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# NARROW OPERATORS ON VECTOR-VALUED SUP-NORMED SPACES

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Abstract. We characterise narrow and strong Daugavet operators on $C(K,E)$-spaces; these are in a way the largest sensible classes of operators for which the norm equation $\|\mathrm{Id}+T\| = 1+\|T\|$ is valid. For certain separable range spaces $E$, including all finite-dimensional spaces and all locally uniformly convex spaces, we show that an unconditionally pointwise convergent sum of narrow operators on $C(K,E)$ is narrow. This implies, for instance, the known result that these spaces do not have unconditional FDDs. In a different vein, we construct two narrow operators on $C([0,1],\ell_1)$ whose sum is not narrow.

## 1. Introduction and preliminaries

This paper is a follow-up contribution to our paper [6], where we defined and investigated narrow operators on Banach spaces with the Daugavet property. Before describing the contents of the present paper, we review some definitions and results from [5] and [6].

A Banach space $X$ is said to have the *Daugavet property* if every rank-1 operator $T\colon X \to X$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{1.1}$$

For instance, $C(K)$ and $L_1(\mu)$ have the Daugavet property provided that $K$ is perfect, i.e., has no isolated points, and $\mu$ does not have any atoms. We shall have occasion to use the following characterisation of the Daugavet property from [5]; the equivalence of (ii) and (iii) results from the Hahn-Banach theorem.

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LEMMA 1.1. *The following assertions are equivalent:*

(i) *$X$ has the Daugavet property.*
(ii) *For all $x \in S(X)$, $x^* \in S(X^*)$ and $\varepsilon > 0$ there exists some $y \in S(X)$ such that $x^*(y) > 1 - \varepsilon$ and $\|x + y\| > 2 - \varepsilon$.*
(iii) *For all $x \in S(X)$ and $\varepsilon > 0$, $B(X) = \overline{\mathrm{co}}\{z \in B(X)\colon \|x + z\| > 2 - \varepsilon\}$.*

It was shown in [5] and [9] that (1.1) automatically extends to wider classes of operators, e.g., weakly compact spaces and, more generally, spaces that do not fix copies of $\ell_1$ or strong Radon-Nikodým operators. (A strong Radon-Nikodým operator maps the unit ball into a set with the Radon-Nikodým property.) In [6] we gave new proofs of these results based on the notions of a strong Daugavet operator and a narrow operator. An operator $T\colon X \to Z$ is said to be a *strong Daugavet operator* if for any two elements $x, y \in S(X)$, the unit sphere of $X$, and for every $\varepsilon > 0$ there is an element $u \in B(X)$, the unit ball of $X$, such that $\|x + u\| > 2 - \varepsilon$ and $\|T(y - u)\| < \varepsilon$. It is almost obvious that a strong Daugavet operator $T\colon X \to X$ satisfies (1.1). The nontrivial task now is to find sufficient conditions on $T$ to be strongly Daugavet. In this vein we could show that, for instance, strong Radon-Nikodým operators and operators not fixing copies of $\ell_1$ are indeed strong Daugavet operators.

For some applications the concept of a strong Daugavet operator is somewhat too wide. Therefore we defined an operator $T\colon X \to Z$ to be *narrow* if for any two elements $x, y \in S(X)$, every $x^* \in X^*$ and every $\varepsilon > 0$ there is an element $u \in B(X)$ such that $\|x+u\| > 2-\varepsilon$ and $\|T(y-u)\| + |x^*(y-u)| < \varepsilon$. It follows that $X$ has the Daugavet property if and only if all rank-1 operators are strong Daugavet operators if and only if there is at least one narrow operator on $X$. We denote the set of all strong Daugavet operators on $X$ by $\mathcal{SD}(X)$ and the set of all narrow operators on $X$ by $\mathcal{NAR}(X)$. Actually, in [6] we took a slightly different point of view in that we declared two operators $T_1\colon X \to Z_1$ and $T_2\colon X \to Z_2$ to be equivalent if $\|T_1 x\| = \|T_2 x\|$ for all $x \in X$. We remark that $\mathcal{SD}(X)$ and $\mathcal{NAR}(X)$ should really denote the sets of the corresponding equivalence classes; however, in this paper we shall not make this point explicitly.

In this paper we continue our investigations of this type of operator, mostly in the setting of vector-valued function spaces $C(K, E)$. One of the drawbacks of the definition of a strong Daugavet operator is that the sum of two such operators need not be a strong Daugavet operator, whereas the definition of a narrow operator has some built-in additivity property. It remained open in [6] whether the sum of any two narrow operators is always narrow, although we could prove that this is true on $C(K)$, and in general we showed that the sum of a narrow operator and an operator not fixing $\ell_1$ is narrow and that the sum of a narrow operator and a strong Radon-Nikodým operator is narrow. (Note that the sum of two strong Radon-Nikodým operators need not be a strong Radon-Nikodým operator [8].) Our work in Section 3, where we completely characterise strong Daugavet and narrow operators on $C(K,E)$, enables us to give counterexamples to the sum problem.

For this purpose we employ a special feature of $\ell_1$ explained in Section 2. This section introduces a class of Banach spaces called *USD-nonfriendly* spaces that are sort of remote from spaces with the Daugavet property; USD stands for uniformly strongly Daugavet. All finite-dimensional spaces and all locally uniformly convex spaces fall within this category, but we have not been able to decide whether a reflexive space must be USD-nonfriendly.

The class of USD-nonfriendly spaces is tailored to our applications in Section 4, where we study pointwise unconditionally convergent series $\sum_{n=1}^{\infty} T_n$ of narrow operators on $C(K,E)$. If $E$ is separable and USD-nonfriendly, we prove that the sum operator must be narrow again. This is new even in the case $E=\mathbb{R}$. To achieve this, we take a detour investigating the related class of $C$-narrow operators, following ideas from [4]. An obvious corollary is the result from [4] that the identity on $C(K)$ is not a pointwise unconditional sum of narrow operators. This implies that $C(K)$ does not admit an unconditional Schauder decomposition into spaces not containing $C[0,1]$.

We conclude this introduction with a technical reformulation of the definition of a strong Daugavet operator. Let

$$D(x,y,\varepsilon)=\{z\in X\colon \|x+y+z\|>2-\varepsilon,\ \|y+z\|<1+\varepsilon\}$$

and

$$\begin{aligned}\mathcal{D}(X) &= \{D(x,y,\varepsilon)\colon x\in S(X),\ y\in S(X),\ \varepsilon>0\},\\ \mathcal{D}_0(X) &= \{D(x,y,\varepsilon)\colon x\in S(X),\ y\in B(X),\ \varepsilon>0\}.\end{aligned}$$

It is easy to see that $T\colon X\to Z$ is a strong Daugavet operator if and only if $T$ is not bounded from below on any set $D\in\mathcal{D}(X)$ [6, Prop. 3.4]. In Section 3 it will be more convenient to work with $\mathcal{D}_0(X)$ instead; the following lemma says that this does not make any difference.

LEMMA 1.2. *An operator $T\colon X\to Z$ is a strong Daugavet operator if and only if $T$ is not bounded from below on any set $D\in\mathcal{D}_0(X)$.*

*Proof.* We have to show that $T\in\mathcal{SD}(X)$ is not bounded from below on $D(x,y,\varepsilon)$ whenever $\|x\|=1$, $\|y\|\le 1$, $\varepsilon>0$. By the above remarks, $T$ is not bounded from below on $D(x,-x,1)$; hence, given $\varepsilon'>0$, for some $\zeta\in S(X)$ we have $\|T\zeta\|<\varepsilon'$. Now pick $\lambda\ge 0$ such that $y+\lambda\zeta\in S(X)$; then there is some $z'\in X$ such that

$$\|x+(y+\lambda\zeta)+z'\|>2-\varepsilon,\ \|(y+\lambda\zeta)+z'\|<1+\varepsilon,\ \|Tz'\|<\varepsilon';$$

i.e., $z:=\lambda\zeta+z'\in D(x,y,\varepsilon)$ and $\|Tz\|<3\varepsilon'$. $\square$

## 2. USD-nonfriendly spaces

In this section we introduce a class of Banach spaces that are geometrically opposite to spaces with the Daugavet property. These spaces will arise naturally in Section 4.

