

## ON THE GEOMETRY OF CONSTANT MEAN CURVATURE ONE SURFACES IN HYPERBOLIC SPACE

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ABSTRACT. We give a geometric classification of regular ends with constant mean curvature 1 and finite total curvature, embedded in hyperbolic space. We prove that each such end is either asymptotic to a catenoid cousin or asymptotic to a horosphere. We also study symmetry properties of constant mean curvature 1 surfaces in hyperbolic space associated to minimal surfaces in Euclidean space. We describe the constant mean curvature 1 surfaces in  $\mathbf{H}^3$  associated to the family of surfaces in  $\mathbf{R}^3$  that is isometric to the helicoid.

### 0. Introduction

The theory of constant mean curvature 1 surfaces in hyperbolic space was created by Robert Bryant in a pioneering paper [Br], in which Bryant gave a holomorphic parametrization of such (simply connected) surfaces, which is a close analogue of the Weierstrass representation for minimal surfaces in  $\mathbf{R}^3$ . Later, Umehara and Yamada, following Bryant's idea, have made important contributions to the theory. In particular, they introduced the notion of regular ends (see Definition 2.1), and they gave some new examples and techniques to construct complete constant mean curvature 1 surfaces with regular and nonregular ends (see [U-Y1], [U-Y2], [U-Y3]).

In this paper we prove that if  $E$  is a regular embedded end into  $\mathbf{H}^3$  with constant mean curvature 1 and finite total curvature, then  $E$  is either asymptotic to a catenoid cousin (embedded or not embedded) or is asymptotic to a horosphere. In the half-space model we prove that, up to an isometry,  $E$  is asymptotic as a *vertical Euclidean graph* to a horosphere or to an end of a catenoid cousin. Thus, we obtain a geometric classification in hyperbolic space of embedded regular ends with finite total curvature. In fact, in the half-space

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Received June 14, 1999; received in final form November 3, 1999.

2000 *Mathematics Subject Classification.* 53C42.

The first author was partially supported by CNPq and PRONEX of Brasil. The second author was supported by FAPERJ of Brasil. The second author is grateful to the Department of Mathematics of PUC-Rio for their hospitality during the preparation of this paper.

model we derive the formulas which give this classification. Clearly, the classification in hyperbolic space does not depend on the model, but we believe that the mentioned Euclidean asymptotic behavior in the half-space model is very surprising. We remark that Lima and Rossman [L-R] independently proved a similar result, but we emphasize that the formulas we derive from the results and techniques of Umehara and Yamada are established in a more explicit fashion.

Recently, Collin, Hauswirth, and Rosenberg [C-H-R] proved that a properly embedded end in  $\mathbf{H}^3$  with constant mean curvature 1 must be regular and has finite total curvature. Using the main result of this paper, they deduced that such an end must be asymptotic to a catenoid cousin or a horosphere end.

Here we introduce a new approach which we use to establish a basic result that describes the helicoids in the half-space model. In fact, we show that, for the half-space model  $\mathbf{H}^3 = \{(u, v, w) \in \mathbf{R}^3, w > 0\}$ , the family in  $\mathbf{H}^3$  given by constant mean curvature 1 surfaces associated to minimal catenoids in  $\mathbf{H}^3$  contains the *catenoid cousins*, the helicoids, and a surface that is invariant under a 1-parameter group of Euclidean horizontal translations. We believe that the fact that this surface belongs to the family of surfaces associated to a minimal catenoid in  $\mathbf{R}^3$  is an unexpected result. This surface (see Example 1.8 with  $e^{i\theta} = -1$  and  $\lambda = 1/4$ ) will be shown to be an *Enneper cousin dual*. In [R-U-Y] a natural geometric notion of “dual” was defined (see Remark 1.11). The Enneper cousin dual has also been described in the thesis of Gomes [Go], using a different terminology, and in a paper by Sato and Rossman [R-S]. We note that the constant mean curvature 1 helicoids were classified by Ordóñez (see [O]), as part of his Doctoral Thesis, by using a complete different method.

Rossman, Umehara, and Yamada [R-U-Y] gave examples of complete constant mean curvature 1 surfaces in hyperbolic space with higher genus, many symmetries, and embedded ends; see also the work of Galvão and Góes [Ga-Go] for another approach on this subject. Finally, we remark that in [SE-T] the authors gave some uniqueness results for constant mean 1 surfaces in hyperbolic space. For instance, we showed that, if  $M$  is such a properly embedded surface contained in the region *inside* a horosphere  $H$  and if  $\partial M$  is a circle lying on  $H$ , then  $M$  is part of an embedded catenoid cousin. In the same paper, we gave a Phragmén-Lindelöf type theorem in the half-space model for unbounded vertical graphs over a horosphere  $\{w = 1\}$ .

