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HULLS OF CLOSED PRIME IDEALS IN H^∞

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Dedicated to the memory of Carroll Guillory

ABSTRACT. We show that hulls of closed prime ideals in H^∞ are "small": they are contained in the k-hulls of single points. Combined with our notion of p-parts, this will prove Alling's conjecture for a large class of prime ideals.

Let H^∞ be the uniform algebra of all bounded analytic functions in the open unit disk $\mathbb{D}$. Its spectrum, or maximal ideal space, is the space $M(H^\infty)$ of all nonzero multiplicative linear functionals on H^∞ endowed with the weak-$*$ topology. It is a compact, connected Hausdorff space. As usual, we identify a function f in H^∞ with its Gelfand transform, $\hat{f}$, defined by $\hat{f}(m) = m(f)$ for $m \in M(H^\infty)$. Moreover, by considering point evaluations, we may look upon $\mathbb{D}$ as a subset of $M(H^\infty)$. By the Corona Theorem (see [Ga]), $M(H^\infty)$ can then be viewed as a compactification of the unit disk: it is the smallest compact T_2-space for which every bounded analytic function on $\mathbb{D}$ has a continuous extension. Intimately related to H^∞ is the uniform algebra $H^\infty + C$ of sums of boundary values of H^∞ functions and continuous complex-valued functions on the circle $T = \{z \in \mathbb{C} : |z| = 1\}$. Its maximal ideal space is exactly the corona $M(H^\infty) \setminus \mathbb{D}$ (see [Ga, p. 377]). Finally we recall that the Shilov boundary of a uniform algebra A is the smallest closed subset of its spectrum $M(A)$ such that (the modulus of) every function $f \in A$ takes its maximum there.

It is well known that the Shilov boundary of H^∞ can be identified with the spectrum, $M(L^\infty)$, of L^∞. Hoffman [Ho1, p. 184] showed that each $m \in M(H^\infty)$ has a unique norm preserving extension to a linear functional on L^∞. Letting $\operatorname{supp} m$ in $M(L^\infty)$ denote the support set of the representing measure

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μ_m for m, one can show (see [Ga, p. 375]) that this extension is given by

$$m(f) = \int_{\operatorname{supp} m} f \, d\mu_m \quad (f \in L^\infty).$$

It follows that each function $f \in L^\infty$ can be thought of as a continuous function on $M(H^\infty)$. This point of view will be adopted throughout this paper and we write $f(m) := m(f)$. We note that this extension to $M(H^\infty)$ of $f \in L^\infty$ coincides on $\mathbb{D}$ with the Poisson integral of f.

In this paper we are interested in the structure of the closed prime ideals in H^∞ and $H^\infty + C$. Of course every maximal ideal is prime and closed. In order to come up with examples of non-maximal closed prime ideals, we need to know something about the analytic structure of $M(H^\infty)$.

For two points x, m in $M(H^\infty)$, we define the pseudohyperbolic distance of x to m by

$$\rho(x,m) = \sup\{|f(m)| : f \in H^\infty, ||f||_\infty \leq 1, f(x) = 0\}.$$

It is well-known that the relation defined on $M(H^\infty)$ by

$$x \sim m \qquad \Longleftrightarrow \qquad \rho(x,m) < 1$$

defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point m is called the Gleason part of m and is denoted by $P(m)$. If the part, $P(m)$, consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. Hoffman's theory [Ho2] shows that for every Gleason part $P(m)$ there is a continuous map L_m of $\mathbb{D}$ onto $P(m)$ with $L_m(0) = m$ such that $f \circ L_m$ is analytic on $\mathbb{D}$ for all $f \in H^\infty$. If (z_α) is a net in $\mathbb{D}$ converging to m, then L_m is given by $L_m(z) = \lim_\alpha (z + z_\alpha)/(1 + \overline{z}_\alpha z)$. One can also view the Gleason parts in $M(A)$, $A = H^\infty$ or $H^\infty + C$, as the connected components of $M(A)$, when endowed with the operator norm topology.

When the Gleason part of m is trivial, L_m is just a constant map. When $P(m)$ is nontrivial, the map L_m is a bijection. The set of all nontrivial points in $M(H^\infty)$ is denoted by G, and the set of all trivial points is denoted by Γ. Since for every $f \in H^\infty + C$ one has $f \circ L_m \in H^\infty$, when $f(m) = 0$ it makes sense to talk about the order of the zero of f at m. For $m \in M(H^\infty + C)$ and $f \in H^\infty + C$ with $f(m) = 0$ we let

$$\operatorname{ord}(f,m) := \sup\left\{ j \in \mathbb{N} : (f \circ L_m)^{(k-1)}(0) = 0 \text{ for } k = 1, 2, \ldots, j \right\}.$$

If $f(m) \neq 0$, we say $\operatorname{ord}(f,m) = 0$. If I is an ideal in H^∞ or $H^\infty + C$, we let $\operatorname{ord}(I,m) = \min\{\operatorname{ord}(f,m) : f \in I\}$. Hoffman showed that if $\operatorname{ord}(f,m) = \infty$ for some $m \in M(H^\infty + C)$, then $f \in H^\infty + C$ vanishes identically on the part $P(m)$ (see [Ho2, p. 79 and p. 101])

An ideal I in H^∞ is said to have inner factor 1, if the weak-$*$ closure of I in H^∞ coincides with H^∞. This is equivalent to the fact that the greatest common divisor of the inner factors of the functions in I is a unimodular constant (see [Ga, p. 83–84]). It is easy to see that every nonmaximal prime ideal in H^∞, different from zero, has inner factor 1.