PROPOSITION 2.1. *The following conditions for a Banach space $E$ are equivalent.*

1. $\mathcal{SD}(E) = \{0\}$.
2. *No nonzero linear functional on $E$ is a strong Daugavet operator.*
3. *For every $x^* \in S(E^*)$ there exist some $\delta > 0$ and $D \in \mathcal{D}(E)$ such that $|x^*(z)| > \delta$ for all $z \in D$.*
4. *Every closed absolutely convex subset $A \subset E$ such that for every $\alpha > 0$ and every $D \in \mathcal{D}(E)$ the intersection $(\alpha A) \cap D$ is nonempty coincides with the whole space $E$.*

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are evident.

(3) $\Rightarrow$ (4): Assume there is a closed absolutely convex subset $A \subset E$ with the property stated in (4) that does not coincide with the whole space $E$. By the Hahn-Banach theorem there is a functional $x^* \in S(E^*)$ and a number $r > 0$ such that $|x^*(a)| \le r$ for every $a \in A$. If $\delta > 0$ and $D \in \mathcal{D}(E)$ are arbitrary, pick $z \in (\frac{\delta}{r}A) \cap D$; this intersection is nonempty by the assumption on $A$. It follows that $|x^*(z)| \le \delta$, and hence (3) fails.

(4) $\Rightarrow$ (1): Suppose $T \in \mathcal{SD}(E)$ and put $A = \{e \in E\colon \|Te\| \le 1\}$. By the definition of a strong Daugavet operator this set $A$ satisfies (4). So $A = E$, and hence $T = 0$. ☐

This proposition suggests the following definition.

DEFINITION 2.2. A Banach space $E$ is said to be an *SD-nonfriendly space* (i.e., strong Daugavet-nonfriendly) if $\mathcal{SD}(E) = \{0\}$. A space $E$ is said to be a *USD-nonfriendly space* (i.e., uniformly strong Daugavet-nonfriendly) if there exists an $\alpha > 0$ such that every closed absolutely convex subset $A \subset E$ which intersects all elements of $\mathcal{D}(E)$ contains $\alpha B(E)$. The largest admissible $\alpha$ is called the *USD-parameter* of $E$.

Proposition 2.1 shows that a USD-nonfriendly space is indeed SD-nonfriendly, but the converse is false as will be shown shortly. Also, SD-nonfriendliness is opposite to the Daugavet property in that the latter is equivalent to the condition that every functional is a strong Daugavet operator.

To further motivate the uniformity condition in the above definition, we prove the following lemma.

LEMMA 2.3. *A Banach space $E$ is USD-nonfriendly if and only if*

(3*) *There exists some $\delta > 0$ such that for every $x^* \in S(E^*)$ there exists $D \in \mathcal{D}(E)$ such that $|x^*(z)| > \delta$ for all $z \in D$.*

*Proof.* It is enough to prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) for the following assertions about a fixed number $\delta > 0$:

(a) There exists a closed absolutely convex set $A \subset E$ not containing $\delta B(E)$ that intersects all $D \in \mathcal{D}(E)$.

(b) There exists a functional $x^* \in S(E^*)$ such that for all $D \in \mathcal{D}(E)$ there exists $z_D \in D$ satisfying $|x^*(z_D)| \le \delta$.

(c) There exists a closed absolutely convex set $A \subset E$ not containing $\delta' B(E)$ for any $\delta' > \delta$ that intersects all $D \in \mathcal{D}(E)$.

To see that (a) implies (b), pick $u \notin A$, $\|u\| \le \delta$. By the Hahn-Banach theorem we can separate $u$ from $A$ by means of a functional $x^* \in S(E^*)$, i.e., for some number $r > 0$ we have $|x^*(z)| \le r$ for all $z \in A$ and $x^*(u) > r$. On the other hand, $x^*(u) \le \|x^*\| \, \|u\| \le \delta$, and hence (b) holds for $x^*$.

If we assume (b), we define $A$ to be the closed absolutely convex hull of the elements $z_D$, $D \in \mathcal{D}(E)$, appearing in (b). Obviously $A$ intersects each $D \in \mathcal{D}(E)$. If $\delta' B(E) \subset A$ for some $\delta' > 0$, then, since $|x^*| \le \delta$ on $A$, we must have $|x^*| \le \delta$ on $\delta' B(E)$, i.e., $\delta' \le \delta$. Therefore $A$ statisfies the property stated in (c). ☐

In Proposition 2.1 and Lemma 2.3 we may replace $\mathcal{D}(E)$ by $\mathcal{D}_0(E)$.

We now turn to some examples.

Proposition 2.4.

(a) *The space $c_0$ is SD-nonfriendly, but not USD-nonfriendly.*

(b) *The space $\ell_1$ is not SD-nonfriendly, and hence not USD-nonfriendly either.*

*Proof.* (a) Theorem 3.5 of [6] implies that $Te_k = 0$ for every unit basis vector $e_k$ if $T \in \mathcal{SD}(c_0)$. (Actually, the theorem quoted is formulated for operators on $C(K)$ for compact $K$, but the theorem holds also on $C_0(L)$ with $L$ locally compact.) Hence $T = 0$ is the only strong Daugavet operator on $c_0$. (Another way to see this is to apply Corollary 3.6.)

To show that $c_0$ is not USD-nonfriendly we exhibit a closed absolutely convex set $A$ intersecting each $D \in \mathcal{D}(c_0)$, yet containing no ball. Let $A = 2B(\ell_1) \subset c_0$, i.e.,

$$A = \left\{ (x(n)) \in c_0 \colon \sum_{n=1}^{\infty} |x(n)| \le 2 \right\},$$

which is closed in $c_0$. Fix $x \in S(c_0)$ and $y \in S(c_0)$. If $|x(k)| = 1$, say $x(k) = 1$, pick $|\beta| \le 2$ such that $y(k) + \beta = 1$. Then $\beta e_k \in D(x, y, \varepsilon) \cap A$ for every $\varepsilon > 0$. Obviously, $A$ does not contain a multiple of $B(c_0)$.

(b) We claim that $x^*_\sigma(x) = \sum_{n=1}^\infty \sigma_n x(n)$ defines a strong Daugavet functional on $\ell_1$ whenever $\sigma$ is a sequence of signs, i.e., if $|\sigma_n| = 1$ for all $n$. Indeed, let $x \in S(\ell_1)$, $y \in S(\ell_1)$ and $\varepsilon > 0$. Pick $N$ such that $\sum_{n=1}^N |x(n)| > 1-\varepsilon$ and define $u \in S(\ell_1)$ by $u(n) = 0$ for $n \le N$ and $u(n) = \sigma_{n-N} y(n-N)/\sigma_n$ for $n > N$. Then $x^*(u) = x^*(y)$ and $\|x+u\| > 2-\varepsilon$; hence $z := u-y \in D(x,y,\varepsilon)$ and $x^*(z) = 0$. $\square$

Next we give some examples of USD-nonfriendly spaces. Recall that a point of local uniform rotundity of the unit sphere of a Banach space $E$ (an LUR-point) is a point $x_0 \in S(E)$ such that $x_n \to x_0$ whenever $\|x_n\| \le 1$ and $\|x_n + x_0\| \to 2$.

Proposition 2.5. *If the unit sphere of $E$ contains an LUR-point, then $E$ is a USD-nonfriendly space with USD-parameter $\ge 1$.*

*Proof.* Let $x_0 \in S(E)$ be an LUR-point and let $A \subset E$ be a closed absolutely convex subset which intersects all elements of $\mathcal{D}(E)$. In particular, for every fixed $y \in S(E)$ the set $A$ intersects all sets $D(x_0, y, \varepsilon) \subset E$, $\varepsilon > 0$. By the definition of an LUR-point this means that all points of the form $x_0 - y$, $y \in S(E)$, belong to $A$, i.e., $B(E) + x_0 \subset A$. But $-x_0$ is also an LUR-point, so $B(E) - x_0 \subset A$, and by the convexity of $A$, $B(E) \subset A$. $\square$

Corollary 2.6. *Every locally uniformly convex space is USD-nonfriendly with USD-parameter 2. In particular, the spaces $L_p(\mu)$ are USD-nonfriendly for $1 < p < \infty$.*

*Proof.* This follows from the previous proposition; that the USD-parameter is 2 is a consequence of the fact that $B(E) + x_0 \subset A$ for all $x_0 \in S(E)$; see the above proof. $\square$

It is clear that no finite-dimensional space enjoys the Daugavet property, but more is true.