We now describe the results of this paper. The paper is divided into three distinct sections. In the first section, we prove some basic results (see, for instance, Proposition 1.7), and we give a simple description of the “helicoids” examples (Example 1.8). In the second section we prove the following result (Theorem 2.3):

A regular embedded end in the half-space model  $\mathbf{H}^3 = \{(u, v, w) \in \mathbf{R}^3, w > 0\}$  with constant mean curvature 1 and finite total curvature is asymptotic to an embedded or nonembedded catenoid cousin or a horosphere.

In fact, we derive an explicit formula for a vertical graph (see Definition 2.1), and we show that the above convergence holds *in the hyperbolic and in the Euclidean sense*, as Euclidean vertical graphs. We introduce a notion of “growth” for embedded vertical ends in  $\mathbf{H}^3$  (that are regular with finite total curvature).

In the final section of this paper we investigate the following problem: Suppose that a simply connected minimal surface  $M$  in  $\mathbf{R}^3$  is invariant by a Euclidean isometry. Is the associated surface in  $\mathbf{H}^3$  (with constant mean curvature 1) also invariant under an isometry of  $\mathbf{H}^3$ ? We will see that the answer is positive if  $M$  has a plane of symmetry (see Proposition 3.2) or if  $M$  is invariant under a rotation whose axis intersects transversally  $M$  (Proposition 3.6). The answer is negative when  $M$  is invariant under reflection about a straight line that is contained (or a part of which is contained) in  $M$  (Proposition 3.3). We also observe that the Schwarz Reflection Principle *about a geodesic, i.e., rotation by an angle  $\pi$  about a line inside a surface*, does not hold for a constant mean curvature 1 surface in  $\mathbf{H}^3$ ; see, for instance, the helicoidal surfaces containing a geodesic line in hyperbolic space (see Remark 3.4).

**Acknowledgement.** We wish to thank the referee for valuable suggestions and for providing the figures in this paper.

### 1. Basic theory and examples

Let us consider the Lorentz space  $\mathbf{L}^4$  defined as the positively oriented vector space  $\mathbf{R}^4$  equipped with the pseudo-metric of signature  $(-, +, +, +)$ . Then the Minkowski model for  $\mathbf{H}^3$ , denoted by  $\mathcal{H}^3$ , is defined by

$$\mathcal{H}^3 = \{(t, x_1, x_2, x_3) \in \mathbf{R}^4, -t^2 + x_1^2 + x_2^2 + x_3^2 = -1, t > 0\}.$$

Let us denote by  $\langle \cdot, \cdot \rangle$  the Riemannian metric induced on  $\mathcal{H}^3$  by the pseudo-metric. Let  $M$  be a Riemann surface and let  $z = x + iy$  be a local coordinate on  $M$ . Let  $Y: M \rightarrow \mathcal{H}^3$  be a smooth immersion whose mean curvature vector is denoted by  $\vec{H}$ . Assuming that  $\vec{H} \neq \vec{0}$ , we will assume in the following that  $(Y, Y_x, Y_y, \vec{H})(z)$  is a positive basis of  $\mathbf{L}^4$  for each  $z \in M$ . Observe that  $\mathcal{H}^3$  has a natural orientation defined by the map  $f: \mathbf{R}^3 \rightarrow \mathcal{H}^3$ ,  $f(x_1, x_2, x_3) = (\sqrt{1 + x_1^2 + x_2^2 + x_3^2}, x_1, x_2, x_3)$ . In view of our convention on  $Y$ , it then follows that  $(Y_x, Y_y, \vec{H})(z)$  is a positive basis of  $\mathcal{H}^3$  for each  $z \in M$ .

The following result follows from Theorem A in [Br] (see also Theorem 1.1 in [U-Y1]).

**THEOREM 1.1.** *Let  $M$  be a Riemann surface and let  $A, B, C,$  and  $D$  be meromorphic functions on  $M$  satisfying*

$$AD - BC \equiv 1 \quad \text{and} \quad dA \cdot dD - dB \cdot dC \equiv 0.$$

*Consider the smooth map  $Y: M \rightarrow \mathcal{H}^3$  defined by  $Y(z) = (t, x_1, x_2, x_3)(z),$  where*

$$(1.1) \quad \begin{aligned} 2t(z) &= (A\bar{A} + B\bar{B} + C\bar{C} + D\bar{D})(z), \\ (x_1 + ix_2)(z) &= (A\bar{C} + B\bar{D})(z), \\ 2x_3(z) &= (A\bar{A} + B\bar{B} - C\bar{C} - D\bar{D})(z). \end{aligned}$$

*Then  $Y$  defines a conformal and constant mean curvature 1 immersion into  $\mathcal{H}^3$  of the open part of  $M$  defined by*

$$|A \cdot dC - C \cdot dA| \left( 1 + \left| \frac{dD}{dC} \right|^2 \right) \neq 0.$$

