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JACOBI FORMS OVER TOTALLY REAL FIELDS AND CODES OVER $\mathbb{F}_p$

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ABSTRACT. In this paper we establish a connection between Jacobi forms over a totally real field $k = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = e^{2\pi i/p}$, and codes over the field $\mathbb{F}_p$. In particular, we derive a theta series, which is a Jacobi form, from the complete weight enumerator or the Lee weight enumerator of a self-dual code over $\mathbb{F}_p$.

1. Introduction

In 1972, Broué and Enguehard [4] studied a map between the space of invariant polynomials for a certain finite group and the ring of modular forms; specifically, they showed that elliptic modular forms of weight $n/2$ can be obtained from the complete weight enumerators $\mathrm{CWe}_C(x, y)$ of binary type II codes C by substituting the theta series $\theta_3(2\tau)$ and $\theta_2(2\tau)$ for x and y, respectively. Because of this map, combinatorial problems in coding theory are closely related to problems studied earlier and independently in pure mathematics. An important problem in algebraic coding theory (see [14]) is to determine, for a given class of self-dual codes, the ring of invariants to which some weight enumerators belong. Not only codes over finite fields, but also codes over finite rings and finite abelian groups have been studied extensively, and polynomial analogues of modular forms of Jacobi form type (see [1],[3],[6],[5]) and of Siegel modular forms (see [15]) have been discovered.

On the other hand, the first connection between Hilbert modular forms and codes was described in [11] by considering Lee weight enumerators of self-dual codes over $\mathbb{F}_p$, where p is an odd prime. In this paper, we extend the relation studied in [11] between the Lee weight enumerator $W_C(X_1, \ldots, X_{(p-1)/2})$ of a self-dual code C over $\mathbb{F}_p$ and a certain Hilbert modular form to a connection between the complete weight enumerators and the Lee weight enumerator of a self-dual code C over $\mathbb{F}_p$ and a Jacobi form $f(\tau_1, \ldots, \tau_{(p-1)/2}, z_1, \ldots, z_{(p-1)/2})$ over $k = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = e^{2\pi i/p}$. Letting $z_1 = z_2 = \cdots = z_{(p-1)/2} = 0$,

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we recover the result given in [11]. More generally, we study a Broué and Enguehard type map from a certain invariant space, in which the complete weight enumerators (or Lee weight enumerators) of self-dual codes over $\mathbb{F}_p$ live, to the space of Jacobi forms over $k = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = e^{2\pi i/p}$.

2. Definitions and notations

In this section we recall the definition of a Jacobi form over a totally real number field. We follow the notations of [16].

Let K be an algebraic number field with finite degree $d = [K : \mathbb{Q}]$. Let $\sigma_1, \dots, \sigma_d \colon K \to \mathbb{C}$ be the different embeddings of K into $\mathbb{C}$, with $\sigma_1 = \mathrm{id}$. The field K is called totally real if $\sigma_j(v) \in \mathbb{R}$ for all $1 \le j \le d$; we set $\sigma_j(v) = v^{(j)}$, $v \in K$. The norm and trace of v are defined by $\mathcal{N}_{K/\mathbb{Q}}(v) = \prod_{j=1}^{d} v^{(j)}$ and $\mathrm{Tr}_{K/\mathbb{Q}}(v) = \sum_{j=1}^{d} v^{(j)}$, respectively.

Next, we define the Jacobi group. The Jacobi group of a totally real number field k of degree d over $\mathbb{Q}$ with ring of integers $\mathcal{O}_k$ is denoted by $\Gamma^J(k)$ and defined as

$$\Gamma^J(k) = SL_2(\mathcal{O}_k) \propto \mathcal{O}_k^2.$$

This group acts on $\mathcal{H}^d \times \mathbb{C}^d$ for each conjugate of the field. The variables of this space are denoted by $(\tau, z) = (\tau_1, \dots, \tau_d, z_1, \dots, z_d)$. The actions of $\Gamma^J(k)$ on the space $\mathcal{H}^d \times \mathbb{C}^d$ are given by

$$\begin{aligned}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(\tau, z) &:= \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \frac{z}{\gamma\tau + \delta}\right)\\ &:= \left(\frac{\alpha^{(1)}\tau_1 + \beta^{(1)}}{\gamma^{(1)}\tau_1 + \delta^{(1)}}, \dots, \frac{\alpha^{(d)}\tau_d + \beta^{(d)}}{\gamma^{(d)}\tau_d + \delta^{(d)}}, \frac{z_1}{\gamma^{(1)}\tau_1 + \delta^{(1)}}, \dots, \frac{z_d}{\gamma^{(d)}\tau_d + \delta^{d}}\right),\end{aligned}$$

where

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(\mathcal{O}_k), \quad (\tau, z) \in \mathcal{H}^d \times \mathbb{C}^d,$$

and

$$\begin{aligned}(\tau, z)[\lambda, \mu] &:= (\tau, z + \lambda\tau + \mu)\\ &:= (\tau_1, \dots, \tau_d, z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}, \dots, z_d + \lambda^{(d)}\tau_d + \mu^{(d)})\end{aligned}$$

for all $[\lambda, \mu] \in \mathcal{O}_k^2$.

Let $f\colon \mathcal{H}^d \times \mathbb{C}^d \to \mathbb{C}$ be a function. Then we define the "slash operators" as follows: for given $\ell \in \mathbb{Z}$ and $m \in \mathcal{O}_k$ we define, for all $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathcal{O}_k)$,

$$\left(f|_{\ell,m}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)(\tau, z) := \left(\prod_{j=1}^{d} e^{-2\pi i m^{(j)} \frac{\gamma^{(j)} z_j^2}{\gamma^{(j)}\tau_j + \delta^{(j)}}}\right) \times \prod_{j=1}^{d} (\gamma^{(j)}\tau_j + \delta^{(j)})^{-\ell} f\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z}{\gamma\tau+\delta}\right)$$

and, for all $[\lambda, \mu] \in \mathcal{O}_k^2$,

$$\left(f|_m[\lambda,\mu]\right)(\tau, z) := \left(\prod_{j=1}^{d} e^{-2\pi i m^{(j)}\left(\lambda^{(j)2}\tau_j + 2\lambda^{(j)} z_j\right)}\right) f(\tau, z + \lambda\tau + \mu).$$

We are now in a position to define a Jacobi form over a totally real number field.

