\Gamma}$. The Γ-map $E\Gamma \longrightarrow \underline{E\Gamma}$ induces a map $B\Gamma \longrightarrow \underline{B\Gamma}$. Recall that $\underline{E\Gamma}$ is a 1-connected Γ-CW-complex, with set of stabilizers of vertices equal to the set of finite subgroups of Γ. By a result of Armstrong [2], it follows that $\pi_1(\underline{E\Gamma}) \cong \Gamma/T_\Gamma$, so that $H_1(\underline{B\Gamma};\mathbb{Z}) \cong H_1(\Gamma/T_\Gamma;\mathbb{Z})$. By the naturality of the map β_1^X, we have a commutative diagram



(The commutativity of the right triangle holds by the construction of φ_*^Γ and by the naturality of the 'induction map'; for the bottom triangle, commutativity follows from the commutativity of the rest of the diagram and of the indicated surjectivity.) This establishes the required injectivity. The statement concerning the injectivity of $\mathcal{U}_\Gamma$ when μ_1^Γ is injective is an immediate consequence.

In [22] and in [9], it is proven that $\nu_1^\Gamma \circ \beta_1^{B\Gamma}$ is rationally injective for any discrete group Γ. We claim that $\Gamma^{ab}\otimes\mathbb{Q} \cong (\Gamma/T_\Gamma)^{ab}\otimes\mathbb{Q}$. Indeed, by Corollary VII.6.4 in [12], there is an exact sequence

$$H_1(T_\Gamma;\,\mathbb{Q})_{\Gamma/T_\Gamma} \longrightarrow H_1(\Gamma;\,\mathbb{Q}) \longrightarrow H_1(\Gamma/T_\Gamma;\,\mathbb{Q}) \longrightarrow 0,$$

and $T_\Gamma^{ab}\otimes\mathbb{Q} = 0$, since T_Γ^{ab} is an abelian group generated by torsion elements. Hence the claim is clear if it is finitely generated; a direct limit argument settles the general case. It follows that the map $\mathcal{U}_\Gamma$ is always rationally injective, and we are done. □

Examples 7.3.

(i) Consider the one-relator group $\Gamma := \langle x,\, y \,|\, (xyx^{-1}y)^3\rangle$. By the Karras-Magnus-Solitar Theorem [30], we have $T_\Gamma = \langle\langle xyx^{-1}y\rangle\rangle_\Gamma$ (the normal closure in Γ), therefore $\Gamma/T_\Gamma \cong \langle x,\, y \,|\, xyx^{-1}y\rangle$, which, by the same theorem, is torsion-free. The Baum-Connes conjecture is known for one-relator groups (cf. [5]), so, by Theorem 7.2, we have

$$(\Gamma/T_\Gamma)^{ab} \cong \langle x,\, y \,|\, [x,\, y],\, y^2\rangle \cong \mathbb{Z}\oplus\mathbb{Z}/2 \hookrightarrow K_1(C_r^*\Gamma).$$

On the other hand, we have $\Gamma^{ab} \cong \langle x,\, y \,|\, [x,\, y],\, y^6\rangle \cong \mathbb{Z}\oplus\mathbb{Z}/6$, so that $\Gamma^{ab}/T_{\Gamma^{ab}} = \langle x\rangle_{\Gamma^{ab}} \cong \mathbb{Z}$. This illustrates the fact that the group $(\Gamma/T_\Gamma)^{ab}$ may contain torsion, and that it is in general bigger than $\Gamma^{ab}/T_{\Gamma^{ab}}$ (the latter always being torsion-free, as is easily seen for Γ finitely generated, and then extended to the general case by a direct limit argument). These observations, together with Theorem 7.2, give a partial answer to a question raised by Bettaieb and Valette in [9].