Proposition 2.7. *Every finite-dimensional Banach space $E$ is a USD-nonfriendly space.*

*Proof.* Assume to the contrary that there is a finite-dimensional space $E$ that is not USD-nonfriendly. By Lemma 2.3 we can find a sequence of functionals $(x_n^*) \subset S(E^*)$ such that $\inf_{z\in D} |x_n^*(z)| \le 1/n$ for each $D \in \mathcal{D}(E)$. By the compactness of the ball we can pass to the limit and obtain a functional $x^* \in S(E^*)$ with the property that $\inf_{z\in D} |x^*(z)| = 0$ for each $D \in \mathcal{D}(E)$.

Set $K = \{e \in B(E)\colon x^*(e) = 1\}$; this is a norm-compact convex set. Let $x_0 \in K$ be an arbitrary point. If we apply the above property to $D(x_0, -x_0, \varepsilon)$ for all $\varepsilon > 0$, we obtain, again by compactness, some $z_0$ such that $\|z_0 - x_0\| = 1$, $\|z_0\| = 2$ and $x^*(z_0) = 0$. We have

$x^*(x_0 - z_0) = 1$, so $x_0 - z_0 \in K$. Therefore

$$2 \geq \operatorname{diam} K \geq \sup_{y \in K} \|x_0 - y\| \geq \|x_0 - (x_0 - z_0)\| = \|z_0\| = 2;$$

hence $\operatorname{diam} K = 2$ and $x_0$ is a diametral point of $K$, i.e.,

$$\sup_{y \in K} \|x_0 - y\| = \operatorname{diam} K.$$

But any compact convex set of positive diameter contains a nondiametral point [3, p. 38]; thus we have reached a contradiction. $\square$

We shall later estimate the worst possible USD-parameter of an $n$-dimensional normed space.

We have not been able to decide whether every reflexive space is USD-nonfriendly. Proposition 2.10 below presents a necessary condition a hypothetical reflexive USD-friendly (= not USD-nonfriendly) space must fulfill.

We first give an easy geometrical lemma.

LEMMA 2.8. *Let $x, h \in E$, $\|x\| \leq 1 + \varepsilon$, $\|h\| \leq 1 + \varepsilon$, $\|x + h\| \geq 2 - \varepsilon$. Let $f \in S(E^*)$ be a supporting functional of $(x + h)/\|x + h\|$. Then $f(x)$ as well as $f(h)$ are bounded from below by $1 - 2\varepsilon$.*

*Proof.* Set $a = f(x)$, $b = f(h)$. Then $\max(a, b) \leq 1 + \varepsilon$ but $a + b \geq 2 - \varepsilon$. So $\min(a, b) = a + b - \max(a, b) \geq 1 - 2\varepsilon$. $\square$

Let $E$ be a reflexive space, and let $x_0^*$ be a strongly exposed point of $S(E^*)$ with strongly exposing evaluation functional $x_0$; i.e., the diameter of the slice $\{x^* \in S(E^*)\colon x^*(x_0) > 1 - \varepsilon\}$ tends to 0 when $\varepsilon$ tends to 0. Set

$$S_{x_0^*} = \{x \in S(E)\colon x_0^*(x) = 1\}.$$

PROPOSITION 2.9. *Let $E$, $x_0^*$, $x_0$ be as above, and let $A$ be a closed convex set which intersects all sets $D(x_0, 0, \varepsilon)$, $\varepsilon > 0$. Then $A$ intersects $S_{x_0^*}$.*

*Proof.* For every $n \in \mathbb{N}$ select $h_n \in A \cap D(x_0, 0, 1/n)$. Then $\|h_n\| \leq 1 + 1/n$, $\|x_0 + h_n\| \geq 2 - 1/n$. Denote by $f_n$ a supporting functional of $(x_0 + h_n)/\|x_0 + h_n\|$. By the previous lemma $f_n(x_0)$ tends to 1 when $n$ tends to infinity. So by the definition of an exposing functional, $f_n$ tends to $x_0^*$. By the same lemma $f_n(h_n)$ tends to 1, so $x_0^*(h_n)$ also tends to 1. Hence every weak limit point of the sequence $(h_n)$ belongs to the intersection of $A$ and $S_{x_0^*}$. Therefore this intersection is nonempty. $\square$

PROPOSITION 2.10. *Let $E$ be a reflexive space.*

(a) *If $E$ is USD-nonfriendly with USD-parameter $< \alpha$, then there exists a functional $x^* \in S(E^*)$ such that for every strongly exposed point $x_0^*$ of $B(E^*)$ the numerical set $x^*(S_{x_0^*})$ contains the interval $[-1 + \alpha, 1 - \alpha]$.*

(b) *If $E$ is not USD-nonfriendly, then for every strongly exposed point $x_0^*$ of $B(E^*)$ the set $S_{x_0^*}$ has diameter 2. Moreover, for every $\delta > 0$ there exists a functional $x^* \in S(E^*)$ such that for every strongly exposed point $x_0^*$ of $B(E^*)$ the numerical set $x^*(S_{x_0^*})$ contains the interval $[-1+\delta, 1-\delta]$.*

*Proof.* (a) Let $A$ be a closed absolutely convex set which intersects all sets $D \in \mathcal{D}(E)$, but does not contain $\alpha B(E)$. By the Hahn-Banach theorem there exists a functional $x^* \in S(E^*)$ such that $|x^*(a)| < \alpha$ for every $a \in A$. We fix $y \in S(E)$ with $x^*(y) = -1$.

Let $x_0^* \in S(E^*)$ be a strongly exposed point of $B(E^*)$. As before, we denote an exposing evaluation functional by $x_0$. Now $A \cap D(x_0, y, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. By Proposition 2.9 and the evident equality $D(x_0, 0, \varepsilon) - y = D(x_0, y, \varepsilon)$ this implies that the set $A+y$ intersects $S_{x_0^*}$. If $z_1$ is an element of this intersection, we see that $x^*(z_1) < \alpha - 1$.

Likewise, since $D(-x_0, 0, \varepsilon) = -D(x_0, 0, \varepsilon)$, we find some $z_2 \in (-A-y) \cap S_{x_0^*}$; hence $x^*(z_2) > -\alpha + 1$. Therefore, $[-1+\alpha, 1-\alpha] \subset x^*(S_{x_0^*})$.

(b) The argument is the same as in (a). □

This proposition allows us to estimate the USD-parameter of finite-dimensional spaces.

PROPOSITION 2.11. *If $E$ is $n$-dimensional, then its USD-parameter is $\geq 2/n$.*

*Proof.* Assume that $\dim(E) = n$ and that its USD-parameter is $< 2/n$; then this parameter is strictly smaller than some $\alpha < 2/n$. Choose $x^*$ as in Proposition 2.10 so that

$$[-1+\alpha, 1-\alpha] \subset x^*(S_{x_0^*}) \tag{2.1}$$

for every strongly exposed functional $x_0^* \in S(E^*)$.

We now claim that in any $\varepsilon$-neighbourhood of $x^*$ there is some $y^* \in B(E^*)$ which can be represented as a convex combination of $\leq n$ strongly exposed functionals. First we observe that the convex hull of the set $\operatorname{stexp} B(E^*)$ of strongly exposed functionals is norm-dense in $B(E^*)$; in fact, this is true of any bounded closed convex set in a separable dual space [1, p. 110]. Hence, for some $\|y_1^* - x^*\| < \varepsilon$, $\lambda_1', \dots, \lambda_r' \geq 0$ with $\sum_{k=1}^r \lambda_k' = 1$ and $x_1^*, \dots, x_r^* \in \operatorname{stexp} B(E^*)$,

$$y_1^* = \sum_{k=1}^r \lambda_k' x_k^*.$$

Let $C = \operatorname{co}\{x_1^*, \dots, x_r^*\}$ and let $y^*$ be the point of intersection of the segment $[y_1^*, x^*]$ with the relative boundary of $C$, i.e., $y^* = \tau x^* + (1-\tau) y_1^*$ with $\tau = \sup\{t \in [0,1] : tx^* + (1-t)y_1^* \in C\}$. Let $F$ be the face of $C$ generated by $y^*$; then $F$ is a convex set of dimension $< n$. Therefore an appeal to Carathéodory's theorem shows that $y^*$ can be represented as a convex combination of no more than $n$ extreme points of $F$. But $\operatorname{ex} F \subset \operatorname{ex} C \subset \{x_1^*, \dots, x_r^*\} \subset \operatorname{stexp} B(E^*)$, and our claim is established.