Definition 2.1. A Jacobi form f of weight ℓ and index $m \in \mathcal{O}_k$ on a totally real number field k is an analytic function $f\colon \mathcal{H}^d \times \mathbb{C}^d \to \mathbb{C}$ satisfying

$$(f|_{\ell,m}M)(\tau, z) = f(\tau, z), \quad M \in SL_2(\mathcal{O}_k) \tag{2.1}$$

and

$$\left(f|_m[\lambda,\mu]\right)(\tau, z) = f(\tau, z), \quad [\lambda, \mu] \in \mathcal{O}_k. \tag{2.2}$$

The form f must have a Fourier series expansion of the form

$$f(\tau, z) = \sum_{N,R \in \delta_k^{-1},\, N \geq 0} c(N,R) \prod_{j=1}^{d} e^{2\pi i\left(N^{(j)}\tau_j + R^{(j)} z_j\right)}, \tag{2.3}$$

where $N \geq 0$ means that N is totally positive or zero, each of the coefficients $c(N,R)$ is constant, and δ_k^{-1} is the inverse different of k.

Remark 2.2.
(1) When $z = 0$, the form $f(\tau, 0)$ is simply a Hilbert modular form on $SL_2(\mathcal{O}_k)$.
(2) Jacobi forms over a totally real field have first been defined and studied in [16].

Next, we recall the definition and some basic facts of linear codes.

A linear code C of length n over the field $\mathbb{F}_p$ is an additive subgroup of $\mathbb{F}_p$. An element of C is called a codeword. We denote by $|C|$ the number of codewords in C. The inner product of x and y in $\mathbb{F}_p^n$ is defined by

$$x \cdot y = x_1y_1 + \cdots + x_ny_n \pmod{p}, \quad x = (x_1, \ldots, x_n),\, y = (y_1, \ldots, y_n) \in \mathbb{F}_p^n.$$

The dual code $C^\perp$ of C is defined as

$$C^\perp = \{y \in \mathbb{F}_p^n \mid x \cdot y = 0 \text{ for all } x \in C\}.$$

If $C = C^\perp$, then C is called self-dual. The complete weight enumerator CWe_C of C over $\mathbb{F}_p$ is defined by

$$\mathrm{CWe}_C\,(X_0, X_1, \ldots, X_{p-2}, X_{p-1}) = \sum_{u \in C} X_0^{n_0(u)} X_1^{n_1(u)} \ldots X_{p-2}^{n_{p-2}(u)} X_{p-1}^{n_{p-1}(u)},$$

where $n_j(u)$, $0 \le j \le p-1$, denotes the number of components of u which are equal to j. The Lee weight enumerator W_C of C over $\mathbb{F}_p$ is defined by

$$W_C\left(X_0, X_1, \ldots, X_{(p-1)/2}\right) = \sum_{u \in C} X_0^{n_0(u)} X_1^{n_1(u)} \ldots X_{(p-1)/2}^{n_{(p-1)/2}(u)},$$

where $n_0(u)$ denotes the number of zero components of u, while $n_j(u), j \neq 0$, denotes the number of components of u which are equal to j or $-j$.

3. Jacobi theta series over number fields

In this section, we study Jacobi theta series. Here we follow the notations of [11].

Let k be a totally real field with degree $r = [k : \mathbb{Q}]$. Consider a vector space V over k of finite dimension $d = \dim_k V$ with a totally positive definite scalar product $\cdot \colon V \times V \to k$ satisfying $(v \cdot v)^{(j)} > 0$ for all $1 \le j \le r$ and all $v \in V - \{0\}$. Let Λ be a k-lattice in V, i.e., a finitely generated $\mathcal{O}_k$-submodule of V which contains a k-basis of V. Let $\{e_1, \ldots, e_{rd}\}$ be a $\mathbb{Z}$-basis of Λ. Then

$$\Delta(\Lambda) := \det\left(\mathrm{Tr}_{k/\mathbb{Q}}(e_i \cdot e_j)\right)$$

is called the discriminant of Λ. The dual lattice Λ^* of Λ is defined as

$$\Lambda^* := \left\{w \in V \mid \mathrm{Tr}_{k/\mathbb{Q}}(w \cdot v) \in \mathbb{Z}, \quad v \in \Lambda\right\}.$$

One can check that Λ^* is also a k-lattice in V (see [11, Prop. 5.6, p. 160]). The lattice Λ is called *unimodular* if $\Lambda^* = \Lambda$, *integral* if $\mathrm{Tr}_{k/\mathbb{Q}}(v \cdot w) \in \mathbb{Z}$ for all $v, w \in \Lambda$, and *even* if $\mathrm{Tr}_{k/\mathbb{Q}}\left(\mathcal{O}_k(v \cdot v)\right) \in 2\mathbb{Z}$ for all $v \in \Lambda$.

Let $K = \mathbb{Q}(\zeta)$ be a cyclotomic field with $\zeta = e^{2\pi i/p}$, where p is an odd prime. Then it is known that the ring of integers $\mathcal{O}_K$ is the set

$$\mathcal{O}_K = \left\{ \alpha = \sum_{j=0}^{p-2} a_j \zeta^j \,\middle|\, a_j \in \mathbb{Z},\, 0 \le j \le p-2 \right\}.$$

Let $\beta := (1 - \zeta)$ be the principal ideal of $\mathcal{O}_K$ generated by the element $1 - \zeta \in \mathcal{O}_K$. It is well-known (see [11, p. 130]) that

$$\mathcal{O}_K/\beta \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p;$$

the map $\rho\colon \mathcal{O}_K \to \mathbb{Z}/p\mathbb{Z}$ sending $\alpha = a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2}$ to $\rho(\alpha) = a_0 + a_1 + \cdots + a_{p-2} \pmod p$ is a homomorphism with kernel β. The map ρ can also be regarded as the reduction map $\pmod \beta$ from $\mathcal{O}_K$ to $\mathbb{Z}/p\mathbb{Z}$.

Next, we consider the totally real subfield k of K, which turns out to be the field $k = \mathbb{Q}(\zeta + \zeta^{-1})$, where $[K : k] = 2$, and $[k : \mathbb{Q}] = (p-1)/2$. For the remainder of this paper, K and k will always be defined in this way, i.e., we set $K = \mathbb{Q}(\zeta)$ and $k = \mathbb{Q}(\zeta + \zeta^{-1})$, where $\zeta = e^{2\pi i/p}$ and p is an odd prime.

We recall the following useful properties (see [11, pp. 132–133]).

Remark 3.1.

(1) For $v, w \in K$, the map $\cdot\colon K \times K \to k$ defined by

$$v \cdot w := v\overline{w} + \overline{v}w,$$

where $\overline{v}$ denotes the complex conjugate of v, is a totally positive definite scalar product on K.

(2) The map $\langle\,,\,\rangle_K\colon K \times K \to \mathbb{R}$ given by

$$\langle x, y\rangle_K = \mathrm{Tr}_{K/\mathbb{Q}}(x \cdot y) = \mathrm{Tr}_{k/\mathbb{Q}}(x\overline{y} + \overline{x}y),$$

is a symmetric bilinear map.