(ii) Generalizing Example (i), if Γ is any one-relator group, the main result of [5] (in degree 1) can be reformulated by saying that

$$\mathcal{U}_\Gamma\colon (\Gamma/T_\Gamma)^{ab} \xrightarrow{\cong} K_1(C_r^*\Gamma).$$

(iii) The group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ is the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. It follows that $T_\Gamma = \Gamma$; in particular, the map

$$\mathbb{Z}/12 \cong H_1(\mathrm{SL}_2(\mathbb{Z});\,\mathbb{Z}) \longrightarrow K_1(C_r^*(\mathrm{SL}_2(\mathbb{Z}))),\ \gamma^{ab} \longmapsto [\gamma]$$

is zero. In fact, $K_1(\mathcal{A}) = 0$ for $\mathcal{A} = C_r^*(\mathrm{SL}_2(\mathbb{Z}))$, $C^*(\mathrm{SL}_2(\mathbb{Z}))$ and $\ell^1(\mathrm{SL}_2(\mathbb{Z}))$ (see [40] and [47]); by [31], $\mathrm{SL}_2(\mathbb{Z})$ satisfies the Baum-Connes conjecture.

(iv) Before Lafforgue's work [32], no infinite group with Kazhdan's property (T) was known to satisfy the Baum-Connes conjecture. Moreover, if a group Γ has property (T), then $H_1(\Gamma;\,\mathbb{Q}) = 0$ (cf. [21, Proposition 1.7]). So, the question of the injectivity of the map $\Gamma^{ab} \longrightarrow K_1(C_r^*\Gamma)$ taking γ^{ab} to $[\gamma]$ is interesting, but delicate. In [15, Theorem 3.1], groups of type-rotating automorphisms of thick $\widetilde{A}_2$-buildings acting simply transitively on the set of vertices are considered and shown to admit a presentation of the form

$$\Gamma = \langle \{\gamma_x\}_{x\in P} \,|\, \gamma_x\gamma_y\gamma_z = 1 \text{ for } (x,\, y,\, z) \in \mathcal{T}\rangle\,,$$

where P is a certain set of vertices of the building (namely, the set of vertices of a given type that are neighbors of a given vertex of another type), and $\mathcal{T}$ is a very specific subset of $P \times P \times P$. By a result of Zuk [53] (see also [17]), such a group Γ has property (T). These groups are known to have Jolissaint's property (RD) ("rapid decay"), by the work of Ramagge, Robertson, and Steger [42]. From this and Lafforgue's results [32], it follows that these groups satisfy the Baum-Connes conjecture. If $\mathcal{T}$ contains no triple (x, x, x), such a group is torsion-free (and most examples given in [16] satisfy this condition). Since Γ^{ab} is finite (by the above remarks), it follows from the remark following [15, Theorem 3.1] that Γ^{ab} contains a copy of $\mathbb{Z}/3$. We conclude that there are examples of infinite groups with property (T), such that

$$\mathbb{Z}/3 \hookrightarrow H_1(\Gamma/T_\Gamma;\, \mathbb{Z}) \overset{\mathcal{U}_\Gamma}{\hookrightarrow} K_1(C_r^*\Gamma).$$

In view of Theorem 7.2 and of the rational injectivity of $\mathcal{U}_\Gamma$, we risk the following conjecture that may survive even if the Baum-Connes conjecture turned out to fail.

Conjecture 7.4. *For any discrete group Γ, the following map is injective:*

$$\mathcal{U}_\Gamma \colon H_1(\Gamma/T_\Gamma;\, \mathbb{Z}) \longrightarrow K_1(C_r^*\Gamma),\ (\gamma T_\Gamma)^{ab} \longmapsto [\gamma].$$

Weaker forms of the conjecture are obtained by replacing the reduced C^*-algebra $C_r^*\Gamma$ by the maximal C^*-algebra $C^*\Gamma$, or by the ℓ^1-algebra $\ell^1\Gamma$. It is worth mentioning that the proof of the rational injectivity of the map $H_1(\Gamma;\mathbb{Z}) \longrightarrow K_1(C^*\Gamma)$, $\gamma^{ab} \longmapsto [\gamma]$, is considerably simpler than for the reduced C^*-algebra; see [7] or [9]. The reason is precisely because the assignments $\Gamma \rightsquigarrow C^*\Gamma \rightsquigarrow K_*(C^*\Gamma)$ are functorial, whereas $\Gamma \rightsquigarrow C_r^*\Gamma$ is *not.* (As already mentioned, it would follow from the Baum-Connes conjecture that $\Gamma \rightsquigarrow K_*(C_r^*\Gamma)$ is functorial; this functoriality is still an open question.)

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