We apply the claim with some $\varepsilon < 2/n - \alpha$ to obtain a convex combination $y^* = \sum_{k=1}^n \lambda_k x_k^*$ of $n$ strongly exposed functionals such that $\|y^* - x^*\| < \varepsilon$. One of the coefficients must be $\geq 1/n$, say $\lambda_n \geq 1/n$. Now if $x \in S_{x_n^*}$, then

$$\begin{aligned} x^*(x) &\geq x^*(y) - \varepsilon = \sum_{k=1}^{n-1} \lambda_k x_k^*(x) + \lambda_n - \varepsilon \\ &\geq -\sum_{k=1}^{n-1} \lambda_k + \lambda_n = -1 + 2\lambda_n - \varepsilon \geq -1 + 2/n - \varepsilon. \end{aligned}$$

By (2.1) we have $-1 + \alpha \geq -1 + 2/n - \varepsilon$ which contradicts our choice of $\varepsilon$. □

For $\ell_\infty^n$ we can say more, namely that its USD-parameter is the worst possible.

Proposition 2.12. *The USD-parameter of $\ell_\infty^n$ is $2/n$.*

*Proof.* In the setting of $\ell_\infty^n$ instead of $c_0$, the argument of Proposition 2.4(a) implies that the USD-parameter of $\ell_\infty^n$ is $\leq 2/n$. The reverse inequality follows from Proposition 2.11. □

## 3. Strong Daugavet and narrow operators in spaces of vector-valued functions

Let $E$ be a Banach space and let $X$ be a subspace of the space of all bounded $E$-valued functions defined on a set $K$, equipped with the sup-norm. It will be convenient to use the following notation: A disjoint pair $(U, V)$ of subsets of $K$ is said to be *interpolating* for $X$ if for all $f, g \in X$ with $\|f\| < 1$ and $\|g\chi_V\| < 1$ there exists $h \in B(X)$ such that $h = f$ on $U$ and $h = g$ on $V$.

For arbitrary $V \subset K$ denote by $X_V$ the subspace of all functions from $X$ vanishing on $V$.

Proposition 3.1. *Let $X$ be as above and let $(U, V)$ be an interpolating pair for $X$. Then for every $f \in X$*

$$\operatorname{dist}(f, X_V) \leq \sup_{t \in V} \|f(t)\|.$$

*Proof.* By the definition of an interpolating pair, for an arbitrary $\varepsilon > 0$ there exists an element $h \in X$, $\|h\| < \sup_{t \in V} \|f(t)\| + \varepsilon$, such that $h = 0$ on $U$ and $h = f$ on $V$. Then the element $f - h$ belongs to $X_V$, so

$$\operatorname{dist}(f, X_V) \leq \|f - (f - h)\| = \|h\| < \sup_{t \in V} \|f(t)\| + \varepsilon,$$

which completes the proof. □

LEMMA 3.2. *Let* $X \subset \ell_\infty(K,E)$, $U,V \subset K$, $f \in S(X_V)$ *and* $\varepsilon > 0$. *Assume that* $U \supset \{t \in K\colon \|f(t)\| > 1-\varepsilon\}$ *and that* $(U,V)$ *is an interpolating pair for* $X$. *If* $T$ *is a strong Daugavet operator on* $X$ *and* $g \in B(X)$, *there is a function* $h \in X_V$, $\|h\| \le 2+\varepsilon$, *satisfying*

$$\|Th\| < \varepsilon,\ \|(g+h)\chi_U\| < 1+\varepsilon \text{ and } \|(f+g+h)\chi_U\| > 2-\varepsilon.$$

*Proof.* Before we begin the proof proper, we formulate a number of technical assertions that are easy to verify and will be needed later.

SUBLEMMA 3.3. *If* $T$ *is a strong Daugavet operator on a Banach space* $X$, *and if* $1-\eta < \|x\| < 1+\eta$ *and* $\|y\| < 1+\eta$, *then there is an element* $z \in X$ *such that*

$$\|x+y+z\| > 2-3\eta,\ \|y+z\| < 1+2\eta,\ \|Tz\| < \eta.$$

*Proof.* Choose $x_0 \in S(X)$ and $y_0 \in B(X)$ such that $\|x_0-x\| < \eta$, $\|y_0-y\| < \eta$ and pick by Lemma 1.2 $z \in D(x_0,y_0,\eta)$ such that $\|Tz\| < \eta$; this element $z$ clearly has the required property. □

SUBLEMMA 3.4. *If* $\|x\| < 1+\eta$, $\|y\| < 1+\eta$ *and* $\|(x+y)/2\| > 1-\eta$ *in a normed space, then* $\|\lambda x + (1-\lambda)y\| > 1-3\eta$ *whenever* $0 \le \lambda \le 1$.

*Proof.* If $\|\lambda x + (1-\lambda)y\| \le 1-3\eta$ for some $0 \le \lambda \le 1/2$, then, since $\lambda_1 x + (1-\lambda_1)(\lambda x + (1-\lambda)y) = (x+y)/2$ for $\lambda_1 = (1/2-\lambda)/(1-\lambda) \in [0,1/2]$, we would have

$$\Big\|\frac{x+y}{2}\Big\| \le \lambda_1(1+\eta) + (1-\lambda_1)(1-3\eta) = 1-(3-4\lambda_1)\eta \le 1-\eta,$$

contradicting the hypothesis of the Sublemma. The case $\lambda > 1/2$ is analogous. □

SUBLEMMA 3.5. *If* $\|y\| < 1+\eta$ *and* $\|x+Ny\|/(N+1) > 1-3\eta$ *in a normed space, then* $\|(x+y)/2\| > 1-(2N+1)\eta$.

*Proof.* If $\|(x+y)/2\| \le 1-(2N+1)\eta$, then we would have

$$\begin{aligned}\Big\|\frac{x+Ny}{1+N}\Big\| &\le \frac{2}{1+N}\Big\|\frac{x+y}{2}\Big\| + \Big(1-\frac{2}{1+N}\Big)\|y\| \\ &\le \frac{2}{1+N}\big(1-(2N+1)\eta\big) + \Big(1-\frac{2}{1+N}\Big)(1+\eta) \\ &= 1-3\eta,\end{aligned}$$

which is a contradiction. □

We now begin the proof of Lemma 3.2. We may assume that $\|T\| = 1$. Fix $N > 6/\varepsilon$ and $\delta > 0$ such that $2(2N+1)9^N\delta < \varepsilon$, and let $\delta_n = 9^n\delta$, so that $(2N+1)\delta_N < \varepsilon/2$. Put $f_1 = f$, $g_1 = g$, and pick $h_1 \in X$ such that

$$\|f_1+g_1+h_1\| > 2-\delta_1,\ \|g_1+h_1\| < 1+2\delta_0,\ \|Th_1\| < \delta_0.$$

We will construct inductively functions $f_n, g_n, h_n \in X$ satisfying

(a) $f_{n+1} = \frac{1}{n+1}(f_1 + \sum_{k=1}^n (g_k + h_k)) = \frac{n}{n+1} f_n + \frac{1}{n+1}(g_n + h_n)$, $1 - 3\delta_n < \|f_{n+1}\| < 1 + \delta_n$;

(b) $g_{n+1} = g_1$ on $U$ and $g_{n+1} = g_n + h_n$ $(= g_1 + h_1 + \cdots + h_n)$ on $V$, $\|g_{n+1}\| < 1 + \delta_n$;

(c) $\|f_{n+1} + g_{n+1} + h_{n+1}\| > 2 - \delta_{n+1}$, $1 - 2\delta_n < \|g_{n+1} + h_{n+1}\| < 1 + 6\delta_n < 1 + \delta_{n+1}$, $\|Th_{n+1}\| < 3\delta_n$.

Suppose that these functions have already been constructed for the indices $1, \ldots, n$, and define $f_{n+1}$ as in (a). Since, by the induction hypothesis, $\|f_n\| < 1 + \delta_{n-1}$ and $\|g_n + h_n\| < 1 + \delta_n$ we clearly have $\|f_{n+1}\| < 1 + \delta_n$. From $\|f_n + g_n + h_n\| > 2 - \delta_n$, we conclude, using Sublemma 3.4 (with $\eta = \delta_n$), that $\|f_{n+1}\| > 1 - 3\delta_n$. Thus (a) holds. To obtain (b) it is enough to use that $(U, V)$ is interpolating along with the induction hypothesis that $\|g_n + h_n\| < 1 + \delta_n$. Finally, (c) follows from Sublemma 3.3 with $\eta = 3\delta_n$.