Let A be a uniform algebra. Then $Z_A(f) = \{m \in M(A) : f(m) = 0\}$ denotes the zero set of a function $f \in A$ and $Z_A(I) = \bigcap_{f \in I} Z_A(f)$ the zero set (or hull) of an ideal I in A. When no ambiguity arises, we write $Z(f)$ for $Z_{H^\infty+C}(f)$ whenever $f \in H^\infty + C$. The zero set of infinite order of a function $f \in H^\infty + C$ is written as $Z^\infty(f)$.

Finally, (weak-$*$) closures of sets E in $M(H^\infty)$ will be denoted by $\overline{E}$. The set of interior points with respect to $M(H^\infty + C)$ of E is given by E^0. Moreover, we write $\{|f| < 1\}$ for sets of the form $\{x \in M(H^\infty + C) : |f(x)| < 1\}$, $f \in H^\infty + C$.

We can now proceed with an "explicit" example of a non-maximal closed prime ideal in H^∞, which was given by N. Alling [A] in 1970:

Let $m \in G$, $m \notin \mathbb{D}$. Then the set

$$I = \{f \in H^\infty : f \text{ vanishes identically on the part } P(m)\}$$

is a nonmaximal closed prime ideal in H^∞.

The hull of I is the closure of the Gleason part $P(m)$ in $M(H^\infty)$, as was shown by Gorkin [Go] in 1988. In [GM2], we proved that every closed prime ideal P in H^∞ is uniquely determined by its hull; that is, each such P has the form

$$P = I(E, H^\infty) := \{f \in H^\infty : f \text{ vanishes identically on } E\}$$

for some closed set $E \subseteq M(H^\infty)$ (which can be chosen to be $Z_{H^\infty}(P)$).

Alling's conjecture now reduces to the following question: Let P be a nonmaximal closed prime ideal in H^∞. Is $Z_{H^\infty}(P) = \overline{P(m)}$ for some $m \in G$?

In Theorem 1.1 we shall first prove that there is a one-to-one correspondence between nonzero, nonmaximal closed prime ideals in H^∞ and $H^\infty + C$. Thus we have a result in the same spirit as that of [GHM], where it was shown that every closed ideal in H^∞ with inner factor 1 can be lifted to a unique closed ideal in $H^\infty + C$. (Note, however, that the H^∞-trace (i.e., the intersection with H^∞) of an *arbitrary* closed ideal in $H^\infty + C$ may very well be $\{0\}$.)

The real importance of Theorem 1.1 now will be that, due to the more flexible behaviour of functions in $H^\infty + C$, we actually can prove results more easily in the setting of $H^\infty + C$ functions, than in the quite narrow class of bounded analytic functions.

After some results on the zero sets of Blaschke products in $M(H^\infty + C)$, we finally prove the main result on the structure of the hull of closed prime ideals. This result, and its proof, uses an important class of subsets of $M(H^\infty + C)$, connected to the spectral synthesis problem: the so-called k-hulls (see p. 526 for the definition).

In the final section we then introduce the concept of p-parts, reminiscent of the well-known notion of p-points in topology (see [GJ]). These p-parts allow us to confirm Alling's conjecture for quite a wide class of prime ideals. For previous results on closed prime ideals and/or related material we refer the reader to [Mo1], [Mo2], [GM2] and [Su].

1. Closed prime ideals

Our first theorem shows that there exists a one-to-one correspondence between nonmaximal closed prime ideals, different from zero, in H^∞ and $H^\infty + C$. Note that the zero ideal is prime in H^∞ but not in $H^\infty + C$.

THEOREM 1.1.

(1) *Let P be a closed prime ideal in $H^\infty + C$. Then $P \cap H^\infty$ is a nonzero closed prime ideal in H^∞. Moreover P is maximal (for $H^\infty + C$) if and only if $P \cap H^\infty$ is maximal (for H^∞).*

(2) *Let Q be a nonzero, closed prime ideal in H^∞, not of the form*

$$M(z_0) = \{f \in H^\infty : f(z_0) = 0\}$$

for some $z_0 \in \mathbb{D}$. Then $P := \overline{Q(H^\infty + C)}$, the closed ideal generated by Q in $H^\infty + C$, is a closed prime ideal in $H^\infty + C$ satisfying

$$\overline{Q(H^\infty + C)} \cap H^\infty = Q.$$

(3) *We have $Z_{H^\infty + C}(P) = Z_{H^\infty}(P \cap H^\infty)$ whenever P is a closed prime ideal in $H^\infty + C$.*

Proof. (1) Let $QC = \{f \in H^\infty + C : \overline{f} \in H^\infty + C\}$ denote the largest C^*-subalgebra contained in $H^\infty + C$. Note that QC can be identified with $C(M(QC))$. It is easy to see that if P is a closed prime ideal in $H^\infty + C$, then $P \cap QC$ is a closed prime ideal in QC. Hence, by Shilov, $P \cap QC$ is a maximal ideal. In particular, we see that the hull of P is entirely contained in a single QC-level set $E_x = \{m \in M(H^\infty + C) : f(m) = f(x) \text{ for every } f \in QC\}$, where $x \in M(QC)$. Let $\lambda = \mathrm{id}(x)$, where $\mathrm{id}(z) = z$. Then $\lambda - z \in P \cap QC \subseteq P$. So $P \cap H^\infty$ is not the zero ideal. Obviously $P \cap H^\infty$ is closed and prime.

