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# REGULARITY OF SOLUTIONS TO THE FREE SCHRÖDINGER EQUATION WITH RADIAL INITIAL DATA

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ABSTRACT. We derive weighted smoothing inequalities for solutions of the free Schrödinger equation. As an application, we give a new proof of the endpoint Strichartz estimates in the radial case. We also consider general dispersive equations and obtain similar estimates in this case.

## 0. Introduction

Consider the homogeneous initial value problem for the free Schrödinger equation

$$
\begin{cases} i\partial_t u - \Delta_x u = 0, & (x,t) \in I\!R^n \times I\!R, \\ u(x,0) = u_0(x), \end{cases} \tag{1}
$$

and denote its solution by $e^{it\Delta}u_0$. The problem of finding values $q, r$ for which the $L^q_t L^r_x$-norm of the solution to (1) is controlled by the $L^2$-norm of the initial data has been extensively studied by several authors. Here $L^q_t L^r_x$ denotes the space of functions $F(x,t)$ such that

$$
\|F\|_{L^q_t L^r_x} = \left( \int_{-\infty}^{+\infty} \left( \int_{I\!R^n} |F(x,t)|^r \, dx \right)^{q/r} dt \right)^{1/q} < +\infty.
$$

In the case $q = r = 2n/(n+2)$, such an estimate was given by R. Strichartz [13], using ideas developed by E. Stein on the restriction properties of the Fourier transform on curved surfaces; see [11, p. 374], and the papers [15] and [16] by P. Tomas.

The estimate was extended to the case $q \neq r$ by J. Ginibre and G. Velo [5], who used these results in an essential manner to study the Initial Value Problem of semilinear perturbations of (1).

To state the known results we need the following definition.

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Definition 1. *The pair $(q,r)$ is an admissible pair if $q,r\geq 2$, $(q,r,n/2)\neq(2,\infty,1)$, and*

$$\frac{2}{q}+\frac{n}{r}=\frac{n}{2}. \tag{2}$$

Theorem 1. *If $(q,r)$ and $(\tilde{q},\tilde{r})$ are admissible, then we have the following estimates:*

$$\|e^{it\Delta}u_0\|_{L_t^qL_x^r}\leq c\,\|u_0\|_{L^2}, \tag{3}$$

$$\left\|\int_{I\!R}e^{-is\Delta}F(\cdot,s)\,ds\right\|_{L^2}\leq c\,\|F\|_{L_t^{q'}L_x^{r'}}, \tag{4}$$

$$\left\|\int_{s<t}e^{i(t-s)\Delta}F(\cdot,s)\,ds\right\|_{L_t^qL_x^r}\leq c\,\|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}. \tag{5}$$

The estimates (3) and (4) are equivalent, and a scaling argument shows that in this case (2) is necessary. For the last inequality the natural restriction is

$$\frac{2}{q}+\frac{n}{r}+\frac{2}{\tilde{q}}+\frac{n}{\tilde{r}}=n,$$

which is weaker than (2), so it is possible that (5) holds for a larger range of pairs. The problem of determining the exact set of pairs for which this estimate holds is still open.

The case when $(q,r)$ or $(\tilde{q},\tilde{r})$ is equal to the critical value

$$P=\left(2,\frac{2n}{n-2}\right)\qquad(n\geq 3)$$

was recently settled by M. Keel and T. Tao [8]. The noncritical case had been solved earlier; see [13] and [5] for the estimates (3) and (4), and [19] and [3] for (5).

When $1\leq q<2$, it is easy to construct an example which shows that (3) and (4) are false. The same counterexample proves that (5) fails when $1/q+1/\tilde{q}>1$. In the case $n=2$ the critical point $P=(2,\infty)$ is not admissible. In fact, (3), (4) and (5) do not hold for this critical value; see [9].