Next we claim that

$$\left\| f_1 + \frac{1}{N} \sum_{k=1}^N (g_k + h_k) \right\| > 2 - \varepsilon/2.$$

This follows from Sublemma 3.5, (c) and (a), and our choice of $\delta$. But for $t \notin U$ we can estimate

$$\left\| f_1(t) + \frac{1}{N} \sum_{k=1}^N (g_k(t) + h_k(t)) \right\| \le 1 - \varepsilon + 1 - \delta_N \le 2 - 2\varepsilon,$$

and therefore, letting $w = \frac{1}{N} \sum_{k=1}^N h_k$,

$$\|(f + g + w)\chi_U\| = \left\| \left( f_1 + \frac{1}{N} \sum_{k=1}^N (g_k + h_k)\chi_U \right) \right\| > 2 - \varepsilon/2.$$

Furthermore we have the estimates

$$\begin{aligned}
\|(g + w)\chi_U\| &= \left\| \frac{1}{N} \sum_{k=1}^N (g_k + h_k)\chi_U \right\| \le 1 + \delta_N < 1 + \varepsilon/2, \\
\|Tw\| &\le \frac{1}{N} \sum_{k=1}^N \|Th_k\| < 3\delta_{N-1} = \frac{1}{3}\delta_N < \varepsilon/2, \\
\|h_k\| &\le \|g_k + h_k\| + \|g_k\| \le 2 + 2\delta_k \le 2 + 2\delta_N \le 2 + \varepsilon/2, \\
\|w\| &\le \frac{1}{N} \sum_{k=1}^N \|h_k\| \le 2 + \varepsilon/2,
\end{aligned}$$

and for $t \in V$

$$\|w(t)\| = \frac{1}{N} \|g_{N+1}(t) - g_1(t)\| \le \frac{2 + \delta_N}{N} < \frac{3}{N} < \varepsilon/2.$$

By Proposition 3.1 and the above remarks we see that $\operatorname{dist}(w, X_V) < \varepsilon/2$. Hence, to complete the proof, it remains to replace $w$ by an element $h \in X_V$, $\|h - w\| \le \varepsilon/2$. □

Let us remark that the conditions of Lemma 3.2 are fulfilled for an arbitrary compact Hausdorff space $K$, any closed subset $V \subset K$, and for $X = C(K, E)$ as well as for $X = C_w(K, E)$. The following corollary gives another example:

Corollary 3.6. *If $X = X_1 \oplus_\infty X_2$ and $T \in \mathcal{SD}(X)$, then $T|_{X_1} \in \mathcal{SD}(X_1)$.*

To see this, let $K = \operatorname{ex} B(X^*)$, $K_1 = \operatorname{ex} B(X_1^*)$, $K_2 = \operatorname{ex} B(X_2^*)$, so that $K = K_1 \cup K_2$ and $X \subset \ell_\infty(K)$ canonically. It remains to apply Lemma 3.2 with the interpolating pair $(K_1, K_2)$. A direct proof of Corollary 3.6 was given in [2].

In the sequel, given an element $y \in E$ we also use the symbol $y$ to denote the constant function in $C(K, E)$ taking that value.

Theorem 3.7. *Let $K$ be a compact Hausdorff space, $E$ a Banach space and $T$ an operator on $X = C(K, E)$. Then the following conditions are equivalent:*

(1) $T \in \mathcal{SD}(X)$.
(2) *For every closed subset $V \subset K$, every $x \in S(E)$, every $y \in B(E)$ and every $\varepsilon > 0$ there exists an open subset $W \subset K \setminus V$, an element $e \in E$ with $\|e + y\| < 1 + \varepsilon$, $\|e + y + x\| > 2 - \varepsilon$, and a function $h \in X_V$, $\|h\| \le 2 + \varepsilon$, such that $\|Th\| < \varepsilon$ and $\|e - h(t)\| < \varepsilon$ for $t \in W$.*
(3) *For every closed subset $V \subset K$, every $x \in S(E)$, every $y \in B(E)$ and every $\varepsilon > 0$ there exists a function $f \in X_V$ such that $\|Tf\| < \varepsilon$, $\|f + y\| < 1 + \varepsilon$, $\|f + y + x\| > 2 - \varepsilon$.*

*If $K$ has no isolated points, then these conditions are equivalent to*

(4) $T \in \mathcal{NAR}(X)$.

*Proof.* The implication (1) ⇒ (2) follows from Lemma 3.2 as follows. Let us apply Lemma 3.2 with $\varepsilon/4 > 0$, $g = \chi_K \otimes y$, $f = f_1 \otimes x \in S(X)$, where $f_1$ is a positive scalar function vanishing on $V$, and $U = \{t \in K\colon \|f(t)\| > 1 - \varepsilon/4\}$, and let $h \in X_V$ be obtained from this lemma. Choose a point $t_0 \in U$ such that $\|(f + g + h)(t_0)\| = \|(f + h)(t_0) + y\| > 2 - \varepsilon/4$. Because $\|h(t_0) + y\| < 1 + \varepsilon/4$ we have $\|f(t_0)\| > 1 - \varepsilon/2$, i.e., $\|f(t_0) - x\| < \varepsilon/2$. Now select an open neighbourhood $W \subset U$ of $t_0$ such that $\|f(\tau) - x\| < \varepsilon/2$ for all $\tau \in W$, and put $e = h(t_0)$.

To prove the implication (2) ⇒ (3) let us fix positive numbers $\varepsilon < 1/10$, $\delta < \varepsilon/4$ and $N > 6 + 2/\varepsilon$. Now apply inductively condition (2) to obtain elements $x_k, y_k, e_k$, $x_1 = x$, $y_k = y$, $k = 1, \dots, N$, open subsets $W_1 \supset W_2 \supset \cdots$, closed subsets $V_{k+1} = K \setminus W_k$, $V_1 = V$, and functions $h_k \in X_{V_k}$, with the following properties:

(a) $x_{n+1} = \dfrac{x + \sum_{k=1}^{n}(y_k + e_k)}{\|x + \sum_{k=1}^{n}(y_k + e_k)\|} \in S(E)$;

(b) $\|e_k + y_k\| < 1 + \delta$, $\|e_k + y_k + x_k\| > 2 - \delta$;

(c) $h_k \in X_{V_k}$, $\|h_k(t) - e_k\| < \varepsilon/4$ for all $t \in W_k$, $\|h_k\| \le 2 + \varepsilon$, and $\|Th_k\| < \varepsilon$.

By an argument similar to that used in the proof of Lemma 3.2, we have with a suitable choice of $\delta$

$$\left\| x + y + \frac{1}{N}\sum_{k=1}^{N} e_k \right\| = \left\| x + \frac{1}{N}\sum_{k=1}^{N}(y_k + e_k)\right\| > 2 - \frac{\varepsilon}{2}.$$

Let us put $f = \frac{1}{N}\sum_{k=1}^{N} h_k$. Then the last inequality and (c) of our construction yield that $f \in X_V$, $\|f + y + x\| > 2 - \varepsilon$, and $\|Tf\| < \varepsilon$. It remains to estimate $\|f + y\|$ from above. If $t \in V$, then $\|f(t) + y\| = \|y\| \le 1$. If $t \in W_n \setminus W_{n+1}$ for some $n$, then

$$\|f(t) + y\| = \left\| \frac{1}{N}\sum_{k=1}^{n} h_k(t) + y \right\| = \left\| \frac{1}{N}\sum_{k=1}^{n} (h_k(t) + y) \right\|.$$

In this sum all summands except for the last one satisfy the inequality $\|h_k(t) + y\| \le 1 + \varepsilon/2$, and the last summand $h_n(t) + y$ is bounded by $3 + \varepsilon$. So

$$\|f(t) + y\| \le \frac{1}{N}\sum_{k=1}^{n-1}\Big(1 + \frac{\varepsilon}{2}\Big) + \frac{1}{N}(3 + \varepsilon) \le 1 + \frac{\varepsilon}{2} + \frac{1}{N}(3 + \varepsilon) \le 1 + \varepsilon.$$

The same estimate holds for $t \in W_N$.