The second assertion follows from the fact that the maximal ideal space of $H^\infty + C$ can be identified with $M(H^\infty) \setminus \mathbb{D}$.

(2) By Theorem 3.8 in [GHM], we see that $\overline{Q(H^\infty + C)} \cap H^\infty = Q$. Next, we show that $P := \overline{Q(H^\infty + C)}$ is a prime ideal in $H^\infty + C$. Let $f, g \in H^\infty + C$ satisfy $fg \in P$. By [Mo1] we may assume without loss of generality that the hull of Q is contained in a single fiber, say $Z_{H^\infty}(Q) \subseteq M_1$, where $M_1 = \{m \in M(H^\infty) : \mathrm{id}(m) = 1\}$. Since functions in $H^\infty + C$ coincide on fibers with bounded analytic functions, we choose $F, G \in H^\infty$ so that $F|_{M_1} = f|_{M_1}$ and $G|_{M_1} = g|_{M_1}$. By combining [He] (see also [Mo1, p. 215]) and [GHM, p. 637–638], any closed ideal in $H^\infty + C$ whose hull is contained in a single fiber

contains all bounded analytic functions vanishing identically on that fiber. Hence for every $q \in H^\infty + C$ with $q|_{M_1} \equiv 0$, one has $(1 - ((1+z)/2)^n)\, q \in P$, and $\lim_n (1 - ((1+z)/2)^n)\, q = q$ uniformly on $M(H^\infty + C)$. Since P is closed, we get that $q \in P$. In particular, $q := FG - fg \in P$. Hence $FG \in P \cap H^\infty = Q$. Since Q is prime, $F \in Q \subseteq P$ or $G \in Q \subseteq P$. Thus, by the same argument as above, $f \in P$ or $g \in P$. Therefore P is prime.

(3) Let $Q = P \cap H^\infty$. Since by (1) and [GHM] we have $P = \overline{Q(H^\infty + C)}$, we immediately get that $Z_{H^\infty + C}(P) = Z_{H^\infty}(Q)$. □

Using, among other results, the fact, proven in [GM2], that any closed prime ideal in H^∞ is an intersection of maximal ideals, we get the following result.

THEOREM 1.2. *Let P be a closed prime ideal in $A = H^\infty$ or $H^\infty + C$. Then P is an intersection of maximal ideals. The ideal P is maximal if the hull of P is contained either in the set of nontrivial points or in the set of trivial points. If the hull of P meets the Shilov boundary, then P is maximal, too. If P is a nonmaximal closed prime ideal, then $Z_A(P)$ is a union of closures of Gleason parts.*

Proof. The result follows from Theorem 1.1 and the corresponding results for H^∞ (see [Mo1] and [GM2]). □

The following important lemma has been proven independently by K. Izuchi [Iz4] and the first author of this paper (see [GM4]). Recall that for a sequence (k_n) of positive integers, a weak power B of a Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z}$$

is given by

$$B(z) = \prod_{n=1}^{\infty} \Big(\frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z} \Big)^{k_n}.$$

LEMMA 1.3. *Let b be a Blaschke product and (K_n) a sequence of closed subsets of $M(H^\infty + C)$ such that $\overline{\{|b| < 1\}} \cap K_n = \emptyset$ for every $n \in \mathbb{N}$. Then there exists a weak power B of b vanishing identically on $\overline{\{|b| < 1\}}$ such that $|B| = 1$ on $\cup_n K_n$.*

PROPOSITION 1.4. *Let B_j, $j = 1, 2, \dots, N$, be Blaschke products and $x \in \bigcap_{j=1}^N Z(B_j)^0$. Then there exists an interpolating Blaschke product b so that $\overline{\{|b| < 1\}} \subseteq \bigcap_{j=1}^N Z(B_j)^0$ and $|b(x)| < 1$.*