In this paper we prove some weighted smoothing inequalities for arbitrary solutions of the free Schrödinger equation. In the first section we consider the homogeneous problem, and in the second section the inhomogeneous problem. Finally, in the last section we recover the endpoint estimates of Strichartz in the radial case from previous estimates by using only a radial version of the Sobolev embedding theorem.

## 1. The homogeneous case

Given a function $f$ and $s\in I\!R$ we define the homogeneous derivative of order $s$ of $f$ by $\widehat{D^sf}(\xi)=c_s|\xi|^s\hat{f}(\xi)$, and the fractional integral of order $s$

of $f$ as $I_s f = D^{-s} f$. For any $\gamma \in I\!R$, we denote the $L^p$-space with measure $|x|^\gamma \, dx \, dt$ by $L^p_{tx}(|x|^\gamma)$, and the $L^p$-space with measure $|x|^\gamma \, dx$ by $L^p_x(|x|^\gamma)$.

The main result of this section is the following theorem.

THEOREM 2. *We have*

$$\|D^s_x e^{it\Delta} u_0\|_{L^2_{tx}(|x|^{-\alpha})} \leq c\,\|u_0\|_{L^2}, \tag{6}$$

$$\left\| D^s_x \left( \int_{I\!R} e^{-i\tau\Delta} F(\cdot,\tau)\, d\tau \right) \right\|_{L^2} \leq c\,\|F\|_{L^2_{tx}(|x|^{\alpha})}, \tag{7}$$

*if and only if* $\alpha = 2(1-s)$, $1 < \alpha < n$ *and* $n \geq 2$.

REMARK 1. Estimates of this type have also been studied in [18].

REMARK 2. We have stated Theorem 2 in terms of the solution to the Schrödinger equation, but the theorem holds in a more general setting. In fact, take $u$ to be the solution to the problem

$$\begin{cases} i\partial_t u + (-\Delta_x)^{a/2} u = 0, & (x,t) \in I\!R^n \times I\!R,\ a > 0, \\ u(x,0) = u_0(x), & \end{cases}$$

which we denote by $e^{it\Delta^{a/2}} u_0$. The result in this case is that

$$\|D^s_x e^{it\Delta^{a/2}} u_0\|_{L^2_{tx}(|x|^{-\alpha})} \leq c\,\|D^\beta u_0\|_{L^2},$$
$$\left\| D^s_x \left( \int_{I\!R} e^{-i\tau\Delta^{a/2}} F(\cdot,\tau)\, d\tau \right) \right\|_{L^2} \leq c\,\|D^\beta F\|_{L^2_{tx}(|x|^{\alpha})},$$

hold if and only if $\alpha = 2(1-s)$, $1 < \alpha < n$, $\beta = 1 - a/2$, and $n \geq 2$.

*Proof of Theorem 2.* By duality, (6) and (7) are equivalent, and because of the scale the restriction $\alpha = 2(1-s)$ is necessary.

In order to prove (6) for $1 < \alpha < n$, we use polar coordinates and a change of variable to write

$$\begin{aligned} D^s_x e^{it\Delta} u_0(x) &= \int_{I\!R^n} e^{ix\cdot\xi} \left(D^s_x e^{it\Delta} u_0\right)^{\widehat{}}(\xi)\, d\xi \\ &= \int_0^{+\infty} e^{itr^2} r^s \left( \int_{S^{n-1}_r} e^{ix\cdot\xi} \widehat{u_0}(\xi)\, d\sigma_r(\xi) \right) dr \\ &= \frac{1}{2} \int_0^{+\infty} e^{itu} \left( \int_{S^{n-1}_{\sqrt{u}}} u^{\frac{s-1}{2}} e^{ix\cdot\xi} \widehat{u_0}(\xi)\, d\sigma_{\sqrt{u}}(\xi) \right) du. \end{aligned}$$