To prove the implication (3) $\Rightarrow$ (1) fix $f, g \in S(X)$ and $0 < \varepsilon < 1/10$. Pick a point $t \in K$ with $\|f(t)\| > 1 - \varepsilon/4$ and a neighbourhood $U$ of $t$ such that

$$\|f(t) - f(\tau)\| + \|g(t) - g(\tau)\| < \frac{\varepsilon}{4} \text{ for all } \tau \in U.$$

Set $x = f(t)/\|f(t)\|$ and $y = g(t)$ and apply condition (3) to obtain a function $h \in X_V$ such that $\|Th\| < \varepsilon$, $\|h + y\| < 1 + \varepsilon/4$, and $\|h + y + x\| > 2 - \varepsilon/4$. For this function $h$ we have $\|h + g\| < 1 + \varepsilon$ and $\|h + g + f\| > 2 - \varepsilon$, so $T \in \mathcal{SD}(X)$.

Let us now consider the case of a perfect compact space $K$. The implication (4) $\Rightarrow$ (1) is evident. The proof of the remaining implication (3) $\Rightarrow$ (4) is similar to that of the implication (3) $\Rightarrow$ (1). Namely, let $f, g \in S(X)$, $x^* \in X^*$, and let $\varepsilon > 0$ be small. We have to show that there is an element $h \in X$ such that

$$\|f + g + h\| > 2 - \varepsilon, \quad \|g + h\| < 1 + \varepsilon \tag{3.1}$$

and

$$\|Th\| + |x^* h| < \varepsilon. \tag{3.2}$$

To this end, let us pick a closed subset $V \subset K$, whose complement $K \setminus V$ we denote by $U$, and a point $t \in U$ such that $\|f(t)\| > 1 - \varepsilon/4$,

$$|x^*|_{X_V} < \frac{\varepsilon}{4}, \tag{3.3}$$

and for every $\tau \in U$

$$\|f(t) - f(\tau)\| + \|g(t) - g(\tau)\| < \frac{\varepsilon}{4}. \tag{3.4}$$

Set $x = f(t)/\|f(t)\|$, $y = g(t)$ and apply condition (3) to obtain a function $h \in X_V$ such that $\|Th\| < \varepsilon/4$, $\|h + y\| < 1 + \varepsilon/4$ and $\|h + y + x\| > 2 - \varepsilon/4$. For this function $h$, (3.1) follows from (3.4), and (3.2) follows from (3.3). □

In [6] we defined the tilde-sum of two operators $T_1\colon X \to Y_1$, $T_2\colon X \to Y_2$ by

$$T_1 \mathbin{\widetilde{+}} T_2\colon X \to Y_1 \oplus_1 Y_2,\ x \mapsto (T_1 x, T_2 x).$$

We proved that the $\widetilde{+}$-sum, and therefore also the ordinary sum, of two narrow operators on $C(K)$ is narrow (another proof will be given in the next section), and we asked whether this is so on any space with the Daugavet property. We are now in a position to provide a counterexample.

Let $T\colon E \to F$ be an operator on a Banach space. Let us denote by $T^K$ the corresponding "multiplication" or "diagonal" operator $T^K\colon C(K,E) \to C(K,F)$ defined by

$$(T^K f)(t) = T(f(t)).$$

PROPOSITION 3.8. *$T^K \in \mathcal{SD}(C(K,E))$ if and only if $T \in \mathcal{SD}(E)$.*

*Proof.* Condition (3) of Theorem 3.7 immediately yields the result. □

Here is the promised counterexample:

THEOREM 3.9. *There exists a Banach space $X$ for which $\mathcal{NAR}(X)$ does not form a semigroup under the operation $\widetilde{+}$; in fact, $C([0,1],\ell_1)$ is such a space.*

*Proof.* The key feature of $\ell_1$ is that $\mathcal{SD}(\ell_1)$ is not a $\widetilde{+}$-semigroup, for we have shown in Proposition 2.4(b) that $x_1^*(x) = \sum_{n=1}^\infty x(n)$ and $x_2^*(x) = x(1) - \sum_{n=2}^\infty x(n)$ define strong Daugavet functionals on $\ell_1$, but $x_1^* + x_2^*\colon x \mapsto 2x(1)$ is not in $\mathcal{SD}(\ell_1)$, and hence $x_1^* \mathbin{\widetilde{+}} x_2^*$ is also not in $\mathcal{SD}(\ell_1)$.

Now if $\mathcal{SD}(E)$ is not a $\widetilde{+}$-semigroup, pick $T_1, T_2 \in \mathcal{SD}(E)$ with $T_1 \mathbin{\widetilde{+}} T_2 \notin \mathcal{SD}(E)$. Put $X = C(K,E)$ for a perfect compact Hausdorff space $K$; then by Proposition 3.8 and Theorem 3.7, $T_1^K, T_2^K \in \mathcal{NAR}(X)$, but $T_1^K \mathbin{\widetilde{+}} T_2^K \notin \mathcal{NAR}(X)$. □

Another example of a space for which $\mathcal{SD}(E)$ is not a $\widetilde{+}$-semigroup is $E = L_1[0,1]$. This is much more subtle than the case of $\ell_1$ and is proved in [6, Th. 6.3]. This example has the additional feature of involving a space with the Daugavet property; by Theorem 3.9, however, $E = C([0,1], \ell_1)$ is another example of this kind.

## 4. Narrow and $C$-narrow operators on $C(K, E)$

The following definition extends the notion of a $C$-narrow operator studied in [4] and [6] to the vector-valued setting.

DEFINITION 4.1. An operator $T \in L(C(K,E), W)$ is called *$C$-narrow* if there is a constant $\lambda$ such that given any $\varepsilon > 0$, $x \in S(E)$, and an open set $U \subset K$ there is a function $f \in C(K,E)$, $\|f\| \le \lambda$, satisfying the following conditions:

(a) $\mathrm{supp}(f) \subset U$;
(b) $f^{-1}(B(x,\varepsilon)) \neq \emptyset$, where $B(x,\varepsilon) = \{z \in E\colon \|z - x\| < \varepsilon\}$;
(c) $\|Tf\| < \varepsilon$.

As the following proposition shows, condition (b) of this definition can be substantially strengthened. In particular, the size of the constant $\lambda$ is immaterial, but introducing this constant in the definition allows for more flexibility in applications. Also, Proposition 4.2 shows that for $E = \mathbb{R}$ the new notion of $C$-narrowness coincides with that given in [6].

PROPOSITION 4.2. *If $T$ is a $C$-narrow operator, then for every $\varepsilon > 0$, every $x \in S(E)$, and any open set $U \subset K$ there is a function $f$ of the form $g \otimes x$, where $g \in C(K)$, $\mathrm{supp}(g) \subset U$, $\|g\| = 1$, and $g$ is nonnegative, such that $\|Tf\| < \varepsilon$.*

*Proof.* Let us fix $\varepsilon > 0$, an open set $U$ in $K$, and $x \in S(E)$. By Definition 4.1 there exists a function $f_1 \in C(K,E)$ as described in this definition corresponding to $\varepsilon$, $U$, and $x$. Put $U_1 = U$ and $U_2 = f_1^{-1}(B(x,1/2))$. As above, there is a function $f_2$ corresponding to $\varepsilon$, $U_2$ and $x$. We set $U_3 = f_2^{-1}(B(x,1/4))$ and continue the process. In the $r$th step we get the set $U_r = f_{r-1}^{-1}(B(x,1/2^{r-1}))$ and apply Definition 4.1 to obtain a function $f_r$ corresponding to $U_r$.