*Furthermore, the induced metric on  $M$  is given by*

$$ds^2 = \left[ |A \cdot dC - C \cdot dA| \left( 1 + \left| \frac{dD}{dC} \right|^2 \right) \right]^2.$$

*Conversely, suppose that  $M$  is simply connected, and consider a conformal and constant mean curvature 1 immersion  $Y: M \rightarrow \mathcal{H}^3.$  Then there exist meromorphic functions  $A, B, C,$  and  $D$  on  $M$  with  $AD - BC \equiv 1, A'D' - B'C' \equiv 0,$  such that  $Y$  is given by (1.1).*

**REMARK 1.2.** With the notations of Theorem A in [Br], we have

$$F(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (z),$$

and  $e_0(F) = F \cdot \bar{F}^t, \alpha = D \cdot dA - B \cdot dC, \beta = A \cdot dC - C \cdot dA, \gamma = D \cdot dB - B \cdot dD.$

**COROLLARY 1.3.** *Consider the (oriented) half-space model of the hyperbolic 3-space  $\mathbf{H}^3 = \{(u, v, w) \in \mathbf{R}^3, w > 0\}$  with the hyperbolic metric  $\langle \cdot, \cdot \rangle_2 = \frac{|du|^2 + |dv|^2 + |dw|^2}{w^2}.$  Let  $M$  be a Riemann surface and let  $A, B, C,$  and  $D$  be meromorphic functions on  $M$  satisfying*

$$AD - BC \equiv 1, \quad dA \cdot dD - dB \cdot dC \equiv 0, \quad \text{and} \quad |A \cdot dC - C \cdot dA| \left( 1 + \left| \frac{dD}{dC} \right|^2 \right) \neq 0.$$

*Consider the immersion  $Y: M \rightarrow \mathcal{H}^3$  defined in Theorem 1.1. Then  $Y$  gives rise to a conformal immersion  $\tilde{X}: M \rightarrow \mathbf{H}^3$  with constant mean curvature 1, defined by  $\tilde{X}(z) = (u, v, w)(z),$  where*

$$(1.2) \quad (u + iv)(z) = \frac{\bar{A}C + \bar{B}D}{|A|^2 + |B|^2}(z), \quad w(z) = \frac{1}{|A|^2 + |B|^2}(z).$$

Furthermore, the induced metric on  $\mathbf{H}^3$  is given by

$$ds^2 = \left[ |A \cdot dC - C \cdot dA| \left( 1 + \left| \frac{dD}{dC} \right|^2 \right) \right]^2,$$

and  $(\tilde{X}_x, \tilde{X}_y, \tilde{H})(z)$  is a positive basis of  $T_{\tilde{X}(z)}\mathbf{H}^3$ , where  $\tilde{H}$  is the mean curvature vector of  $\tilde{X}$ .

*Proof.* Let us equip the unit 3-ball  $\mathbf{B}^3$  with the hyperbolic metric  $\langle \cdot, \cdot \rangle_1 = 4 \frac{|dX|^2}{(1-|X|^2)^2}$ , where  $X = (X_1, X_2, X_3) \in \mathbf{B}^3$ . Then the stereographic projection  $F: (\mathcal{H}^3, \langle \cdot, \cdot \rangle) \rightarrow (\mathbf{B}^3, \langle \cdot, \cdot \rangle_1)$  defined by  $F(t, x_1, x_2, x_3) = (X_1, X_2, X_3)$ , where

$$X_1 + iX_2 = \frac{x_1 + ix_2}{1+t} \text{ and } X_3 = \frac{x_3}{1+t},$$

is an orientation preserving isometry. Thus one gets the geometrical image of the surface  $Y(M)$  in the ball model of the hyperbolic 3-space. Now observe that the map  $J: (\mathbf{B}^3, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathbf{H}^3, \langle \cdot, \cdot \rangle_2)$ , defined by

$$J(X_1, X_2, X_3) = 2 \frac{(X_1, -X_2, X_3 + 1)}{|(X_1, X_2, X_3 + 1)|^2} - (0, 0, 1),$$

is an orientation preserving isometry. We conclude that

$$(J \circ F)(t, x_1, x_2, x_3) = \frac{1}{t+x_3}(x_1, -x_2, 1).$$

We then see that the constant mean curvature 1 immersion of  $M$  into  $\mathbf{H}^3$ ,  $\tilde{X}(z) = (J \circ F \circ Y)(z) = (u, v, w)(z)$ , is given by (1.2). Moreover, as  $J \circ F$  is an orientation preserving isometry between  $\mathcal{H}^3$  and  $\mathbf{H}^3$ , we deduce that  $(\tilde{X}_x, \tilde{X}_y, \tilde{H})(z)$  is a positive basis of  $T_{\tilde{X}(z)}\mathbf{H}^3$  for each  $z \in M$ .  $\square$

REMARK 1.4.