Using this identity together with Plancherel's identity in the variable $t$, we have

$$
\begin{aligned}
&\|e^{it\Delta}u_0\|_{L^2_{tx}(|x|^{-\alpha})} \\
&\quad = c\left(\int_{I\!R^n}|x|^{-\alpha}\int_0^{+\infty}\left|\int_{S^{n-1}_{\sqrt{u}}} u^{\frac{s-1}{2}}e^{ix\cdot\xi}\widehat{u_0}(\xi)\,d\sigma_{\sqrt{u}}(\xi)\right|^2 du\,dx\right)^{1/2} \\
&\quad = c\left(\int_{I\!R^n}|x|^{-\alpha}\int_0^{+\infty}\left|\int_{S^{n-1}_r} r^{s-1}e^{ix\cdot\xi}\widehat{u_0}(\xi)\,d\sigma_r(\xi)\right|^2 r\,dr\,dx\right)^{1/2} \\
&\quad = c\left(\int_0^{+\infty}\left(\int_{I\!R^n}|x|^{-\alpha}\left|\int_{S^{n-1}_r} e^{ix\cdot\xi}\widehat{u_0}(\xi)\,d\sigma_r(\xi)\right|^2 dx\right) r^{2s-1}\,dr\right)^{1/2}.
\end{aligned}
$$

Hence it is enough to prove that

$$
\int_{I\!R^n}|x|^{-\alpha}\left|\int_{S^{n-1}_r} e^{ix\cdot\xi}\widehat{u_0}(\xi)\,d\sigma_r(\xi)\right|^2 dx \le c\,r^{1-2s}\int_{S^{n-1}_r}|\widehat{u_0}(\xi)|^2\,d\sigma_r(\xi).
$$

Since $\alpha = 2(1-s)$, this inequality is invariant under dilations, so we may assume without loss of generality that $r=1$. Then,

$$
\begin{aligned}
&\int_{I\!R^n}|x|^{-\alpha}\left|\int_{S^{n-1}} e^{ix\cdot\xi}g(\xi)\,d\sigma(\xi)\right|^2 \\
&\quad = \int_{|x|\le 1}|x|^{-\alpha}\left|\int_{S^{n-1}} e^{ix\cdot\xi}g(\xi)\,d\sigma(\xi)\right|^2 dx \\
&\qquad + \int_{|x|>1}|x|^{-\alpha}\left|\int_{S^{n-1}} e^{ix\cdot\xi}g(\xi)\,d\sigma(\xi)\right|^2 dx = \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

When $\alpha < n$, we have

$$
\begin{aligned}
\mathrm{I} &\le \left\|\int_{S^{n-1}} e^{ix\cdot\xi}g(\xi)\,d\sigma(\xi)\right\|^2_{L^\infty_x}\int_{|x|\le 1}|x|^{-\alpha}\,dx \\
&\le c\left(\int_{S^{n-1}}|g(\xi)|\,d\sigma(\xi)\right)^2 \\
&\le c\int_{S^{n-1}}|g(\xi)|^2\,d\sigma(\xi).
\end{aligned}
$$

To estimate the integral II, we need the following lemma.

Lemma 1 (Trace lemma; see [1]). *We have*

$$
\sup_{x_0,R}\frac{1}{R}\int_{B(x_o,R)}\left|\int_{S^{n-1}_r} e^{ix\cdot\xi}f(\xi)\,d\sigma_r(\xi)\right|^2 dx \le c\int_{S^{n-1}_r}|f(\xi)|^2\,d\sigma_r(\xi),
$$

*where the constant $c$ is independent of $r$, $S_r^{n-1}$ is the Euclidean sphere of radius $r$, $d\sigma_r$ is the surface measure, $x_o \in \mathbb{R}^n$, and $R, r > 0$.*