(3) Let β be the principal ideal generated by $(1-\zeta)$ of the ring of integers $\mathcal{O}_K$ of K. Then β is a k-lattice in K. Furthermore, we have

$$\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{x \cdot y}{p}\right) \in \mathbb{Z}, \quad x, y \in \beta$$

and

$$\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{x \cdot x}{p}\right) \in 2\mathbb{Z}, \quad x \in \beta.$$

(4) Let β^* be the dual lattice of β. Then $\beta^* = \mathcal{O}_K$. Since $\mathcal{O}_K/\beta \simeq \mathbb{F}_p$, we have $\beta^* = \bigcup_{j=0}^{p-1}(j + \beta)$.

For each j, $0 \le j \le p-1$, and a fixed element $y \in \mathcal{O}_K$, define a theta function $\theta_{j,y}(\tau, z) := \theta_{j+\beta,y}\colon \mathcal{H}^{(p-1)/2} \times \mathbb{C}^{(p-1)/2} \to \mathbb{C}$ by

$$\theta_{j,y}(\tau, z) := \theta_{j+\beta,y}(\tau, z) := \sum_{v \in j+\beta} e^{\pi i \,\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{v\cdot v}{p}\tau\right)} e^{2\pi i \,\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{v\cdot y}{p}z\right)}, \tag{3.1}$$

where

$$\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{v \cdot v}{p}\tau\right) := 2\sum_{s=1}^{(p-1)/2} \frac{(v\overline{v})^{(s)}}{p}\tau_s,$$

$$\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{v \cdot y}{p}z\right) := \sum_{s=1}^{(p-1)/2} \frac{(v\overline{y} + \overline{v}y)^{(s)}}{p} z_s$$

and $\tau_i \in \mathcal{H}$, $z_i \in \mathbb{C}$.

REMARK 3.2. The transformation properties of $\theta_{j+\beta}(\tau,0)$ as a function over $\mathcal{H}^{(p-1)/2}$ have been studied in [11, p. 162]. The above theta series generalizes that given in [11] to a function defined on $\mathcal{H}^{(p-1)/2}\times\mathbb{C}^{(p-1)/2}$.

We next derive a transformation formula for this theta series.

LEMMA 3.3. *Let β be the principal ideal generated by $(1-\zeta)$ in $\mathcal{O}_K$. For each j, $0\le j\le p-1$, and a fixed $y\in\mathcal{O}_K$ we have*

$$\theta_{j+\beta,y}\left(\frac{-1}{\tau},\frac{z}{\tau}\right)=\frac{1}{\sqrt{\Delta(\beta)}}e^{\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{y\cdot y}{p}\frac{z^2}{\tau}\right)}\prod_{s=1}^{(p-1)/2}\left(\frac{\tau_s}{i}\right)\sum_{\ell=0}^{p-1}e^{2\pi i\frac{j\ell}{p}}\theta_{\ell+\beta,y}(\tau,z).$$

Proof. We modify the argument of [11, p. 162]. Let

$$Z=\begin{pmatrix}\tau_1\mathbb{E}_2 & & & \\ & \tau_2\mathbb{E}_2 & & \\ & & \cdots & \\ & & & \tau_{(p-1)/2}\mathbb{E}_2\end{pmatrix},\quad W=\begin{pmatrix}z_1\mathbb{E}_2 & & & \\ & z_2\mathbb{E}_2 & & \\ & & \cdots & \\ & & & z_{(p-1)/2}\mathbb{E}_2\end{pmatrix},$$

where $\mathbb{E}_2=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. For given $x_0,Y\in\mathbb{R}^p$ let

$$g_{Y,x_0}(x):=e^{-\pi i(x+x_0)Z^{-1}(x+x_0)^t}e^{2\pi i(x+x_0)Z^{-1}WY^t}.$$

For $v\in\mathbb{R}^p$ let

$$\hat{g}_{Y,x_0}(v):=\int_{\mathbb{R}^p}g_{Y,x_0}(x)e^{-2\pi ixv^t}\,dx.$$

Then

$$\hat{g}_{Y,x_0}(v)=e^{-2\pi ix_0v^t}\int_{\mathbb{R}^p}e^{-\pi ixZ^{-1}x^t}e^{2\pi ixZ^{-1}WY^t}e^{-2\pi ixv^t}\,dx,$$

upon substituting $x-x_0$ by x.

We have

$$\begin{aligned}&\int_{\mathbb{R}^p}e^{-\pi ixZ^{-1}x^t}e^{2\pi ixZ^{-1}WY^t}e^{-2\pi ixv^t}\,dx\\
&\quad=\int_{\mathbb{R}^p}e^{-\pi i\left(\frac{x_1^2}{\tau_1}+\frac{x_2^2}{\tau_1}\cdots+\frac{x_{p-1}^2}{\tau_{(p-1)/2}}+\frac{x_p^2}{\tau_{(p-1)/2}}\right)}\\
&\qquad\times e^{2\pi i\left(\frac{x_1z_1y_1}{\tau_1}+\frac{x_2z_1y_2}{\tau_1}\cdots+\frac{x_{p-1}z_{(p-1)/2}y_{p-1}}{\tau_{(p-1)/2}}+\frac{x_pz_{(p-1)/2}y_p}{\tau_{(p-1)/2}}\right)}\\
&\qquad\times e^{-2\pi i(x_1v_1+x_2v_2+\cdots+x_pv_p)}\,dx_1\ldots dx_p\\
&\quad=\det\left(\frac{Z}{i}\right)^{1/2}e^{\pi i\left(y_1^2\frac{z_1^2}{\tau_1}+y_2^2\frac{z_1^2}{\tau_1}+\cdots+y_p^2\frac{z_{(p-1)/2}^2}{\tau_{(p-1)/2}}\right)}\\
&\qquad\times e^{\pi i(v_1^2\tau_1+v_2^2\tau_1+\cdots+v_{p-1}^2v_p^2\tau_{(p-1)/2})}\\
&\qquad\times e^{-2\pi i\left(v_1z_1y_1+v_2z_1y_1+\cdots+v_{p-1}z_{(p-1)/2}y_{p-1}+v_pz_{(p-1)/2}y_p\right)},\end{aligned}$$

where $x = (x_1, \dots, x_p)$, $Y = (y_1, \dots, y_p)$, $v = (v_1, \dots, v_p) \in \mathbb{R}^p$. From this identity we obtain

$$\hat{g}_{Y,x_0}(v) = \det\left(\frac{Z}{i}\right)^{1/2} e^{-2\pi i x_0 v^t} e^{\pi i Y Z^{-1} W Y^t} e^{\pi i v Z v^t} e^{-2\pi i v W Y^t}.$$