(1) Suppose now that  $M \subset \mathbf{C}$  is a simply connected domain. Let  $g$  (resp.,  $\omega$ ) be a meromorphic function (resp., holomorphic form) on  $M$  such that for any  $z_0 \in M$ ,  $z_0$  is a pole of  $g$  if and only if  $z_0$  is a zero of  $\omega$  with multiplicity twice that of the pole of  $g$  at  $z_0$ . Then  $(g, \omega)$  defines a conformal and minimal immersion  $X: M \rightarrow \mathbf{R}^3$  by setting

$$X(z) = \operatorname{Re} \int ((1 - g^2)\omega, i(1 + g^2)\omega, 2g\omega).$$

The pair  $(g, \omega)$  is called the *Weierstrass representation* of the immersion  $X$ ; see [Os, Chapter 8]. The induced metric is  $ds^2 = [(1 + |g|^2)|\omega|]^2$ . Consider  $\Pi = -2 \operatorname{Re}(\omega dg)$ , the second fundamental form of  $X$ . Setting  $\tilde{\Pi} = \Pi + ds^2$ , it is easily seen that  $ds^2$  and  $\tilde{\Pi}$  satisfy the Gauss and Codazzi equations in  $\mathbf{H}^3$ . It follows that there exists a conformal immersion  $\tilde{X}: M \rightarrow \mathbf{H}^3$  whose

induced metric is  $ds^2$  and whose second fundamental form is  $\tilde{\Pi}$ . Furthermore,  $\tilde{X}$  has constant mean curvature 1.

Conversely, consider a conformal and constant mean curvature 1 immersion  $\tilde{X}: M \rightarrow \mathbf{H}^3$ . Let  $ds^2$  be the induced metric and  $\tilde{\Pi}$  the second fundamental form of  $\tilde{X}$ . Setting  $\Pi = \tilde{\Pi} - ds^2$ , the forms  $ds^2$  and  $\Pi$  satisfy the Gauss and Codazzi equations in  $\mathbf{R}^3$ . Therefore, they define a conformal and minimal immersion  $X: M \rightarrow \mathbf{R}^3$ .

We will call the immersions  $\tilde{X}$  and  $X$  (or the surfaces  $\tilde{X}(M)$  and  $X(M)$ ) *associated*; see [La].

(2) In the context of item (1), let  $A, B, C$ , and  $D$  denote the meromorphic functions on  $M$  defining the immersion  $\tilde{X}: M \rightarrow \mathbf{H}^3$ . Consider the Weierstrass representation  $(g, \omega)$  of the associated immersion  $X: M \rightarrow \mathbf{R}^3$ . Then we have

$$(1.3) \quad F^{-1} \cdot dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega,$$

where

$$F(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (z).$$

In particular, we have  $g = \frac{A'D - BC'}{AC' - A'C} = -\frac{D'}{C'}$  and  $\omega = AC' - A'C$  (see Section 1 of [U-Y1]). We call  $(g, \omega)$  the Weierstrass representation associated to the immersion  $\tilde{X}$ . Observe that, given  $A, B, C$ , and  $D$ , relation (1.3) shows how we can find the Weierstrass representation of  $\tilde{X}$ . Using this relation and setting  $\omega = \eta dz$ , Umehara and Yamada showed that  $A$  and  $C$  are independent solutions of the differential equation

$$(1.4) \quad X'' - \frac{\eta'}{\eta} X' - \eta g' X = 0,$$

and that  $B$  and  $D$  are independent solutions of

$$(1.5) \quad Y'' - \frac{(g^2 \eta)'}{g^2 \eta} Y' - g' \eta Y = 0;$$

see [U-Y1, Lemma 2.1]. Observe that, if  $A$  and  $C$  were not independent, the conditions on  $A, B, C$  and  $D$  would force  $A$  and  $C$  to be constant. Thus, the associated minimal surface would be plane, since (1.4) implies that  $g$  is constant. Hence, the constant mean curvature 1 surface in  $\mathbf{H}^3$  is a horosphere.

EXAMPLE 1.5. In [Br, Example 2, p. 341] Bryant called *catenoid cousins* the constant mean curvature 1 surfaces in  $\mathbf{H}^3$  defined by the functions  $A, B, C$ , and  $D$  on  $\mathbf{C}^*$ , where (writing  $\mu$  for  $\alpha$  and  $z$  for  $z^{-1}$  in [Br])

$$F(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\sqrt{2\alpha + 1}} \begin{pmatrix} (\alpha + 1)z^{-\alpha} & \alpha z^{\alpha+1} \\ \alpha z^{-\alpha-1} & (\alpha + 1)z^\alpha \end{pmatrix},$$

where  $\alpha$  is a real number satisfying  $\alpha > -1/2$  and  $\alpha \neq 0$ . This implies

$$(F^{-1} \cdot F')(z) = \frac{\alpha(\alpha + 1)}{2\alpha + 1} \begin{pmatrix} -z^{-1} & z^{2\alpha} \\ -z^{-2\alpha-2} & z^{-1} \end{pmatrix}.$$