Dividing the range of integration in II diadically, we can write

$$\begin{aligned} \mathrm{II} &= \sum_{j=0}^{+\infty} \int_{2^j<|x|\le 2^{j+1}} |x|^{-\alpha} \left| \int_{S^{n-1}} e^{ix\cdot\xi} g(\xi)\, d\sigma(\xi) \right|^2 dx \\ &\le c \sum_{j=0}^{+\infty} 2^{-j(\alpha-1)} \frac{1}{2^{j+1}} \int_{|x|\le 2^{j+1}} \left| \int_{S^{n-1}} e^{ix\cdot\xi} g(\xi)\, d\sigma(\xi) \right|^2 dx \\ &\le c \int_{S^{n-1}} |g(\xi)|^2 \, d\sigma(\xi). \end{aligned}$$

The last inequality is a consequence of the trace lemma and the fact that $\alpha > 1$.

When $n = 1$, the estimate (6) fails. To see this, take $u_0$ such that $\widehat{u_0}$ is an even function; then (6) does not hold because $\cos x \notin L_x^2(|x|^{-\alpha})$ for any $\alpha \in \mathbb{R}$. When $n \ge 2$, (6) fails whenever $\alpha \le 1$ or $\alpha \ge n$. To see this, take $u_0$ such that $\widehat{u_0}$ is a radial function; then (6) fails because $\widehat{d\sigma} \notin L_x^2(|x|^{-\alpha})$, where $d\sigma$ denotes the surface measure of the unit Euclidean sphere. $\square$

Remark 3. When $\alpha = 1$ and $s = 1/2$, then (6) also fails. However, in this case we have the following substitute of this estimate:

$$\sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_{-\infty}^{+\infty} \left| D_x^{1/2} e^{it\Delta} u_0(x) \right|^2 dt\, dx \le c \, \|u_0\|_{L^2}^2.$$

This Kato type smoothing estimate was proved in [4], [10] and [17].

## 2. The inhomogeneous case

In this section we consider the inhomogeneous Initial Value Problem

$$\begin{cases} i\partial_t u - \Delta_x u = F(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = 0. \end{cases} \tag{8}$$

Using Duhamel's formula we can write the solution of (8) in the form

$$u(x,t) = \frac{1}{i} \int_0^t e^{i(t-\tau)\Delta} F(x,\tau)\, d\tau.$$

Our main result is the following theorem.

Theorem 3. *The solution to the IVP (8) satisfies*

$$\|D_x^{s_1} u\|_{L^2_{tx}(|x|^{-\alpha_1})} \le c \, \|I_{s_2} F\|_{L^2_{tx}(|x|^{\alpha_2})}, \tag{9}$$

*whenever $\alpha_i = 2(1 - s_i)$, $1 < \alpha_i < n$ ($i = 1, 2$), and $n \ge 2$.*

Remark 4. As in the homogeneous case, Theorem 3 can be formulated in a more general setting for the solution of the IVP (8), where $-\Delta_x$ is replaced by $(-\Delta_x)^{a/2}$. In this case we have the estimate

$$\|D_x^{s_1} u\|_{L^2_{tx}(|x|^{-\alpha_1})} \le c\,\|D_x^{\beta} I_{s_2} F\|_{L^2_{tx}(|x|^{\alpha_2})},$$

whenever $\alpha_i = 2(1-s_i)$, $\beta = 1-a/2$, $1<\alpha_i<n$ $(i=1,2)$ and $n \ge 2$.

*Proof of Theorem 3.* We follow the argument used in Theorem 2.3 of [7] and formally write the solution of (8) in the form

$$u(x,t) = v(x,t) - \left(e^{it\Delta} v(\cdot,0)\right)(x),$$

where

$$v(x,t) = \int_{I\!R\times I\!R^n} \frac{1}{|\xi|^2-\tau} \widehat{F}^{x,t}(\xi,\tau) e^{it\tau + ix\cdot\xi}\, d\xi\, d\tau.$$

Here $\widehat{F}^{x,t}$ denotes the Fourier transform of $F$ in both variables.