Choose $n \in \mathbb{N}$ so that $(\lambda+2)/n < \varepsilon$ and put $f = \frac{1}{n}(f_1+f_2+\cdots+f_n)$. By the Urysohn Lemma we can find a continuous function $g$ satisfying $\frac{k-1}{n} \le g(t) \le \frac{k}{n}$ for all $t \in U_k$, $k = 1,\dots,n$, $\|g\| = 1$, and vanishing outside $U_1$. We claim that $\|f - g \otimes x\| < \varepsilon$. Indeed, by our construction, if $t \in K \setminus U_1$, then $\|(f - g\otimes x)(t)\| = 0$, and if $t \in U_k \setminus U_{k+1}$ (with the understanding that $U_{n+1}$ stands for $\emptyset$), then

$$\begin{aligned}\|(f - g\otimes x)(t)\| &= \Big\|\frac{1}{n}(f_1+\cdots+f_k)(t) - g(t)\cdot x\Big\| \\ &\le \Big\|\frac{1}{n}\big((f_1(t)-x)+\cdots+(f_{k-1}(t)-x)+f_k(t)\big)\Big\| + \frac{1}{n} \\ &\le \frac{1}{n}\Big(\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}+\lambda\Big)+\frac{1}{n} < \frac{\lambda+2}{n} < \varepsilon.\end{aligned}$$

Moreover,

$$\|Tf\| \le \frac{1}{n}\left(\|Tf_1\|+\|Tf_2\|+\cdots+\|Tf_n\|\right) < \varepsilon.$$

Thus $\|T(g\otimes x)\| < \varepsilon + \varepsilon\|T\|$, and since $\varepsilon$ was chosen arbitrarily, we are done. ☐

Another way to express this proposition is to say that $T\colon C(K,E)\to W$ is $C$-narrow if and only if, for each $x\in E$, the restriction $T_x\colon C(K)\to W$, $T_x(g) = T(g\otimes x)$, is $C$-narrow.

PROPOSITION 4.3.

(a) *Every $C$-narrow operator on $C(K,E)$ is a strong Daugavet operator. Hence, in the case of a perfect compact space $K$ every $C$-narrow operator on $C(K,E)$ is narrow.*

(b) *If $E$ is a separable USD-nonfriendly space, then every strong Daugavet operator on $C(K,E)$ is $C$-narrow.*

(c) *If every strong Daugavet operator on $C(K,E)$ is $C$-narrow, then $E$ is SD-nonfriendly.*

*Proof.* (a) Let $T$ be $C$-narrow. We will use condition (3) of Theorem 3.7. Let $F\subset K$ be a closed subset, $x\in S(E)$, $y\in B(E)$, and $\varepsilon>0$. According to Proposition 4.2 there exists a function $f$ vanishing on $F$ of the form $g\otimes(x-y)$, where $g\in C(K)$, $\|g\|=1$, and $g$ is nonnegative, such that $\|Tf\|<\varepsilon$. Evidently this function $f$ satisfies all requirements of condition (3) in Theorem 3.7.

(b) Let $T$ be a strong Daugavet operator, and suppose $E$ is separable. Let $U\subset K$ be a non-empty open subset. Given $x,y\in S(E)$ and $\varepsilon'>0$, we define

$$\begin{aligned}O(x,y,\varepsilon') \;=\; \{t\in U\colon\ &\exists f\in C(K,E)\colon \operatorname{supp} f\subset U,\ \|f+y\|<1+\varepsilon', \\ &\|f(t)+y+x\|>2-\varepsilon',\ \|Tf\|<\varepsilon'\}.\end{aligned}$$

This is an open subset of $K$, and by Theorem 3.7(3) it is dense in $U$. Now pick a countable dense subset $\{(x_n,y_n)\colon n\in\mathbb{N}\}$ of $S(E)\times S(E)$ and a null sequence $(\varepsilon_n)$. Then, by Baire's theorem, $G:=\bigcap_n O(x_n,y_n,\varepsilon_n)$ is nonempty.

Let $\varepsilon>0$, and fix $t_0\in G$. We denote by $A(U,\varepsilon)$ the closure of

$$\{f(t_0)\colon f\in C(K,E),\ \|f\|<2+\varepsilon,\ \|Tf\|<\varepsilon,\ \operatorname{supp} f\subset U\};$$

this is an absolutely convex set. We claim that $A(U,\varepsilon)$ intersects each set $D(x,y,\varepsilon')\in\mathcal{D}(E)$. Indeed, if $\|x_n-x\|<\varepsilon'/4$, $\|y_n-y\|<\varepsilon'/4$, $\varepsilon_n<\varepsilon'/2$

and $\varepsilon_n < \varepsilon$, then for a function $f_n$ as given in the definition of $O(x_n, y_n, \varepsilon_n)$ we have $f_n(t_0) \in A(U, \varepsilon) \cap D(x_n, y_n, \varepsilon_n) \subset A(U, \varepsilon) \cap D(x, y, \varepsilon')$.

Since $E$ is USD-nonfriendly, say with parameter $\alpha$, the set $A(U, \varepsilon)$ contains $\alpha B(E)$. This implies that $T$ satisfies the definition of a $C$-narrow operator with constant $\lambda = 3/\alpha$.

(c) Let $T \in \mathcal{SD}(E)$; then by Proposition 3.8 $T^K$ is a strong Daugavet operator on $C(K, E)$. But

$$\big(T^K(g \otimes e)\big)(t) = T\big((g \otimes e)(t)\big) = g(t)Te.$$

Hence $T^K$ is not $C$-narrow unless $T = 0$. $\square$

The example $E = c_0$ shows that the converse of (b) is false. We have already pointed out in Proposition 2.4(a) that $c_0$ fails to be USD-nonfriendly; yet every strong Daugavet operator on $C(K, c_0)$ is $C$-narrow. To see this we first remark that it is enough to verify the condition of Proposition 4.2 for $x$ belonging to a dense subset of $S(E)$. In our context we may therefore assume that the sequence $x$ vanishes eventually, say $x(n) = 0$ for $n > N$. If we write $c_0 = \ell_\infty^N \oplus_\infty Z$, where $Z$ is the space of null sequences supported on $\{N+1, N+2, \dots\}$, we also have $C(K, c_0) = C(K, \ell_\infty^N) \oplus_\infty C(K, Z)$. By Corollary 3.6 the restriction of any strong Daugavet operator $T$ on $C(K, c_0)$ to $C(K, \ell_\infty^N)$ is again a strong Daugavet operator, and hence it is $C$-narrow, since $\ell_\infty^N$ is USD-nonfriendly (Proposition 2.7). This implies that $T$ is $C$-narrow.

We do not know whether (c) is actually an equivalence.

One of the fundamental properties of $C$-narrow operators is stated in our next theorem.

THEOREM 4.4. *Suppose that the operators $T$, $T_n \in L(C(K, E), W)$ are such that the series $\sum_{n=1}^\infty w^*(T_n f)$ converges absolutely to $w^*(Tf)$, for every $w^* \in W^*$ and $f \in C(K, E)$. If all $T_n$ are $C$-narrow, then so is $T$. In particular, the sum of two $C$-narrow operators is a $C$-narrow operator.*

COROLLARY 4.5. *A pointwise unconditionally convergent sum of narrow operators on $C(K, E)$ is a narrow operator itself if $E$ is separable and USD-nonfriendly.*

Indeed, this follows from Theorem 4.4 and Proposition 4.3; note that $K$ is perfect if there exists a narrow operator defined on $C(K, E)$ in case $E$ fails the Daugavet property. To see the latter, assume that $K = \{k\} \cup K'$ for some isolated point $k$. If there exists a narrow operator on $C(K, E) \cong E \oplus_\infty C(K', E)$, then this space has the Daugavet property, and so has $E$ [5, Lemma 2.15].

We remark that the case of a sum of two narrow operators on $C(K)$ was treated earlier in [4] and [6], but the assertion about infinite sums is new even in this case. In [5] it was shown that a pointwise unconditionally convergent sum $T = \sum_{n=1}^{\infty} T_n$ on a space with the Daugavet property satisfies

$$\|\mathrm{Id} + T\| \geq 1$$

whenever $\|\mathrm{Id} + S\| = 1 + \|S\|$ for every $S$ in the linear span of the $T_n$. In the context of Theorem 4.4 we have, in fact,

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{4.1}$$

in the case when all $T_n$ are narrow on $C(K)$. In particular, the identity on $C(K)$ cannot be represented as an unconditional sum of narrow operators, since obviously (4.1) fails for $T = -\mathrm{Id}$. This last consequence shows that for an unconditional Schauder decomposition $C(K) = X_1 \oplus X_2 \oplus \ldots$ with corresponding projections $P_1, P_2, \ldots$ one of the $P_n$ must be non-narrow, since $\mathrm{Id} = \sum_{n=1}^{\infty} P_n$ pointwise unconditionally. Hence one of the $X_n$ must be infinite-dimensional if $K$ is a perfect compact Hausdorff space. In fact, one of the $X_n$ must contain a copy of $C[0,1]$ and therefore, by a theorem of Pełczyński [7], be isomorphic to $C[0,1]$ if $K$ is in addition metrisable; see [4] and [5] for more results along these lines.