On the other hand, for each s with $1 \le s \le (p-1)/2$, $\mathbb{R}$ becomes a k-module, which we will denote by k_s, via the map

$$k \times \mathbb{R} \ni (x, \rho) \to \sigma_s(x) \cdot \rho.$$

Then the space $K_s := K \otimes_k k_s$ is a real vector space of dimension 2, which contains K via the map $K \ni v \to v \otimes 1$. The scalar product

$$K \times K \to k_s, \quad (v, w) \to \sigma_s(v \cdot w)$$

can be extended to a scalar product on K_s that is $\mathbb{R}$-bilinear. Since K_s can be considered as an Euclidean vector space, we have an isomorphism $L_s \colon K_s \to \mathbb{R}^2$ with $\sigma_s(v \cdot w) = L_s(v) \cdot L_s(w)$ for all $v, w \in K_s$. Next, using the embedding

$$\begin{gathered} L \colon K \to \mathbb{R}^p, \\ v \to L(v) := \big(L_1(v), L_2(v), \dots, L_{(p-1)/2}(v)\big) \end{gathered}$$

one can realize $L(\beta)$ as a lattice in $\mathbb{R}^p$, with $L(\beta^*) = L(\beta)^*$. Hence the theta series can be written as

$$\begin{aligned} \theta_{j+\beta,y}(\tau, z) &= \sum_{v \in j+\beta} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{v \cdot v}{p}\tau\right)} e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{v \cdot y}{p} z\right)} \\ &= \sum_{V \in L(j+\beta)} e^{\pi i \frac{V \cdot V}{p}\tau} e^{2\pi i \frac{V \cdot Y}{p} z} \\ &= \sum_{V \in L(\beta)} e^{\pi i \frac{(V + L(j)) \cdot (V + L(j))}{p}\tau} e^{2\pi i \frac{(V + L(j)) \cdot Y}{p} z}. \end{aligned}$$

Applying the Poisson summation formula, we obtain, on letting $Y = L(y) = (L_1(y), \dots, L_{(p-1)/2}(y))$,

$$\begin{aligned} \theta_{j+\beta,y}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sum_{V \in L(\beta)} g_{\frac{Y}{\sqrt{p}}, \frac{L(j)}{p}}\left(\frac{V}{\sqrt{p}}\right) \\ &= \frac{1}{\sqrt{\Delta(\beta)}} \sum_{V \in L(\beta)^*} \hat{g}_{\frac{Y}{\sqrt{p}}, \frac{L(j)}{\sqrt{p}}}\left(\frac{V}{\sqrt{p}}\right) \\ &= \frac{1}{\sqrt{\Delta(\beta)}} \sum_{V \in L(\beta)^*} \hat{g}_{\frac{Y}{\sqrt{p}}, \frac{L(j)}{\sqrt{p}}}\left(\frac{-V}{\sqrt{p}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\Delta(\beta)}} \det\left(\frac{Z}{i}\right)^{1/2} e^{\pi i \frac{YWZ^{-1}W^tY^t}{p}} \\
&\quad \times \sum_{V \in L(\beta^*)} e^{2\pi i \frac{L(j)V^t}{p}} e^{\pi i \frac{VZV^t}{p}} e^{2\pi i \frac{VWY^t}{p}} \\
&= \frac{1}{\sqrt{\Delta(\beta)}} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{y\cdot y}{p}\frac{z^2}{\tau}\right)} \prod_{s=1}^{(p-1)/2} \left(\frac{\tau_s}{i}\right) \\
&\quad \times \sum_{v \in \beta^*} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{j\cdot v}{p}\right)} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{v\cdot v}{p}\tau\right)} e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{v\cdot y}{p} z\right)} \\
&= \frac{1}{\sqrt{\Delta(\beta)}} \prod_{s=1}^{(p-1)/2} \left(\frac{\tau_s}{i}\right) \\
&\quad \times e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{y\cdot y}{p}\frac{z^2}{\tau}\right)} \sum_{\ell=0}^{p-1} e^{2\pi i \frac{j\ell}{p}} \theta_{\ell+\beta,y}(\tau, z). \qquad \square
\end{aligned}
$$

4. Lattices induced by codes over $\mathbb{F}_p$

We first recall the construction of a lattice from a code over $\mathbb{F}_p$.

Let $\rho\colon \mathcal{O}_K^\ell \to \mathbb{F}_p^\ell$ be the natural homomorphism defined by the reduction modulo the principal ideal $\beta = (1-\zeta)$ in each coordinate (see Section 3). Then the lattice induced by a code C over $\mathbb{F}_p$ is defined as

$$\Lambda_C := \frac{1}{\sqrt{p}} \rho^{-1}(C).$$

Let $x, y \in \mathcal{O}_K^\ell$, $x = (x_1, \ldots, x_\ell)$, $y = (y_1, \ldots, y_\ell)$, with $x_i, y_i \in \mathcal{O}_K, 1 \leq i \leq \ell$. The symmetric bilinear form $\langle\, ,\, \rangle_K$ defined in Remark 3.1 induces a symmetric bilinear form on $\frac{1}{\sqrt{p}}\mathcal{O}_K^\ell$ by

$$\langle x\, ,\, y \rangle_K := \sum_{i=1}^{\ell} \operatorname{Tr}_{k/\mathbb{Q}}\left(\frac{x_i \cdot y_i}{p}\right), \quad x_i, y_i \in \mathcal{O}_K. \tag{4.1}$$

The following properties have been derived in [11].

Lemma 4.1. *Let $C \subset \mathbb{F}_p^\ell$ be a code of dimension t with $C \subset C^\perp$. Then the following properties hold:*

1. *If C is self-dual, then $\ell(p-1) \equiv 0 \pmod 8$.*
2. *The lattice Λ_C, endowed with the symmetric bilinear form $\langle x\, ,\, y\rangle_K$ defined in (4.1), is an even integral lattice of rank $\ell(p-1)$ and discriminant $p^{\ell-2t}$. If C is self-dual, then Λ_C is unimodular.*
3. *$\Lambda_C^* = \Lambda_{C^\perp}$.*

Proof. See Lemma 5.5, Proposition 5.2, and Corollary 5.1 in [11]. $\square$

We now study the theta series over the lattice Λ_C induced by the code C over $\mathbb{F}_p$.