Hence the Weierstrass representation of catenoid cousins is  $g(z) = z^\mu$  and  $\omega = \frac{1-\mu^2}{4\mu} z^{-1-\mu} dz$ , where  $z \in \mathbf{C}^*$  and  $\mu = 1 + 2\alpha$ . Observe that  $\mu > 0$  and  $\mu \neq 1$ . It is known that the catenoid cousin is embedded if and only if  $0 < \mu < 1$ ; see [Br] and [U-Y1]. To determine the minimal surface in  $\mathbf{R}^3$  that is locally associated to the catenoid cousin, we make the change of variable  $u = z^\mu$ . Thus, in terms of the  $u$ -variable,  $g(u) = u$  and  $\omega = \frac{1-\mu^2}{4\mu^2} u^{-2} du$ . Therefore, we get the Weierstrass representation of the Euclidean catenoid. We conclude that catenoid cousins in  $\mathbf{H}^3$  are locally associated to Euclidean catenoids of  $\mathbf{R}^3$ . However, the converse is not true: in fact, consider the Weierstrass representation  $g(u) = u$  and  $\omega = au^{-2} du$ , where  $a < -1/4$ , which gives catenoids in  $\mathbf{R}^3$ . We show in Example 1.8 that the constant mean curvature 1 surfaces that are locally associated in  $\mathbf{H}^3$  are not rotational surfaces, but there are surfaces that are invariant under a one parameter group of hyperbolic translations. Observe that  $\frac{1-\mu^2}{4\mu^2}$  assumes every real value  $a$  with  $a > -1/4$  and  $a \neq 0$  (because  $\mu > 0$  and  $\mu \neq 1$ ). We know from [Br] and [U-Y1] that catenoid cousins are rotational surfaces, and that there are no other rotational surfaces in  $\mathbf{H}^3$  with constant mean curvature 1.

REMARK 1.6.

(1) If  $M$  is not simply connected we can still carry out the construction given in Remark 1.4. However, in this case the functions  $A, B, C$ , and  $D$  may be multi-valued, and so the immersion into  $\mathcal{H}^3$  given by (1.1) also may be multi-valued.

(2) Suppose  $M$  is simply connected, and suppose that  $A, B, C, D$ , and  $(g, \omega)$  on  $M$  satisfying (1.3) and  $AD - BC \equiv 1$  are given. Choose any constant matrix  $T \in \mathbf{SL}(2, \mathbf{C})$  and set  $\hat{F}(z) = T \cdot F(z)$ . We call  $\tilde{X}$  (resp.,  $\hat{X}$ ) the conformal and constant mean curvature 1 immersion of  $M$  into  $\mathbf{H}^3$  defined by  $F$  (resp.,  $\hat{F}$ ). Observe that  $F^{-1} \cdot dF = \hat{F}^{-1} \cdot d\hat{F}$ . Hence  $\tilde{X}$  and  $\hat{X}$  have the same induced metric,  $ds^2 = [(1 + |g|^2) |\omega|]^2$ , and the same second fundamental form,  $\tilde{\Pi} = -2 \operatorname{Re}(\omega dg) + ds^2$ . Therefore the two immersions are equal up to a global isometry of  $\mathbf{H}^3$ . That is, there exists an isometry  $S: \mathbf{H}^3 \rightarrow \mathbf{H}^3$  such that  $\hat{X} = S \circ \tilde{X}$ .

PROPOSITION 1.7. *Let  $M \subset \mathbf{C}$  be a simply connected domain. Consider a conformal and minimal immersion  $X: M \rightarrow \mathbf{R}^3$ . Let  $(g, \omega)$  be the Weierstrass representation of  $X$ . Let  $A$  be any nonconstant solution of (1.4) (where  $\omega =$*

$\eta dz$ ) and let  $H$  be a meromorphic function on  $M$  satisfying  $H' = \eta/A^2$ . Set

$$B = -gA + \frac{A'}{\eta}, \quad C = HA, \quad D = \frac{1}{A} - gHA + H\frac{A'}{\eta}.$$

Then  $A$  and  $C$  (resp.,  $B$  and  $D$ ) are independent solutions of (1.4) (resp., of (1.5)). Furthermore  $A, B, C,$  and  $D$  satisfy  $AD - BC \equiv 1$  and (1.3). In particular, the function  $\tilde{X}: M \rightarrow \mathbf{H}^3$  defined by (1.2) is a conformal immersion and has constant mean curvature 1.