We now turn to the proof of Theorem 4.4, for which we need an auxiliary concept. A similar idea was used in [4].

Definition 4.6. Let $G$ be a closed $G_\delta$-set in $K$ and let $T \in L(C(K), W)$. We say that $G$ is a *vanishing set* of $T$ if there is a sequence of open sets $(U_i)_{i\in\mathbb{N}}$ in $K$ and a sequence of functions $(f_i)_{i\in\mathbb{N}}$ in $S(C(K))$ such that

(a) $G = \bigcap_{i=1}^{\infty} U_i$;
(b) $\mathrm{supp}(f_i) \subset U_i$;
(c) $\lim_{i\to\infty} f_i = \chi_G$ pointwise;
(d) $\lim_{i\to\infty} \|Tf_i\| = 0$.

The collection of all vanishing sets of $T$ is denoted by $\mathrm{van}\, T$.

Let $T \in L(C(K), W)$. By the Riesz Representation Theorem, $T^*w^*$ can be viewed as a regular measure on the Borel subsets of $K$ whenever $w^* \in W^*$. For convenience, we denote this regular measure also by $T^*w^*$.

Lemma 4.7. *Suppose $G$ is a closed $G_\delta$-set in $K$ and $T \in L(C(K), W)$. Then $G \in \mathrm{van}\, T$ if and only if $T^*w^*(G) = 0$ for all $w^* \in W^*$.*

*Proof.* Let $G \in \mathrm{van}\, T$, and pick functions $(f_i)_{i\in\mathbb{N}}$ as in Definition 4.6. Then by the Lebesgue Dominated Convergence Theorem, for any given $w^* \in W^*$ we have

$$T^*w^*(G) = \int_K \chi_G \, dT^*w^* = \lim_{i\to\infty} \int_K f_i \, dT^*w^* = \lim_{i\to\infty} w^*(Tf_i) = 0.$$

Conversely, let $(U_i)_{i\in\mathbb{N}}$ be a sequence of open sets in $K$ such that $\overline{U}_{i+1} \subset U_i$ and $G = \bigcap_{i=1}^{\infty} U_i$. By the Urysohn Lemma there exist functions $(f_i)_{i\in\mathbb{N}}$ having the following properties: $0 \leq f_i(t) \leq 1$ for all $t \in K$, $\operatorname{supp}(f_i) \subset U_i$, and $f_i(t) = 1$ if $t \in \overline{U}_{i+1}$. Clearly, $\lim_{i\to\infty} f_i = \chi_G$ pointwise, and

$$\lim_{i\to\infty} w^*(Tf_i) = \lim_{i\to\infty} T^*w^*(f_i) = T^*w^*(G) = 0$$

whenever $w^* \in W^*$. This means that the sequence $(Tf_i)_{i\in\mathbb{N}}$ is weakly null. Applying the Mazur Theorem we finally obtain a sequence of convex combinations of the functions $(f_i)_{i\in\mathbb{N}}$ which satisfies all conditions of Definition 4.6.

This completes the proof. □

LEMMA 4.8. *An operator $T \in L(C(K), W)$ is $C$-narrow if and only if every non-empty open set $U \subset K$ contains a non-empty vanishing set of $T$. Moreover, if $(T_n)_{n\in\mathbb{N}} \subset L(C(K), W)$ is a sequence of $C$-narrow operators, every open set $U \neq \emptyset$ contains a set $G \neq \emptyset$ that is simultaneously a vanishing set for all $T_n$.*

*Proof.* We first prove the more general "moreover" part. Put $U_{1,1} = U$. By the definition of a $C$-narrow operator and Proposition 4.2 there is a function $f_{1,1} \subset S(C(K))$ with $\operatorname{supp}(f_{1,1}) \subset U_{1,1}$, $U_{1,2} := f_{1,1}^{-1}(1/2, 1] \neq \emptyset$ and $\|T_1 f_{1,1}\| < 1/2$. Obviously, $\overline{U}_{1,2} \subset f_{1,1}^{-1}[1/2, 1] \subset U_{1,1}$. Again applying the definition we find $f_{1,2} \in S(C(K))$ with $\operatorname{supp}(f_{1,2}) \subset U_{1,2}$, $U_{2,1} = f_{1,2}^{-1}(2/3, 1] \neq \emptyset$ and $\|T_1 f_{1,2}\| < 1/3$. As above $\overline{U}_{2,1} \subset U_{1,2}$.

In view of the $C$-narrowness of $T_2$ there exists a function $f_{2,1} \in S(C(K))$ with $\operatorname{supp}(f_{2,1}) \subset U_{2,1}$, $U_{1,3} = f_{2,1}^{-1}(2/3, 1] \neq \emptyset$ and $\|T_2 f_{2,1}\| < 1/3$. In the next step we construct $f_{1,3} \in S(C(K))$ such that $U_{2,2} = f_{1,3}^{-1}(3/4, 1] \neq \emptyset$ and $\|T_1 f_{1,3}\| < 1/4$.

Proceeding in the same way, in the $n$th step we find a set of functions $(f_{k,l})_{k+l=n} \subset S(C(K))$ and nonempty open sets $(U_{k,l})_{k+l=n}$ in $K$ such that $\operatorname{supp}(f_{k,l}) \subset U_{k,l}$, $\|T_k f_{k,n-k}\| < \frac{1}{n}$ and $U_{k,l} = f_{k-1,l+1}^{-1}(\frac{n-1}{n}, 1]$, if $k \neq 1$. Then we put $U_{1,n} = f_{n-1,1}^{-1}(\frac{n-1}{n}, 1]$ to begin the next step.

It remains to show that the set $G = \bigcap_{k,l\in\mathbb{N}} U_{k,l} = \bigcap_{k,l\in\mathbb{N}} \overline{U}_{k,l}$ is as desired. Indeed, $G$ is clearly a nonempty closed $G_\delta$-set and $G = \bigcap_{i=1}^{\infty} U_{n,i}$ for every $n \in \mathbb{N}$. It is easily seen that the sequences $(f_{n,i})_{i\in\mathbb{N}}$ and $(U_{n,i})_{i\in\mathbb{N}}$ meet the conditions of Definition 4.6 for the operator $T_n$. Hence, $G \in \operatorname{van} T_n$ for every $n \in \mathbb{N}$.

To prove the converse, let $U \neq \emptyset$ be any open set in $K$ and let $\varepsilon > 0$. By the assumption on $\operatorname{van} T$ we can find a closed $G_\delta$-set $\emptyset \neq G \subset U$, $G \in \operatorname{van} T$. Consider the open sets $(U_i)_{i\in\mathbb{N}}$ and the functions $(f_i)_{i\in\mathbb{N}}$ provided by Definition 4.6. For sufficiently large $i \in \mathbb{N}$ we have $U_i \subset U$ and $\|Tf_i\| < \varepsilon$ so that $f_i$ may serve as the function required in Definition 4.1.

This finishes the proof. □

We are now in a position to prove Theorem 4.4.

*Proof of Theorem 4.4.* By virtue of Proposition 4.2 we may assume that $E = \mathbb{R}$. By Lemma 4.8 it suffices to show that $\bigcap_{n=1}^{\infty} \operatorname{van} T_n \subset \operatorname{van} T$.

Suppose $G \in \bigcap_{n=1}^{\infty} \operatorname{van} T_n$. According to Lemma 4.7 we need to prove that $T^* w^*(G) = 0$ for all $w^* \in W^*$. By the hypothesis of the theorem, the series $\sum_{n=1}^{\infty} T_n^* w^*$ is weak*-unconditionally Cauchy and hence weakly unconditionally Cauchy. Since $C(K)^*$ does not contain a copy of $c_0$, it is actually unconditionally norm convergent by the Bessaga-Pełczyński Theorem. This implies that the bounded sequence of functions $(f_i)_{i\in\mathbb{N}}$ satisfying $f_i \to \chi_G$ pointwise, which was constructed in the proof of Lemma 4.7, satisfies

$$\begin{aligned} T^* w^*(G) &= \lim_{i\to\infty} T^* w^*(f_i) = \lim_{i\to\infty} \sum_{n=1}^{\infty} T_n^* w^*(f_i) \\ &= \sum_{n=1}^{\infty} T_n^* w^*(\chi_G) = \sum_{n=1}^{\infty} T_n^* w^*(G) = 0. \end{aligned}$$

This completes the proof. □

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