LEMMA 4.2. *Let C be a code of length ℓ over $\mathbb{F}_p$ and let Λ_C be a lattice induced by C. Given $Y \in \Lambda_C$, define the theta series $\Theta_{C,Y} : \mathcal{H}^{(p-1)/2} \times \mathbb{C}^{(p-1)/2} \to \mathbb{C}$ by*

$$\Theta_{C,Y}(\tau, z) := \sum_{x \in \Lambda_C} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}((x \cdot x)\tau)} e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}((x \cdot Y)z)},$$

where

$$\operatorname{Tr}_{K/\mathbb{Q}}((x \cdot x)\tau) := 2 \sum_{j=1}^{(p-1)/2} (x\overline{x})^{(j)} \tau_j$$

and

$$\operatorname{Tr}_{K/\mathbb{Q}}((x \cdot Y)Z) := \sum_{j=1}^{(p-1)/2} (x\overline{Y} + \overline{x}Y)^{(j)} z_j.$$

Then

$$\Theta_{C,Y}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{\sqrt{\Delta(\Lambda_C)}} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left((Y \cdot Y)\frac{z^2}{\tau}\right)} \prod_{s=1}^{(p-1)/2} \left(\frac{\tau_s}{i}\right)^{\ell} \times \sum_{v \in \Lambda_C^*} e^{\pi i \operatorname{Tr}_{K/\mathbb{Q}}((v \cdot v)\tau)} e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}((v \cdot Y)z)}.$$

Proof. The functional equation can be established by applying the Poisson summation formula. The argument is similar to the proof of Lemma 3.3, so we omit the details. □

Using the above properties, we now derive the following result.

THEOREM 4.3. *Let C be a self-dual code over $\mathbb{F}_p$ of length ℓ. Let $Y = \frac{1}{\sqrt{p}}(y, y, \ldots, y) \in \Lambda_C$ be such that $Y\overline{Y} \in \mathcal{O}_k$. Then $\Theta_{C,Y}(\tau, z)$ is a Jacobi form over the field k with weight ℓ and index $Y\overline{Y} = \ell y\overline{y}/p$ over the full group $SL_2(\mathcal{O}_k)$.*

Proof. Since $SL_2(\mathcal{O}_k)$ is generated by the transformations $\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right)$, $\gamma \in \mathcal{O}_k$, and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see, for example, [9],[10],[13],[17]), to show the modularity of the Jacobi form it is enough to establish the appropriate transformation formula for these two types of transformations. First, we have

$$\left(\Theta_{C,Y}|_{\ell, Y\overline{Y}} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}\right)(\tau, z) = \Theta_{C,Y}(\tau + \gamma, z) = \Theta_{C,Y}(\tau, z),$$

because $\mathrm{Tr}_{K/\mathbb{Q}}((x\cdot x)\gamma)$ is even for all $x\in\Lambda_C$ and $\gamma\in\mathcal{O}_K$. Second, using Lemma 4.2 and the fact that $\Lambda_C^*=\Lambda_{C^\perp}$, we have

$$\begin{aligned}
&\left(\Theta_{C,Y}|_{\ell,Y\overline{Y}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)(\tau,z)\\
&\quad= e^{-2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(Y\overline{Y}z^2/\tau\right)}\prod_{s=1}^{(p-1)/2}(\tau_s)^{-\ell}\Theta_{C,Y}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)\\
&\quad= i^{-\ell(p-1)/2}\Delta(\Lambda_C)^{-1/2}\Theta_{C^\perp,Y}(\tau,z)=\Theta_{C,Y}(\tau,z),
\end{aligned}$$

since $\ell(p-1)\equiv 0 \pmod 8$ and the self-duality of C implies the unimodularity of Λ_C (see Lemma 4.1, (2)) and, therefore, $\Delta(\Lambda_C)=1$.

Next, we establish the second transformation formula. For any $[\lambda,\mu]\in\mathcal{O}_k^2$,

$$\begin{aligned}
(\Theta_{C,Y}|_{Y\overline{Y}}[\lambda,\mu])(\tau,z) &= e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}\left(\left(\frac{Y\cdot Y}{2}\right)(\lambda^2\tau+2\lambda z)\right)}\Theta_{C,Y}(\tau,z+\lambda\tau+\mu)\\
&= e^{\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((Y\cdot Y)\lambda^2\tau)}e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((Y\cdot Y)\lambda z)}\\
&\qquad\times\sum_{v\in\Lambda_C}e^{\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((v\cdot v)\tau)}e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((v\cdot Y)z)}\\
&\qquad\times e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((v\cdot Y)\lambda\tau)}e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((v\cdot Y)\mu)}\\
&= \sum_{v\in\Lambda_C}e^{\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((v+\lambda Y)\cdot(v+\lambda Y)\tau)}e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}(((v+\lambda Y)\cdot Y)z)}\\
&= \Theta_{C,Y}(\tau,z).
\end{aligned}$$

This follows from the fact that Λ_C is integral lattice, i.e., $\mathrm{Tr}_{k/\mathbb{Q}}(\mu(v\cdot Y))\in\mathbb{Z}$, and that $v+\lambda Y\in\Lambda_C$ for any $v,Y\in\Lambda_C$ and $\mu,\lambda\in\mathcal{O}_k$.

Finally, we show that $\Theta_{C,Y}$ has the required Fourier series expansion:

$$\begin{aligned}
\Theta_{C,Y}(\tau,z) &= \sum_{x\in\Lambda_C}e^{\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((x\cdot x)\tau)}e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}((x\cdot Y)z)}\\
&= \sum_{x\in\Lambda_C}e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}(x\overline{x})\tau)}e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}((x\overline{Y}+\overline{x}Y)z)}\\
&= \sum_{x\in\Lambda_C}q_1^{(x\overline{x})^{(1)}}\cdots q_{(p-1)/2}^{(x\overline{x})^{((p-1)/2)}}\zeta_1^{(x\overline{Y}+\overline{x}Y)^{(1)}}\cdots\zeta_{(p-1)/2}^{(x\overline{Y}+\overline{x}Y)^{((p-1)/2)}}\\
&= \sum_{N,R\in\delta_k^{-1},\,N\geq 0}c(N,R)q_1^{N^{(1)}}\cdots q_{(p-1)/2}^{N^{((p-1)/2)}}\zeta_1^{R^{(1)}}\cdots\zeta_{(p-1)/2}^{R^{(p-1)/2}},
\end{aligned}$$

where $q_s:=e^{2\pi i\tau_s}$, $\zeta_s:=e^{2\pi i z_s}$, $1\leq s\leq(p-1)/2$. The last step follows from the fact that $N^{(j)}:=(x\overline{x})^{(j)}>0$ for all $x\in\Lambda_C\setminus\{0\}$ and for each j, and thus, $N:=x\overline{x}\geq 0$ for all $x\in\Lambda_C$. Also, note that $R:=x\overline{x}$ and $x\overline{Y}+\overline{x}Y\in\mathcal{O}_k^*=\{v\in k\mid\mathrm{Tr}_{k/\mathbb{Q}}(v\mathcal{O}_k)\subset\mathbb{Z}\}=\delta_k^{-1}$. The Fourier coefficient $c(N,R)$ is defined by

$$c(N,R):=\#\left\{x\in\Lambda_C\mid x\overline{x}=N,\quad x\overline{Y}+\overline{x}Y=R\right\}. \qquad\square$$

5. The complete weight enumerators

In this section we exhibit a connection between Jacobi forms over the field $k = \mathbb{Q}(\zeta + \zeta^{-1})$ and the complete weight enumerators of the code C over $\mathbb{F}_p$.