To estimate the second term, we use Theorem 2. To control the first term, we rewrite this term as

$$v(x,t) = \int_{-\infty}^{+\infty} T_\tau\left(\widehat{F}^t(\tau)\right)(x) e^{it\tau}\, d\tau,$$

where $T_\tau$ is the Helmholtz operator defined by

$$\widehat{T_\tau f}(\xi) = \frac{1}{|\xi|^2-\tau}\hat{f}(\xi). \tag{10}$$

Using Plancherel's identity in the variable $t$ we have

$$\begin{aligned}
\|D_x^{s_1} v\|_{L^2_{tx}(|x|^{-\alpha_1})} &= \left\| \left\| \left(D_x^{s_1} T_\tau\left(\widehat{F}^t(\tau)\right)\right)^{\wedge t}(t) \right\|_{L^2_t} \right\|_{L^2_x(|x|^{-\alpha_1})} \\
&= \left\| \left\| D_x^{s_1} T_\tau\left(\widehat{F}^t(\tau)\right) \right\|_{L^2_\tau} \right\|_{L^2_x(|x|^{-\alpha_1})} \qquad (11)\\
&= \left\| \left\| D_x^{s_1} T_\tau\left(\widehat{F}^t(\tau)\right) \right\|_{L^2_x(|x|^{-\alpha_1})} \right\|_{L^2_\tau}.
\end{aligned}$$

The following proposition will allow us to complete the proof.

Proposition 1. *The Helmholtz operator $T_\tau$ defined by* (10) *satisfies*

$$\|D_x^{s_1} T_\tau f\|_{L^2(|x|^{-\alpha_1})} \le c\,\|I_{s_2} f\|_{L^2(|x|^{\alpha_2})}, \tag{12}$$

*whenever $\alpha_i = 2(1-s_i)$, $1<\alpha_i<n$ $(i=1,2)$, and $n\ge 2$. Here $c$ is a constant independent of $\tau$.*

Using this proposition in (11) and the Plancherel identity in the $t$ variable, we have

$$\begin{aligned} \|D_x^{s_1} v\|_{L^2_{tx}(|x|^{-\alpha_1})} &\le c \left\| \left\| I_{s_2} \widehat{F}^t(\tau) \right\|_{L^2_x(|x|^{\alpha_2})} \right\|_{L^2_\tau} \\ &= c \left\| \left\| (I_{s_2} F)^t(\tau) \right\|_{L^2_\tau} \right\|_{L^2_x(|x|^{\alpha_2})} \\ &= c \left\| \|I_{s_2} F\|_{L^2_t} \right\|_{L^2_x(|x|^{\alpha_2})} \\ &= c \, \|I_{s_2} F\|_{L^2_{tx}(|x|^{\alpha_2})}. \end{aligned}$$

The above formal process can be justified by applying it to the equation

$$i\partial_t u - \Delta_x u + i\varepsilon u = F(x,t), \qquad (x,t) \in I\!R^n \times I\!R, \quad \varepsilon > 0.$$

In this case the estimate (9) holds uniformly in $\varepsilon$ and the result follows on letting $\varepsilon \to 0$. □

To prove the proposition we need the following two lemmas.

LEMMA 2 ([12]). *Let $0 < \beta < n$, $1 < p < \infty$ and $p\beta - n < r < n(p-1)$. Then*

$$\|I_\beta f\|_{L^p(|x|^{r-p\beta})} \le c \, \|f\|_{L^p(|x|^r)}.$$

LEMMA 3 ([6], [7]). *Let $\varphi \in C^\infty(I\!R)$ with* supp $\varphi \subseteq [-1,1]$, *$\varphi = 1$ on the interval $[-1/2, 1/2]$ and $0 \le \varphi \le 1$. Given $s \in I\!R$, define the operator $S$ by*

$$\widehat{Sf}(\xi) = \frac{|\xi|^s}{|\xi|^2 - 1} \, \varphi(2(|\xi| - 1)) \hat{f}(\xi).$$

*If $f$ has compact support, then*

$$R^{-1} \, \|Sf\|^2_{L^2(B(0,R)))} \le c \, d(\text{supp } f) \|f\|^2_{L^2},$$

*where $d(\text{supp } f)$ is the diameter of the support of $f$ and $R > 0$.*

*Proof of Proposition 1.* By a scaling argument it is enough to prove (12) when $\tau = \pm 1$.