Proof. Let U be an open set in $M(H^\infty)$ so that $x \in U$ and

$$\overline{U} \cap M(H^\infty + C) \subseteq \bigcap_{j=1}^{N} Z(B_j)^0.$$

If x is a trivial point—necessarily outside the Shilov boundary—then we choose according to [GM1] a nontrivial point $m \in U$ satisfying $\operatorname{supp} m \subseteq \operatorname{supp} x$. Take an interpolating Blaschke product b with $b(m) = 0$ satisfying $Z(b) \subseteq U$. Obviously $|b(x)| < 1$. Since every zero of b in $M(H^\infty + C)$ is a zero of infinite order of B_j (note that $Z(B_j)^0 \subseteq Z^\infty(B_j)$), we may apply the result in [AG] and [GIS] to conclude that for every $j \in \{1, \dots, N\}$, the Blaschke product B_j is divisible in $H^\infty + C$ by all powers of b and hence $\{|b| < 1\} \subseteq Z(B_j)^0$. But actually more holds. In fact, let $\{|B_j| > 0\} = \bigcup_{n\in\mathbb{N}}\{|B_j| \geq 1/n\}$ and $K_n^j = \{|B_j| \geq 1/n\}$. Since $\overline{\{|b| < 1\}} \subseteq Z(B_j)$, we see that for every n and $j \in \{1, \dots, N\}$ we have $K_n^j \cap \overline{\{|b| < 1\}} = \emptyset$. Hence, by Lemma 1.3, there exists a weak power b^* of b so that $|b^*| = 1$ on $\bigcup_{j=1}^N \bigcup_n K_n^j$. Thus, for every $j \in \{1, \dots, N\}$, $|b^*| = 1$ on $\overline{\{|B_j| > 0\}}$, which is the complement of $Z(B_j)^0$. Now b^* vanishes identically on $\overline{\{|b| < 1\}}$. Hence $\overline{\{|b| < 1\}} \subseteq \bigcap_{j=1}^N Z(B_j)^0$. If $x \in G$ we simply let in the above proof $m = x$. □

Remark. The proof shows that if $x \in G$, then b can be taken so that, additionally, $b(x) = 0$.

By a result of Izuchi [Iz2, p. 57] we know that the zero set of infinite order of a Blaschke product is a G_δ set. Here we obtain additional information.

Theorem 1.5. *Let B be a Blaschke product. Then the following assertions hold:*

(1) $\overline{Z(B)^0} = Z^\infty(B)$.
(2) $Z(B)^0$ *is a union of closures of Gleason parts.*

Proof. (1) First we note that, due to the analytic structure of the Gleason parts in $H^\infty + C$, the inclusion $Z(B)^0 \subseteq Z^\infty(B)$ is clear. To prove the denseness, let $x \in Z^\infty(B)$ and let U be a neighborhood of x in $M(H^\infty + C)$. Choose an open set V in $M(H^\infty)$ so that $x \in V$ and such that $M(H^\infty + C) \cap \overline{V} \subseteq U$. By the Corona Theorem there exists a net (z_α) in $V \cap \mathbb{D}$ converging to x. Actually (z_a) may be chosen to be a subnet of a certain *sequence* (z_n) in $V \cap \mathbb{D}$. By Hoffman [Ho2], $f \circ L_{z_\alpha} \to f \circ L_x$ uniformly on $\{z \in \mathbb{D} : |z| \leq r\}$ for every $0 < r < 1$ and every $f \in H^\infty$. Choose an interpolating subsequence $(z_{n_k})_{k\in\mathbb{N}}$ of (z_n) so that $\sup_{|z|\leq r} |B \circ L_{z_{n_k}}| \to 0$ if $k \to \infty$. Let b be the interpolating Blaschke product associated with $\{z_{n_k} : k \in \mathbb{N}\}$. Then $Z(b) \subseteq Z^\infty(B)$. By [AG] or [GIS] every power of b divides B in $H^\infty + C$. Thus $B \equiv 0$ on $\{|b| < 1\}$. In particular, $Z(b) \subseteq Z(B)^0$. Noticing that $Z(b) \subseteq M(H^\infty + C) \cap \overline{V} \subseteq U$, gives the result that $Z(B)^0$ is dense in $Z^\infty(B)$.

(2) Let $x \in Z(B)^0$. We show that $\overline{P(x)} \subseteq Z(B)^0$. By Proposition 1.4 there exists an interpolating Blaschke product b so that $|b(x)| < 1$ and $\overline{\{|b| < 1\}} \subseteq Z(B)^0$. But $\overline{P(x)} \subseteq \overline{\{|b| < 1\}}$. This yields the desired conclusion. $\square$

Remark. An analysis of the proof shows that the result also holds for arbitrary functions in $H^\infty + C$. See [GMS] for further generalizations to Douglas algebras.

For a subset S in $M(L^\infty)$, we denote its complement $M(L^\infty) \setminus S$ by S^c. The characteristic function associated with a clopen (that is, closed and open) subset S of $M(L^\infty)$ is given by χ_S. Obviously, $\chi_S \in L^\infty$. As explained in the introduction, we will look upon χ_S as a continuous function defined on the whole maximal ideal space of H^∞.

Lemma 1.6 ([GM3, Lemma 2.2]). *Let $f \in H^\infty + C$ and let E be a clopen subset of $M(L^\infty)$. Then*

$$f\chi_{E^c} \in H^\infty + C \iff f \equiv 0 \text{ on } \{0 < \chi_E < 1\}.$$

Moreover, if $S(E) = \{\varphi \in M(H^\infty + C) : \operatorname{supp} \varphi \subseteq E\}$, then both statements imply that

$$Z(f\chi_{E^c}) = S(E) \cup \{0 < \chi_E < 1\} \cup \big(Z(f) \cap S(E^c)\big),$$

with an analogous formula if Z is replaced by Z_∞. In particular, $Z(f) \subseteq Z(f\chi_{E^c})$ and $Z_\infty(f) \subseteq Z_\infty(f\chi_{E^c})$.