Let Λ_C be the lattice induced by C in the sense of Section 4.

THEOREM 5.1. *Let $C \subset \mathbb{F}_p^\ell$ be a code with $C \subset C^\perp$ and $Y = \frac{1}{\sqrt{p}}(y, \ldots, y) \in \Lambda_C$. Let $\Theta_{C,Y}(\tau, z)$ be the theta series defined in Lemma 4.2. Then the following identity holds:*

$$\Theta_{C,Y}(\tau, z) = \mathrm{CWe}_C\left(\theta_{0,y}(\tau, z), \ldots, \theta_{p-1,y}(\tau, z)\right).$$

Here $\theta_{j,y}(\tau, z)$ is the theta function defined in (3.1).

Proof. Let u be a codeword of C. Setting $Y = \frac{1}{\sqrt{p}}Y_0 = \frac{1}{\sqrt{p}}(y, \ldots, y)$, we have

$$\begin{aligned}
&\sum_{x \in \frac{1}{\sqrt{p}}\rho^{-1}(u)} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}((x\overline{x})\tau)} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}((x\overline{Y} + \overline{x}Y)z)} \\
&= \sum_{x \in \rho^{-1}(u)} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}\left(\left(\frac{x\overline{x}}{p}\right)\tau\right)} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{Y_0} + \overline{x}Y_0}{p} z\right)} \\
&= \left(\sum_{x \in \beta} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{x}}{p}\tau\right)} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{y} + \overline{x}y}{p} z\right)}\right)^{n_0(u)} \cdots \\
&\quad \cdots \left(\sum_{x \in \beta + p - 1} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{x}}{p}\tau\right)} e^{2\pi i \operatorname{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{y} + \overline{x}y}{p} z\right)}\right)^{n_{p-1}(u)} \\
&= \theta_{0,y}(\tau, z)^{n_0(u)} \cdots \theta_{p-1,y}(\tau, z)^{n_{p-1}(u)},
\end{aligned}$$

where $n_j(u)$ denotes the number of components of u that are equal to j. Using the relation $\Lambda_C = \frac{1}{\sqrt{p}}\rho^{-1}(C)$ and summing over all codewords in C yields Theorem 5.1. □

6. The Lee weight enumerators

In this section we derive a connection between a Jacobi form over the field $k = \mathbb{Q}(\zeta + \zeta^{-1})$ and the Lee weight enumerator of a code C over $\mathbb{F}_p$. Using the correspondence between Jacobi forms over k and complete weight enumerators of codes over $\mathbb{F}_p$ obtained in the previous section, we derive the main result of the paper by a particular choice of the value y in the theta series. This is the analogue of the theory given in [11] for functions on $\mathcal{H}^{(p-1)/2} \times \mathbb{C}^{(p-1)/2}$.

THEOREM 6.1. *Let $C \subset \mathbb{F}_p^\ell$ be a code with $C \subset C^\perp$. Let $Y = \frac{1}{p}(y, \ldots, y) \in \Lambda_C$ be such that $\theta_{j,y}(\tau, z) = \theta_{p-j,y}(\tau, z)$ for each $j \in \{0, 1, \ldots, p-1\}$. Then* *the following identity holds:*

$$\Theta_{C,Y}(\tau,z) = W_C\left(\theta_{0,y}(\tau,z),\ldots,\theta_{(p-1)/2,y}(\tau,z)\right).$$

Here $\theta_{j,y}(\tau,z)$ *is defined by* (3.1).

Proof. Using the relation $\theta_{j,y}(\tau,z) = \theta_{p-j,y}(\tau,z)$, the result follows from Theorem 5.1. We omit the details. □

REMARK 6.2. In particular, if one chooses $Y = \frac{1}{\sqrt{p}}(y,y,\ldots,y)$ with $\overline{y} = -y$, then $\theta_{j,y}(\tau,z) = \theta_{p-j,y}(\tau,z)$. To see this, note that

$$\begin{aligned}\theta_{j,y}(\tau,z) &= \sum_{x\in\beta+j} e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{x}}{p}\tau\right)} e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{\overline{x}y+x\overline{y}}{p}z\right)} \\ &= \sum_{x\in\beta+j} e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{(-x)(-\overline{x})}{p}\tau\right)} e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{(-\overline{x})(-y)+(-x)(-\overline{y})}{p}z\right)} \\ &= \sum_{x\in\beta-j} e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{x}}{p}\tau\right)} e^{2\pi i\,\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{x\overline{y}+\overline{x}y}{p}z\right)} \\ &= \theta_{p-j,y}(\tau,z),\end{aligned}$$

since $-\overline{x}\in\beta-j$ and $\overline{y}=-y$.

EXAMPLE 6.3. Let $p=3$, $y=1+2\zeta$, $\zeta=e^{2\pi i/3}$. In this case,

$$\theta_{1,y}(\tau,z) = q^{1/3}\sum_{(a,b)\in\mathbb{Z}^2} q^{a^2+b^2-ab+a+1}\xi^{-a+2b}$$

and

$$\theta_{2,y}(\tau,z) = q^{\frac{4}{3}}\sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+n^2-mn+2m}\xi^{-m+2n}.$$

Replacing m by $a+2$ and n by $b+1$, one can verify that $\theta_{1,y}(\tau,z) = \theta_{2,y}(\tau,z)$.

The theta functions $\Theta_{C,Y}(\tau,0)$, corresponding to the case $z=0$, have been used to obtain a connection between Hilbert modular forms over $SL_2(\mathcal{O}_k)$ and Lee weight enumerators of the code C over $\mathbb{F}_p$, which can be stated as follows (see [11, p. 141]).