*Proof.* Using the fact that  $A$  is a solution of (1.4) and using the definition of the function  $H$ , we find

$$\begin{aligned} B' &= -gA' & \text{and} & & B'' &= -g'A' - gA'', \\ C' &= \frac{\eta}{A} + HA' & \text{and} & & C'' &= \frac{\eta'}{A} + HA'', \\ D' &= -g\frac{\eta}{A} - gHA' & \text{and} & & D'' &= -g'\frac{\eta}{A} - g\frac{\eta'}{A} - g'HA' - gHA''. \end{aligned}$$

From this we deduce that  $A$  and  $C$  are two independent solutions of (1.4) and that  $B$  and  $D$  are two independent solutions of (1.5). Also, a straightforward computation shows that  $AD - BC \equiv 1$ . Furthermore, setting

$$F(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (z),$$

we have

$$F^{-1} \cdot F' = \begin{pmatrix} A'D - BC' & DB' - BD' \\ AC' - CA' & AD' - CB' \end{pmatrix}.$$

Hence the functions  $A, B, C,$  and  $D$  satisfy (1.3). The last assertion of the proposition follows from Corollary 1.3. □

**EXAMPLE 1.8.** We determine the constant mean curvature 1 surfaces in  $\mathbf{H}^3$  locally associated to the family of minimal surfaces of  $\mathbf{R}^3$  that are isometric to the helicoid. We know that the Weierstrass representation of the helicoids and the isometric minimal surfaces is given by  $g(z) = e^z, \omega = \lambda e^{i\theta} \cdot e^{-z} dz$ , where  $z \in \mathbf{C}, \lambda > 0$  and  $\theta \in [0, 2\pi[$ . When  $e^{i\theta} = \pm 1$  we get the catenoids, and when  $e^{i\theta} = \pm i$  we get the helicoids. Let us find the functions  $A, B, C,$  and  $D$  associated to  $g$  and  $\omega$ . Remark 1.4(2) shows that  $A$  and  $C$  are solutions of the second order differential equation:

$$(*) \quad X'' + X' - \lambda e^{i\theta} X = 0.$$

The characteristic equation of this differential equation is

$$(**) \quad \gamma^2 + \gamma - \lambda e^{i\theta} = 0.$$

Let  $\gamma$  be a solution of the characteristic equation. Then the function  $A(z) = e^{\gamma z}$  is a solution of (\*).



Suppose that  $1 + 2\gamma \neq 0$ . In the notation of Proposition 1.7, we set  $H(z) = -\frac{\lambda e^{i\theta}}{1+2\gamma} e^{-(1+2\gamma)z}$ . Then,

$$\begin{aligned} A(z) &= e^{\gamma z}, & B(z) &= -\frac{\gamma}{1+\gamma} e^{(1+\gamma)z}, \\ C(z) &= -\frac{\gamma(1+\gamma)}{1+2\gamma} e^{-(1+\gamma)z}, & D(z) &= \frac{(1+\gamma)^2}{1+2\gamma} e^{-\gamma z}. \end{aligned}$$

By (1.2) it follows that (with  $z = x + iy$ )

$$\begin{aligned} (u + iv)(z) &= -e^{2\gamma_2 y} \cdot e^{-x(1+2\gamma_1)} \cdot e^{-i(y+2Im(\gamma z))} \\ &\quad \times \frac{(1+\gamma)^2}{1+2\gamma} \cdot \frac{\bar{\gamma}(1+\gamma)e^x + \gamma(1+\bar{\gamma})e^{-x}}{|1+\gamma|^2 e^{-x} + |\gamma|^2 e^x} \end{aligned}$$

and

$$w(z) = |1+\gamma|^2 \frac{e^{2\gamma_2 y - 2\gamma_1 x}}{|1+\gamma|^2 + |\gamma|^2 e^{2x}},$$

where  $\gamma = \gamma_1 + i\gamma_2$ .

If  $\gamma_2 = 0$ , we easily verify that each vertical line  $\{x = x_0\}$  is mapped under the immersion onto a horizontal circle whose center lies on the geodesic line  $\{u = v = 0\}$  of  $\mathbf{H}^3$ . Hence the whole surface is a rotational surface and therefore is a catenoid cousin (embedded or non-embedded); see Example 1.5, [Br] or [U-Y1].