We proceed with another lemma, due to D. Sarason. Since we could not locate its proof in the literature, we will present it for the reader's convenience.

Lemma 1.7. *Let u be an inner function and let $x \in M(H^\infty + C) \setminus M(L^\infty)$. Then either u is a unimodular constant on $\operatorname{supp} x$ or $u(\operatorname{supp} x) = \partial\mathbb{D}$.*

Proof. Suppose that $u|_{\operatorname{supp} x}$ is not constant. Without loss of generality let $1 \in u(\operatorname{supp} x)$.

Case 1: $u(x) = 0$. Assume that $u(\operatorname{supp} x) \neq \partial\mathbb{D}$. Since $u(\operatorname{supp} x)$ is compact, there exists an arc (α, β) on the circle $\partial\mathbb{D}$ which is disjoint from $u(\operatorname{supp} x)$. Choose $n \in \mathbb{N}$ so that the image of $\partial\mathbb{D} \setminus (\alpha, \beta)$ with respect to an appropriate branch of $z^{1/n}$ is contained within the arc $(-\pi/4, \pi/4)$. By Runge's approximation theorem, there exists a sequence of polynomials p_j converging uniformly to $z^{1/n-1}$ on $\partial\mathbb{D} \setminus (\alpha, \beta)$. Hence $zp_j(z)$ converges uniformly to $z^{1/n}$ on $\partial\mathbb{D} \setminus (\alpha, \beta)$. This implies that $\operatorname{Re}(zp_j) \to \operatorname{Re} z^{1/n}$. However, $\operatorname{Re} z^{1/n} \geq \delta > 0$ on $\partial\mathbb{D} \setminus (\alpha, \beta)$. Thus there exists a polynomial p with $p(0) = 0$ such that $\operatorname{Re} p > 0$ on $\partial\mathbb{D} \setminus (\alpha, \beta) \supseteq u(\operatorname{supp} x)$. This implies that

$$0 < \int\limits_{\operatorname{supp} x} \operatorname{Re}(p \circ u)\, d\mu_x = \operatorname{Re} \int\limits_{\operatorname{supp} x} (p \circ u)\, d\mu_x = \operatorname{Re}(p \circ u)(x) = p(0) = 0,$$

an obvious contradiction. This shows that $u(\operatorname{supp} x) = \partial\mathbb{D}$.

Case 2: $u(x) \neq 0$. Since u is not constant on $\operatorname{supp} x$, we have that $|u(x)| < 1$. Let $v = \big(u - u(x)\big)/\big(1 - \overline{u(x)}u\big)$. Then v is an inner function with $v(x) = 0$. Case 1 applied to v now yields the result that $v(\operatorname{supp} x) = \partial\mathbb{D}$ and therefore $u(\operatorname{supp} x) = \partial\mathbb{D}$, too. $\square$

We will now recall the concept of k-hulls, which we introduced in [GM4] to study spectral synthesis problems in the spectrum of $H^\infty + C$. Let A be a uniform algebra and let $E \subseteq M(A)$ be closed. Define the ideals

$$I(E, A) = \{f \in A : f \text{ vanishes identically on } E\},$$

and

$$J(E, A) = \{f \in A : f \text{ vanishes identically in a neighborhood of } E\}.$$

If the context is clear, we simply write $I(E)$ and $J(E)$. As usual, we define the hull $h(E)$ of E to be the hull kernel closure of E in $M(A)$, that is, $h(E) = Z_A(I(E))$. If $E = h(E)$, we say that E is a hull. Finally, we let $k(E) = Z_A(J(E))$; this is the k-hull of E.

In H^∞, we obviously have that $J(E) = \{0\}$, so that is an uninteresting case. But in $H^\infty + C$, there are a lot of functions vanishing on open sets in the spectrum. For example if $E \subseteq M(H^\infty + C) \setminus M(L^\infty)$, then $k(E)$ does not meet the Shilov boundary, either. Moreover we know that $k(k(E)) = k(E)$ for any closed set $E \subseteq M(H^\infty + C)$. See [GM4] for a detailed exposition. In general, $k(E)$ is strictly bigger than E. It is a union of closures of Gleason parts (see [GM4]). If $E \subseteq M(H^\infty + C) \setminus M(L^\infty)$, then the k-hull, $k(E)$, of E can be represented as

$$k(E) = \bigcap_{c \in J(E)} Z(c) = \bigcap_{b \in I(E)} \overline{\{|b| < 1\}}, \tag{1}$$

where b and c are Blaschke products (see [GM4]).

In the case we are interested in here, $E = \{x\}$ is a singleton. We know that $\overline{P(x)} \subseteq k(x)$. In general this inclusion is strict (see [GM4]). In the second section, we present examples of nontrivial points x, for which $k(x) = \overline{P(x)}$.

The following lemma is implicitly contained in [GM4].