COROLLARY 6.4. *Let C be a self-dual code of length ℓ over $\mathbb{F}_p$ and let $Y = \frac{1}{\sqrt{p}}(y,\ldots,y)\in\Lambda_C$. Then $\Theta_{C,Y}(\tau,0)$ is a Hilbert modular form of weight ℓ on the full group $SL_2(\mathcal{O}_k)$, where $k=\mathbb{Q}(\zeta+\zeta^{-1})$, $\zeta=e^{2\pi i/p}$.*

Proof. Note that if $z=(z_j)=0$, then $\theta_{j,y}(\tau,0)=\theta_{p-j,y}(\tau,0)$ for each $j\in\{0,1,\ldots,p-1\}$, since the theta series $\Theta_{C,Y}(\tau,0)$ is independent of Y. In this case, it is known that a Jacobi form over k is simply a Hilbert modular form over $SL_2(\mathcal{O}_k)$. □

7. Invariant spaces, self-dual codes over $\mathbb{F}_p$, and Hilbert-Jacobi forms over $k = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = e^{2\pi i/p}$

In this section we give a homomorphism between a certain invariant space, to which the complete weight enumerators of the self-dual codes over $\mathbb{F}_p$ belong, and a ring of Jacobi forms over the field $k = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = e^{2\pi i/p}$.

The invariant space under the action of the group G is the set $\mathbb{C}[X_1, X_2, \ldots, X_t]^G$ of homogeneous polynomials satisfying $L \cdot f(X) = f(X)$ for all $L \in G$, where $X = (X_0, \ldots, X_t)$. Here, a $t \times t$ matrix L acts on the polynomial ring $\mathbb{C}[X_1, \ldots, X_t]$ by

$$L \cdot f(X_1, \ldots, X_t) = f\left(\sum_{1 \le j \le t} b_{1j} X_j, \ldots, \sum_{1 \le j \le t} b_{\ell j} X_j\right),$$

where $f \in \mathbb{C}[X_1, \ldots, X_t]$ and $B = (b_{ij})$.

We recall the well-known MacWilliams identity for the complete weight enumerators (or Lee weight enumerators) for codes over $\mathbb{F}_p$.

LEMMA 7.1 (MacWilliams Identity). *Let $C \subset \mathbb{F}_p^\ell$ be a code with $C \subset C^\perp$. Then the following identity for the complete weight enumerators of codes over $\mathbb{F}_p$ holds:*

$$\mathrm{CWe}_{C^\perp}(X_0, X_1, \ldots, X_{p-1}) = \mathrm{CWe}_C((X_0, X_1, \ldots, X_{p-1})M_p).$$

Here,

$$M_p := \frac{1}{\sqrt{p}} \begin{pmatrix} 1 & 1 & \ldots & \ldots & \ldots & 1 \\ 1 & \zeta & \zeta^2 & \ldots & \ldots & \zeta^{p-1} \\ 1 & \zeta^2 & \zeta^4 & \ldots & \ldots & \zeta^{2(p-2)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \zeta^p & \zeta^{2p} & \ldots & \ldots & \zeta^{(p-1)^2} \end{pmatrix}.$$

Proof. This is well known; see, for instance, [14, pp. 143–144]. Alternatively, one can derive this identity using the transformation formula of the theta series $\Theta_{C,Y}(\tau, z)$ given in Lemma 4.2 and the connection of this series to the weight enumerators of the code over $\mathbb{F}_p$ given in Theorem 5.1. $\square$

LEMMA 7.2. *Let C be a self-dual code of length ℓ over $\mathbb{F}_p$. Let $G_p(\gamma)$ be a group generated by M_p, $N_p(\gamma)$, for all $\gamma \in \mathcal{O}_k$, where M_p is as in Lemma 7.1 and*

$$N_p(\gamma) := \mathrm{Diagonal}\left(e^{2\pi i j^2 \gamma/p}\right), \quad 0 \le j \le p-1.$$

Then the complete weight enumerators $\mathrm{CWe}_C(X_0, X_1, \ldots, X_{p-1})$ are contained the invariant space

$$\mathbb{C}[X_0, X_1, \ldots, X_{p-1}]^{G_p(\gamma)}.$$

Proof. Let $Y_0 = \frac{1}{\sqrt{p}}(y, y, \ldots, y) \in \mathcal{O}_k$ and consider the theta series $\Theta_{C,Y_0}(\tau, z)$, defined in Lemma 4.2, on the lattice Λ_C induced by the code C. Then for all $\gamma \in \mathcal{O}_k$ we have, using the periodicity of $\Theta_{C,Y_0}(\tau, z)$,

$$\begin{aligned}
&\mathrm{CWe}_C\left(\theta_{0,y}(\tau, z), \ldots, \theta_{p-1,y}(\tau, z)\right)\\
&\qquad = \Theta_{C,Y_0}(\tau, z)\\
&\qquad = \Theta_{C,Y_0}(\tau + \gamma, z)\\
&\qquad = \mathrm{CWe}_C(\theta_{0,y}(\tau + \gamma, z), \ldots, \theta_{p,y}(\tau + \gamma, z))\\
&\qquad = \mathrm{CWe}_C\left(\theta_{0,y}(\tau, z), \ldots, e^{2\pi i \frac{j^2\gamma}{p}}\theta_{j,y}(\tau, z), \ldots, e^{2\pi i \frac{(p-1)^2\gamma}{p}}\theta_{p-1,y}(\tau, z)\right)\\
&\qquad = \mathrm{CWe}_C\left((\theta_{0,y}(\tau, z), \ldots, \theta_{p-1,y}(\tau, z))N_p(\gamma)\right),
\end{aligned}$$

since C is self-dual. From the algebraic independence of the theta series $\theta_{0,y}(\tau, z)$, $\theta_{1,y}(\tau, z), \ldots, \theta_{p-1,y}(\tau, z)$ it follows that

$$N_p(\gamma) \cdot \mathrm{CWe}_C(X_0, \ldots, X_{p-1}) = \mathrm{CWe}_C(X_0, \ldots, X_{p-1}), \quad \gamma \in \mathcal{O}_k.$$

On the other hand, the self-duality of C combined with the MacWilliams identity of Lemma 7.1 imply that

$$M_p \cdot \mathrm{CWe}_C(X_0, \ldots, X_{p-1}) = \mathrm{CWe}_C(X_0, \ldots, X_{p-1}).$$

This proves the theorem. □

More generally, the following theorem shows the existence of a ring homomorphism from the invariant space of $G_p(\gamma)$ to a ring of Jacobi forms over the field $k = \mathbb{Q}(\zeta + \zeta^{-1})$.