On the other hand, suppose that  $\gamma_2 \neq 0$ . Then, for any real numbers  $y, t$ , and  $x_0$  we have

$$(u + iv, w)(x_0, y + t) = e^{2\gamma_2 t} (e^{-it(1+2\gamma_1)}(u + iv), w)(x_0, y).$$

Thus, each vertical line  $\{x = x_0\}$  is mapped onto a *helix* (in the hyperbolic sense) about the vertical geodesic line  $\{u = v = 0\}$  in  $\mathbf{H}^3$ . This means that the associated surface  $M(\lambda, \theta)$  is a helicoidal surface in  $\mathbf{H}^3$  whose axis is the geodesic line  $\{u = v = 0\}$ . Observe also that the axis stays on  $M(\lambda, \theta)$  if and only if the vertical line  $\{x = 0\}$  is mapped onto the geodesic line  $\{u = v = 0\}$ ; this occurs if and only if  $\text{Re}(\bar{\gamma}(1+\gamma)) = 0$ . In fact, there exists a family of such surfaces containing the axis. We obtain an element of this family of surfaces by choosing any real number  $\gamma_1 \in ]-1, 0[$  and setting  $\gamma = \gamma_1 \pm i\sqrt{-\gamma_1(1+\gamma_1)}$ . We clearly have  $\text{Re}(\bar{\gamma}(1+\gamma)) = 0$ , and we then set  $\lambda = |\gamma^2 + \gamma|$  and  $\theta = \text{Arg}(\gamma^2 + \gamma)$ .

Finally, by computing the discriminant of (\*\*), we find for the case  $1 + 2\gamma \neq 0$  that  $\gamma_2 = 0$  if and only if  $e^{i\theta} = 1$  or  $e^{i\theta} = -1$  and  $\lambda < 1/4$ . By the change of variables  $u = e^z$  (assuming  $\gamma_2 = 0$ ), we obtain, in terms of the  $u$ -variable,  $g(u) = u$  and  $\omega = au^{-2} du$ , where  $a = \lambda e^{i\theta} \in \mathbf{R}$  (since  $e^{i\theta} = \pm 1$ ). We recognize the catenoids, which we will denote by  $C_a$ . Hence we conclude that, when  $a > -1/4$ , the constant mean curvature 1 surface in  $\mathbf{H}^3$  that is locally associated to the catenoid  $C_a$  is a catenoid cousin, and thus

a rotational surface. When  $a < -1/4$ , the catenoid  $C_a$  is locally associated to a (nonrotational) surface in  $\mathbf{H}^3$  that is invariant under a one parameter group of hyperbolic translations, with constant mean curvature 1; see Figure 1 for the case when  $\lambda = 1/2$ ,  $e^{i\theta} = -1$ , and Figure 2 below for the case when  $\lambda = 5/4$ ,  $e^{i\theta} = -1$ . Observe that the remaining case  $a = -1/4$  corresponds to  $\lambda = 1/4$  and  $e^{i\theta} = -1$ ; that is,  $1 + 2\gamma = 0$ . We will consider this case later. Conversely, we see from Example 1.5 that the minimal surface of  $\mathbf{R}^3$  that is locally associated to a catenoid cousin is a catenoid  $C_a$  with  $a > -1/4$ .

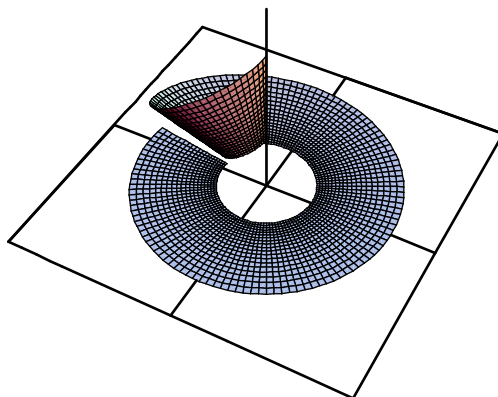


FIGURE 1

Suppose now that  $1 + 2\gamma = 0$ ; that is,  $\lambda = 1/4$  and  $e^{i\theta} = -1$ . We have  $A(z) = e^{-z/2}$ , and we choose  $H(z) = -z/4$ . From Proposition 1.7 we obtain  $B(z) = e^{z/2}$ ,  $C(z) = (-z/4)e^{-z/2}$ , and  $D(z) = (1 - z/4)e^{z/2}$ . We get

$$(u + iv)(z) = \frac{e^x}{e^x + e^{-x}} - \frac{x}{4} - i\frac{y}{4},$$

$$w(z) = \frac{1}{e^x + e^{-x}}.$$

Now, each vertical line  $\{x = x_0\}$  is mapped onto a (Euclidean) horizontal straight line that is parallel to the  $v$ -axis. Hence the surface in  $\mathbf{H}^3$  that is locally associated to the catenoid  $C_{-1/4}$  is a complete and constant mean curvature 1 surface that is invariant under the horizontal translations  $X \rightarrow X + (0, v, 0)$  for any  $v \in \mathbf{R}$ , where  $X \in \mathbf{H}^3$ ; see Figure 3 below.