Lemma 1.8. *Let $x \in M(H^\infty + C) \setminus M(L^\infty)$ and let $y \in M(H^\infty + C) \setminus k(x)$. Then there exists a Blaschke product B vanishing in a neighborhood of x such that $y \notin \overline{\{|B| < 1\}}$.*

Proof. By [GM4], $k(x) = \bigcap \overline{\{|b| < 1\}}$, where the intersection is taken over all Blaschke products b vanishing at x. Hence there exists b vanishing at x such that $y \notin \overline{\{|b| < 1\}}$. Let K be a compact neighborhood of y disjoint from $\overline{\{|b| < 1\}}$. Now apply Lemma 1.3 for $K_n = K$ to get a weak power B of b which vanishes in a neighborhood of x but for which $y \notin \overline{\{|B| < 1\}}$. $\square$

We are now ready to prove our main result.

Theorem 1.9. *Let A denote either the algebra H^∞ or $H^\infty + C$ and let P be a nonzero, nonmaximal closed prime ideal in A. Suppose that $x \in Z(P)$. Then $Z(P) \subseteq k(x)$.*

Proof. In view of Theorem 1.1 it is sufficient to prove the result for closed prime ideals in $H^\infty + C$. This has the advantage that we can use $H^\infty + C$ techniques which are more flexible than those for H^∞.

So let P be a closed prime ideal in $H^\infty + C$. Using the nonmaximality of P and Theorem 1.2, we may assume that $x \in M(H^\infty + C) \setminus M(L^\infty)$. Let $y \in M(H^\infty + C) \setminus k(x)$. We are going to prove that $y \notin Z(P)$.

By Lemma 1.8 there exists a Blaschke product B vanishing identically on a neighborhood of x such that

$$y \notin \overline{\{|B| < 1\}}. \tag{2}$$

Choose according to Proposition 1.4 an interpolating Blaschke product b such that $|b(x)| < 1$ and

$$\overline{\{|b| < 1\}} \subseteq Z(B)^0. \tag{3}$$

Without loss of generality $b(y) = 1$. Consider for every $n \in \mathbb{N}$ the sets

$$E_n = \{\eta \in M(L^\infty) : |b(\eta) - 1| > 1/n\}.$$

Since $M(L^\infty)$ is extremely disconnected, the closures $\overline{E_n}$ of the sets E_n are clopen subsets of $M(L^\infty)$. Let $\chi_{\overline{E}_n}$ be the associated characteristic functions. Let $e^{i\theta_{1,n}}$ and $e^{i\theta_{2,n}}$ be the two intersection points of the unit circle with the circle $\{z \in \mathbb{C} : |z - 1| = 1/n\}$. We claim that the functions

$$(b - e^{i\theta_{1,n}})(b - e^{i\theta_{2,n}})B\chi_{\overline{E}_n} \quad \text{and} \quad (b - e^{i\theta_{1,n}})(b - e^{i\theta_{2,n}})B\chi_{\overline{E}_n^c}$$

are in $H^\infty + C$. To see this it suffices to show, by Lemma 1.6, that whenever $m \in M(H^\infty + C)$ is such that

$$\operatorname{supp} m \cap \overline{E_n} \neq \emptyset \text{ and } \operatorname{supp} m \cap \overline{E_n}^c \neq \emptyset, \tag{4}$$

then $f_n := (b - e^{i\theta_{1,n}})(b - e^{i\theta_{2,n}})B$ vanishes at m. So, assume that (4) holds. If $\operatorname{supp} m \cap E_n \neq \emptyset$ and $\operatorname{supp} m \cap \{\eta \in M(L^\infty) : |b(\eta) - 1| < 1/n\} \neq \emptyset$, then $b|_{\operatorname{supp} m}$ cannot be constant. Hence $|b(m)| < 1$. Thus, by (3), $B(m) = 0$. If, on the other hand, either of the sets $\operatorname{supp} m \cap E_n$ or $\operatorname{supp} m \cap \{\eta \in M(L^\infty) : |b(\eta) - 1| < 1/n\}$ is empty, then, by Sarason's Lemma 1.7, $b|_{\operatorname{supp} m}$ must be constant. Since b is unimodular on $M(L^\infty)$, we deduce from (4) that this constant is either $e^{i\theta_{1,n}}$ or $e^{i\theta_{2,n}}$. Thus $b(m) = e^{i\theta_{1,n}}$ or $b(m) = e^{i\theta_{2,n}}$. In any case, we obtain that $f_n(m) = 0$. This proves our claim that $f_n\chi_{\overline{E}_n} \in H^\infty + C$ and $f_n\chi_{\overline{E}_n^c} \in H^\infty + C$.

We obviously have $0 = (f_n\chi_{\overline{E}_n})(f_n\chi_{\overline{E}_n^c}) \in P$. Since P is prime, we have the following, not necessarily disjoint, alternatives:

$$f_n\chi_{\overline{E}_n^c} \in P \text{ for some } n \text{ or } f_n\chi_{\overline{E}_n} \in P \text{ for every } n.$$

So suppose that for some n, $f_n\chi_{\overline{E}_n^c} \in P$. Since $b(y) = 1$ implies that $\operatorname{supp} y \subseteq \overline{E_n}^c$, we deduce that $\chi_{\overline{E}_n^c}(y) = 1$. Because $e^{i\theta_{1,n}}$ and $e^{i\theta_{2,n}}$ are different from 1, we obtain from (2) that $f_n\chi_{\overline{E}_n^c}(y) \neq 0$. Hence $y \notin Z(P)$.