In the case $\tau = -1$ we have no singularity. Therefore $|D_x^{s_1} T_{-1} f(x)| \le c \, I_{2-s_1} |f|(x)$, and the result follows from Lemma 2.

When $\tau = 1$, we take $\varphi \in C_0^\infty(I\!R)$ with supp $\varphi \subseteq [-1,1]$, $\varphi = 1$ on $[-1/2, 1/2]$ and $0 \le \varphi \le 1$. Let $\varphi_1(\xi) = \varphi(2(|\xi| - 1))$ and $\varphi_2(\xi) = 1 - \varphi_1(\xi)$, and define the operators $T_{1,1}$ and $T_{1,2}$ by

$$(T_{1,i} f)\widehat{\ }(\xi) = m_i(\xi) \hat{f}(\xi) = \frac{|\xi|^{s_1}}{|\xi|^2 - 1} \, \varphi_i(\xi) \hat{f}(\xi), \quad i = 1, 2.$$

Then we have

$$\|D_x^{s_1} T_1 f\|_{L^2(|x|^{-\alpha_1})} \le \|T_{1,1} f\|_{L^2(|x|^{-\alpha_1})} + \|T_{1,2} f\|_{L^2(|x|^{-\alpha_1})}.$$

The second term can be controlled as in the case $\tau = -1$ because $m_2$ has no singularity. To control the first term, we replace the homogeneous weights $|x|^{-\alpha_1}$ and $|x|^{\alpha_2}$ with the inhomogeneous weights $\langle x\rangle^{-\alpha_1}$ and $\langle x\rangle^{\alpha_2}$, respectively, where $\langle x\rangle = \left(1+|x|^2\right)^{1/2}$. This is possible, by the Littlewood-Paley localization and the estimates

$$\begin{aligned} \|\Delta_0 f\|_{L^2(|x|^{-\alpha_1})} &\le c\,\|f\|_{L^2(\langle x\rangle^{-\alpha_1})}, \\ \|\Delta_0 f\|_{L^2(\langle x\rangle^{\alpha_2})} &\le c\,\|f\|_{L^2(|x|^{\alpha_2})}, \end{aligned}$$

for $0 \le \alpha_i < n$ $(i = 1,2)$, where $\Delta_0$ is the Littlewood-Paley projection to frequencies $|\xi| \sim 1$.

We now divide $I\!R^n$ and decompose $f$ into

$$I\!R^n = \bigcup_{j=0}^{+\infty} X_j, \qquad f = \sum_{k=0}^{+\infty} f_k,$$

where $X_0 = \{x : |x| \le 1\}$, $X_j = \{x : 2^{j-1} < |x| \le 2^j\}$ for $j \ge 1$ and $f_k = f\chi_{X_k}$. Using these decompositions, Lemma 3, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|T_{1,1}f\|_{L^2(\langle x\rangle^{-\alpha_1})} &\le \sum_{k=0}^{+\infty} \|T_{1,1}f_k\|_{L^2(\langle x\rangle^{-\alpha_1})} \\ &\le \sum_{k=0}^{+\infty} \left(\sum_{j=0}^{+\infty} 2^{-j\alpha_1}\,\|T_{1,1}f_k\|^2_{L^2(B(0,2^j))}\right)^{1/2} \\ &\le c\sum_{k=0}^{+\infty} \left(\sum_{j=0}^{+\infty} 2^{-j(\alpha_1-1)}\,2^k\,\|f_k\|^2_{L^2}\right)^{1/2} \\ &\le c\sum_{k=0}^{+\infty} 2^{k/2}\,\|f_k\|_{L^2} \\ &\le c\left(\sum_{k=0}^{+\infty} 2^{k\alpha_2}\,\|f_k\|^2_{L^2}\right)^{1/2} \\ &\le c\,\|f\|_{L^2(\langle x\rangle^{\alpha_2})}, \end{aligned}$$