THEOREM 7.3. *Let $f(X_0, \ldots, X_{p-1})$ be a homogeneous polynomial of degree ℓ, satisfying $\ell(p-1) \equiv 0 \pmod 8$, in the invariant space $\mathbb{C}[X_0, X_1, \ldots, X_{p-1}]^{G_p(\gamma)}$, for all $\gamma \in \mathcal{O}_k$. Let $y \in \mathcal{O}_K, K = \mathbb{Q}(\zeta)$, be such that $\ell y\overline{y}/p \in \mathcal{O}_k$. Then*

$$f\left(\theta_{0,y}(\tau, z), \theta_{1,y}(\tau, z), \ldots, \ldots, \theta_{p-1,y}(\tau, z)\right)$$

is in the space of Jacobi forms of weight ℓ and index $\ell y\overline{y}/p$ over the field $k = \mathbb{Q}(\zeta + \zeta^{-1})$, via the map

$$f(X_0, \ldots, X_{p-1}) \longrightarrow f\left(\theta_{0,y}(\tau, z), \ldots, \theta_{p-1,y}(\tau, z)\right).$$

Proof. Since $SL_2(\mathcal{O}_k)$ is generated by the transformations $\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right)$, $\gamma \in \mathcal{O}_k$, and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see [9],[10],[13],[17]), the transformation formulas of theta series together with the MacWilliams identity in Lemma 7.1 give the modularity. To obtain the elliptic property, a direct computation shows that, for each $j = 0, \ldots, p-1$,

$$e^{2\pi i \frac{y\overline{y}}{p}(z + \lambda\tau)}\theta_{j,y}(\tau, z + \lambda\tau + \mu) = \theta_{j,y}(\tau, z), \quad (\lambda, \mu) \in \mathbb{Z}^2.$$

This implies that

$$\begin{aligned} &e^{2\pi i\ell\frac{y\overline{y}}{p}(z+\lambda\tau)} f\left(\theta_{0,y}(\tau, z+\lambda\tau+\mu), \ldots, \theta_{p-1,y}(\tau, z+\lambda\tau+\mu)\right) \\ &\qquad = f\left(\theta_{0,y}(\tau,z), \ldots, \theta_{p-1,y}(\tau,z)\right). \end{aligned}$$

Finally, the Fourier expansion can again be checked directly, using the expansions of $\theta_{j,y}(\tau,z)$. □

Next, as a special case of the above theorem, we have the following relation.

LEMMA 7.4 (MacWilliams Identity). *Let $C \subset \mathbb{F}_p^\ell$ be a code with $C \subset C^\perp$. Let $y \in \mathcal{O}_K$, $K = \mathbb{Q}(\zeta)$, be such that $\theta_{j,y}(\tau,z) = \theta_{p-j,y}(\tau,z)$, for each $j \in \{1, 2, \ldots, (p-1)/2\}$. Then the following identity for the Lee weight enumerators of codes over $\mathbb{F}_p$ holds:*

$$W_{C^\perp}\left(X_0, X_1, \ldots, X_{(p-1)/2}\right) = \mathrm{CWe}_C\left(\left(X_0, X_1, \ldots, X_{(p-1)/2}\right) A_p\right)$$

Here,

$$A_p := \frac{1}{\sqrt{p}} \begin{pmatrix} 1 & 2 & \ldots & \ldots & \ldots & 2 \\ 1 & \zeta + \zeta^{p-1} & \zeta^2 + \zeta^{p-2} & \ldots & \ldots & \zeta^{\frac{p-1}{2}} + \zeta^{\frac{p+1}{2}} \\ 1 & \zeta^2 + \zeta^{2(p-2)} & \ldots & \ldots & \ldots & \zeta^{2\frac{p-1}{2}} + \zeta^{2\frac{p+1}{2}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \zeta^{p-1} + \zeta^{(p-1)^2} & \ldots & \ldots & \ldots & \zeta^{(p-1)\frac{p-1}{2}} + \zeta^{(p-1)\frac{p+1}{2}} \end{pmatrix}.$$

Proof. This is well known; see, for instance, [14, pp. 145–146]. Alternatively, one can derive the result using the relation $\theta_{j,y}(\tau,z) = \theta_{p-j,y}(\tau,z)$ and Lemma 7.1. □

LEMMA 7.5. *Let C be a self-dual code of length ℓ over $\mathbb{F}_p$. Let $H_p(\gamma)$ be a group generated by $A_p, B_p(\gamma)$, $\gamma \in \mathcal{O}_k$, where A_p is as in Lemma 7.4 and*

$$B_p(\gamma) := \mathrm{Diagonal}\left(e^{2\pi i j^2\gamma/p}\right), \quad 1 \le j \le \frac{p-1}{2}.$$

Then the Lee weight enumerator $W_C\left(X_0, X_1, \ldots, X_{(p-1)/2}\right)$ is in the invariant space

$$\mathbb{C}\left[X_0, X_1, \ldots, X_{(p-1)/2}\right]^{H_p(\gamma)}.$$

Proof. This is an immediate consequence of Lemma 7.2. □

More generally, the following corollary shows the existence of a ring homomorphism from an invariant space of $H_p(\gamma)$ to a ring of Jacobi forms over k.

COROLLARY 7.6. *Let $g(X_0, \ldots, X_{(p-1)/2})$ be a homogeneous polynomial of degree ℓ, satisfying $\ell(p-1) \equiv 0 \pmod 8$, in the invariant space $\mathbb{C}[X_0, X_1, \ldots, X_{(p-1)/2}]^{H_p(\gamma)}$, for all $\gamma \in \mathcal{O}_k$. Let $y \in \mathcal{O}_K$, $K = \mathbb{Q}(\zeta)$, be such that $\ell y\overline{y} \in \mathcal{O}_k$ and $\theta_{j,y}(\tau,z) = \theta_{p-j,y}(\tau,z)$, for each $j \in \{1, 2, \ldots, (p-1)/2\}$. Then*

$$g(\theta_{0,y}(\tau,z), \theta_{1,y}(\tau,z), \ldots, \theta_{(p-1)/2,y}(\tau,z))$$

is in the space of Jacobi forms of weight ℓ *over* $k = \mathbb{Q}(\zeta + \zeta^{-1})$*, via the map*

$$g(X_0, \ldots, X_{p-1}) \longrightarrow g\left(\theta_{0,y}(\tau, z), \ldots, \theta_{(p-1)/2,y}(\tau, z)\right).$$

Proof. This is an immediate consequence of Theorem 7.3. □

8. Conclusion

This is the first attempt at obtaining a connection between Jacobi forms over a totally real number field $k = \mathbb{Q}(\zeta + \zeta^{-1})$, $\zeta = e^{2\pi i/p}$, and codes defined over $\mathbb{F}_p$. Our results generalize those given in [11], which provide a connection between the codes over $\mathbb{F}_p$ and Hilbert modular forms. Using results from coding theory, one may obtain number theoretic information such as interesting identities among theta series. We hope to continue this investigation in the near future.

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