Now suppose that $f_n\chi_{\overline{E}_n} \in P$ for every n. Choose any $\psi \in M(H^\infty + C) \setminus Z(B)$ satisfying $b(\psi) \neq 1$. By (3), we have $|b(\psi)| = 1$. Since for $n \to \infty$ we have $e^{i\theta_{1,n}} \to 1$ and $e^{i\theta_{2,n}} \to 1$, there exists n_0, depending on ψ, so that for $n \geq n_0$, $b(\psi) \notin (e^{i\theta_{1,n}}, e^{i\theta_{2,n}})$, the arc on the circle containing 1 with endpoints $e^{i\theta_{1,n}}$ and $e^{i\theta_{2,n}}$.

Thus, for $n \geq n_0$, $\operatorname{supp}\psi \subseteq E_n$. (Note that b takes only the value $b(\psi)$ on $\operatorname{supp}\psi$.) Therefore

$$\left(f_n\chi_{\overline{E}_n}\right)(\psi) = f_n(\psi) = \left(b(\psi) - e^{i\theta_{1,n}}\right)\left(b(\psi) - e^{i\theta_{2,n}}\right)B(\psi) \neq 0.$$

Hence $\psi \notin Z(P)$. Thus we showed that

$$Z(P) \subseteq Z(B) \cup \{b = 1\}.$$

Hence $(1-b)B$ vanishes identically on $Z(P)$, the hull of P. Since P is a closed prime ideal, we have, by Theorem 1.2, that P is an intersection of maximal ideals. Hence $(1-b)B \in P$. Since $|b(x)| < 1$ and $x \in Z(P)$, we see that $1 - b \notin P$. Hence, since P is prime, $B \in P$. In particular, by (2), $y \notin Z(P)$. This finishes the proof of the theorem. $\square$

We do not know whether every nonmaximal, closed prime ideal P in $A = H^\infty$ or $H^\infty + C$ is given by

$$P = \{f \in A : f \equiv 0 \text{ on } P(m)\}$$

for some nontrivial point m (which precisely is Allling's conjecture). In order to find counterexamples, good candidates could be the ideals $I(k(x))$. Therefore we ask the following question.

Question Q1. For which $x \in M(H^\infty + C)$ is the set $I(k(x))$ a closed prime ideal?

Note that, in view of Theorem 1.2, we can restrict our search for counterexamples to Alling's conjecture to prime ideals whose hull does not meet the Shilov boundary and whose hull is not entirely contained in the set of trivial points.

We also have the following consequences of Theorem 1.9.

Corollary 1.10. *Suppose that $I(k(x))$ is a (closed) prime ideal. Then $k(x)$ is a minimal k-hull.*

Proof. Let $P = I(k(x))$. Obviously $Z(P) = k(x)$. Let $y \in k(x)$. Then, by Theorem 1.9, $Z(P) \subseteq k(y)$. Hence $k(x) \subseteq k(y) \subseteq k(x)$. □

Let us point out that Keiji Izuchi [Iz5] recently showed that for every $x \in M(H^\infty + C)$, the ideals $J(x)$ are prime in $H^\infty + C$. Note also that by [GM4] one has $\overline{J(x)} = I(k(x))$. As a further corollary of Theorem 1.9 we obtain:

Corollary 1.11. *Let P be a nonzero closed prime ideal in $H^\infty + C$. Suppose that $x \in Z(P)$. Then $J(x) \subseteq P$.*

Proof. As noted above, $\overline{J(x)} = I(k(x))$. But Theorem 1.9 implies that $Z(P) \subseteq k(x)$. Therefore $I(k(x)) \subseteq I(Z(P))$. Since P is closed and prime, by Theorem 1.2 we know that $P = I(Z(P))$. Hence $J(x) \subseteq P$. □

2. p-parts

We shall now present a class of nontrivial Gleason parts for which Alling's conjecture is true. Recall that a point x in a topological Hausdorff space X is said to be a p-point, if any continuous function on X is constant in a neighborhood of x (see [GJ, p. 63])

In analogy to this, we give the following definition:

Definition. A part $P(m)$ in $M(H^\infty + C)$ is called a p-part, if any function $f \in H^\infty + C$ vanishing identically on $P(m)$ vanishes in a neighborhood of $P(m)$.

The first question which comes to mind is the following: do there exist p-parts in $M(H^\infty + C)$? The following result gives an affirmative answer, provided we assume the continuum hypothesis.