whenever $1 < \alpha_i < n$ $(i = 1,2)$. □

## 3. Application

From Theorems 2 and 3 we can derive the Strichartz estimates (3), (4) and (5) in the critical case and for radial initial data whenever $n \ge 3$, using only the following radial version of the Hardy-Littlewood-Sobolev theorem.

LEMMA 4. *Let $f$ be a radial function. Then*

$$\|I_s f\|_{L^q(\mathbb{R}^n)} \leq c\,\|f\|_{L^p(|x|^{-\alpha})},$$

*whenever*

$$\frac{1}{p} - \frac{1}{q} \leq s \leq n\left(\frac{1}{p} - \frac{1}{q}\right), \quad \alpha = p\left[n\left(\frac{1}{p} - \frac{1}{q}\right) - s\right], \quad 1 < p < q < \infty.$$

This lemma can be proved using the ideas in [12]. The fact that $f$ is radial allows us to reduce $s$ to $1/p - 1/q$.

Given a radial initial data $u_0$, $e^{it\Delta}u_0$ is radial too, so we can apply Lemma 4 with $p = 2$ and $r = 2n/(n-2)$ $(n \geq 3)$ to obtain

$$\|e^{it\Delta}u_0\|_{L^r_x} \leq c\,\|D^s_x e^{it\Delta}u_0\|_{L^2_x(|x|^{-\alpha})},$$

whenever $1/n \leq s \leq 1$, $\alpha = 2(1-s)$. Taking the $L^2$-norm in time and using Theorem 2, we get the estimates

$$\|e^{it\Delta}u_0\|_{L^2_t L^r_x} \leq c\,\|D^s_x e^{it\Delta}u_0\|_{L^2_{tx}(|x|^{-\alpha})} \leq c\,\|u_0\|_{L^2}, \tag{13}$$

for $1/n \leq s < 1/2$.

The dual version of (13) is

$$\left\|\int_{\mathbb{R}} e^{-is\Delta}F(\cdot, s)\,ds\right\|_{L^2} \leq c\,\|I_s F\|_{L^2_{tx}(|x|^{\alpha})} \leq c\,\|F\|_{L^2_t L^{r'}_x},$$

and the analogous result for the solution $u$ of the inhomogeneous problem is

$$\|u\|_{L^2_t L^r_x} \leq c\,\|D^s_x u\|_{L^2_{tx}(|x|^{-\alpha})} \leq c\,\|I_s F\|_{L^2_{tx}(|x|^{\alpha})} \leq c\,\|F\|_{L^2_t L^{r'}_x}. \tag{14}$$

Here $F$ is a radial function in the $x$-variable, and $I_s F$ denotes the fractional integral in the $x$ variable.

When $n = 2$ and $r = \infty$, this method fails because Lemma 4 is false for $p = 2$ and $q = \infty$. However, Tao [14] recently showed that the estimates (3), (4) and (5) hold in this case for radial data whenever $(\tilde{q}, \tilde{r})$ is an admissible pair.

Estimates similar to (14) have recently been used by Bourgain [2] to prove the global existence for the defocusing quintic nonlinear Schrödinger equation with radial data and arbitrary large energy norm. In particular, Bourgain used estimates such as (14) to prove that solutions which cease to exist in finite time must concentrate. This property has not been established for dimensions $n \geq 3$, and data in $L^2$, even in the radial case. We shall study these questions elsewhere.

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