Proposition 2.1. *The set of points $m \in M(H^\infty + C)$ for which $P(m)$ is a nontrivial p-part is dense in $M(H^\infty + C)$.*

Proof. Assume the continuum hypothesis. Let U be any open set in $M(H^\infty + C)$. We show that U contains a nontrivial point m such that $P(m)$ is a p-part. Choose an open set U^* in $M(H^\infty)$ so that $\overline{U^*} \cap M(H^\infty + C) \subseteq U$. Let (z_n) be any interpolating sequence in $\mathbb{D} \cap U^*$, b the associated interpolating Blaschke product, and let $E = \overline{\{z_n : n \in \mathbb{N}\}} \setminus \{z_n : n \in \mathbb{N}\}$. We may assume that $E \subseteq U$. We know that E is homeomorphic to $S := \beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. A result in topology (see [GJ, p. 100]), tells us that, under the continuum hypothesis, S has p-points. This implies that every function $f \in H^\infty + C$, viewed as a continuous function on E, is constant in a relative neighborhood of some m in E. Moreover, $m \in U$. We claim that $P(m)$ is a p-part. To see this, let $f \in H^\infty + C$ vanish identically on $P(m)$ and let $V_1 \subseteq E$ be a neighborhood of m in E on which f is identically zero. Since E is totally disconnected, we may also assume that V_1 is a clopen set in E. Hence, by [Iz1] there exists a subproduct b_1 of b satisfying $Z(b) \cap V_1 = Z(b_1)$. Thus $Z(b_1) \subseteq Z(f)$. By [AG] and [GIS], $f_1 := f\overline{b}_1 \in H^\infty + C$. Moreover, since f vanishes identically on the part $P(m)$, we see that f_1 has the same property. Continuing this procedure, we get a sequence of neighborhoods V_n of m in E, $V_{n+1} \subseteq V_n$, and a sequence of interpolating Blaschke products b_n dividing $f(\overline{b}_1 \dots \overline{b}_{n-1})$. Since m is a p-point, every G_δ set (in E) containing m is a neighborhood of m (see [GJ, p. 63]); this holds in particular for $V := \bigcap_n V_n$. Choose a clopen set W of E satisfying $m \in W \subseteq V$ and let b_0 be the factor of b satisfying $Z(b_0) = Z(b) \cap W$. It is clear that b_0 divides each b_n (in $H^\infty + C$). Hence b_0^n divides f in $H^\infty + C$ for every $n \in \mathbb{N}$. Thus, by [AG] and [GIS], $\{|b_0| < 1\} \subseteq Z(f)$. Noticing that $b_0(m) = 0$ we obtain that $m \in Z(f)^0$. Hence, by Theorem 1.5 and the remark following it, $\overline{P(m)} \subseteq Z(f)^0$. Thus $P(m)$ is a p-part. □

Proposition 2.2. *Let $P(m)$ be a nontrivial p-part. Then the following assertions hold:*

(a) $k(m) = \overline{P(m)}$.
(b) $\overline{P(m)}$ *is a strong synthesis set.*
(c) $P(m)$ *is a maximal part; that is, if* $\overline{P(m)} \subseteq \overline{P(x)}$ *for some* $x \in M(H^\infty + C)$, *then* $\overline{P(m)} = \overline{P(x)}$.
(d) $\operatorname{supp} m$ *is a maximal support set.*
(e) *Alling's conjecture is true for m; i.e., if I is a closed prime ideal in $A = H^\infty$ (or $A = H^\infty + C$) such that its hull contains m, then either $I = \operatorname{Ker} m$, and hence I is maximal, or $I = \{f \in A : f \equiv 0$ on $\overline{P(m)}\}$.*

Proof. (a) This follows from the definition of a p-part and the fact that $\overline{P(m)}$ is, by [Go], hull-kernel closed.

(b) Follows from (a) and [GM4, Theorem 3.1].

(c) Follows from (a) and formula (1) (see Example 5 in [GM4]).

(d) Suppose there exists $y \in M(H^\infty + C)$ such that $\operatorname{supp} m \subseteq \operatorname{supp} y$. Then it follows from $k(m) = \bigcap_{B(m)=0} \overline{\{|B| < 1\}}$ that $y \in k(m)$. By (a), we then obtain that $y \in \overline{P(m)}$. Hence $\operatorname{supp} y \subseteq \operatorname{supp} m$. Therefore $\operatorname{supp} y = \operatorname{supp} m$, which shows the maximality of the support set for m.

(e) Let I be a closed prime ideal in H^∞ and suppose that $m \in Z(I)$. If I contains an interpolating Blaschke product b, necessarily vanishing at m, then by [Mo1, Theorem 3.1], I is maximal, so $I = \operatorname{Ker} m$. If I does not contain an interpolating Blaschke product, then, by [Mo1, Proposition 3.4], for every $f \in I$ we have $\operatorname{ord}(f, m) = \infty$. Hence $\overline{P(m)} \subseteq Z(I)$. By Theorem 1.9, $Z(I) \subseteq k(m)$. Since $P(m)$ is a p-part, we have by (a) that $k(m) = \overline{P(m)}$. Hence $Z(I) = \overline{P(m)}$. Since by Theorem 1.2 I is an intersection of maximal ideals, we obtain that

$$I = \{f \in H^\infty : f \equiv 0 \text{ on } \overline{P(m)}\}.$$

The result for $A = H^\infty + C$ follows from this and Theorem 1.1. $\square$

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