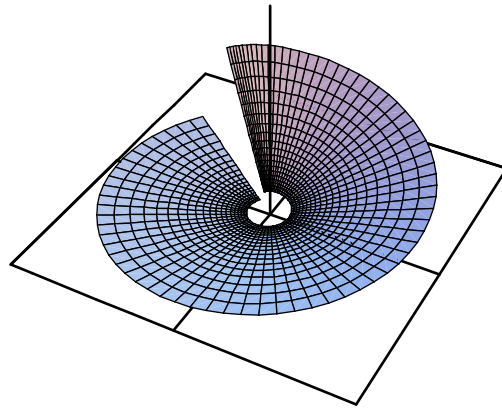


FIGURE 2

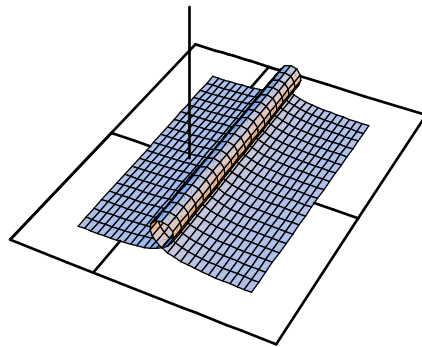


FIGURE 3

We now resume our discussion of the surfaces in Example 1.8. In the case when  $e^{i\theta} = 1$ , or  $e^{i\theta} = -1$  and  $\lambda < 1/4$ , the associated surface in  $\mathbf{H}^3$  is

rotational. We deduce from Example 1.5 that this surface is embedded in  $\mathbf{H}^3$  if and only if  $e^{i\theta} = 1$ .

In the case when  $e^{i\theta} = -1$  and  $\lambda = 1/4$ , the associated surface is invariant under a Euclidean horizontal translation in  $\mathbf{H}^3$  and is an Enneper cousin dual (see Remark 1.11). (We observe that these Euclidean horizontal translations are indeed parabolic isometries of hyperbolic space.)

When  $e^{i\theta} = -1$  and  $\lambda > 1/4$ , the associated surface is invariant by a one parameter group of hyperbolic translations. In all other cases, the associated surface is a helicoidal surface.

DEFINITION 1.9. Let  $M$  be a Riemann surface, and let  $\tilde{X}: M \rightarrow \mathbf{H}^3$  be a conformal immersion such that  $(\tilde{X}_x, \tilde{X}_y, \tilde{H})(z)$  is a positive basis of  $T_{\tilde{X}(z)}\mathbf{H}^3$  for any  $z \in M$ , where  $\tilde{H}$  is the mean curvature vector of  $\tilde{X}$ . Geometrically, the *hyperbolic Gauss map*,  $G$ , is constructed as follows (see [Br]): For a given point  $p \in S = \tilde{X}(M)$ , consider the geodesic through  $p$  orthogonal to  $S$ , oriented by  $\tilde{H}(p)$ . Then  $G(p) \in \partial_\infty\mathbf{H}^3$  is the limit point of this geodesic.

Under the assumption that  $S$  is not totally umbilic, Bryant showed that  $G$  is meromorphic if and only if  $S$  has constant mean curvature 1; see [Br, Proposition 1] and also [Ga-Go, Theorem 1]. We have:

LEMMA 1.10. *In the notation of Corollary 1.3 and its proof, the hyperbolic Gauss map of  $\tilde{X}(M)$  is given by*

$$G = \frac{dC}{dA} = \frac{dD}{dB}.$$

*Proof.* We recall that the Klein model  $\mathcal{D}$  for the hyperbolic 3-space is obtained from  $\mathcal{H}^3$  via the projection  $\Phi: \mathcal{H}^3 \rightarrow \mathcal{D}$ ,  $\Phi(t, x_1, x_2, x_3) = (1, \frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t})$ . Observe that  $\mathcal{D} = \{(1, a, b, c) \in \mathbf{R}^4; a^2 + b^2 + c^2 < 1\}$ . We equip  $\mathcal{D}$  with the metric and the orientation induced by  $\Phi$ . Then  $I = J \circ F \circ \Phi^{-1}: \mathcal{D} \rightarrow \mathbf{H}^3$  is an orientation preserving isometry and we have  $I(1, a, b, c) = \frac{1}{1+c}(a, -b, \alpha)$ , where  $\alpha^2 = 1 - (a^2 + b^2 + c^2)$ . Clearly, we can extend  $I$  continuously to the asymptotic boundary of  $\mathcal{D}$ , by setting  $I_\infty: \partial_\infty\mathcal{D} \rightarrow \partial_\infty\mathbf{H}^3 = \mathbf{C} \cup \{\infty\}$ , with  $I_\infty(1, a, b, c) = \frac{a-ib}{1+c}$ . (Observe that  $a^2 + b^2 + c^2 = 1$ .)

Now, let  $Y: M \rightarrow \mathcal{H}^3$  be the conformal and constant mean curvature 1 immersion which gives rise to  $\tilde{X}$ ; that is,  $\tilde{X} = (J \circ F) \circ Y$ . Let  $n: M \rightarrow \partial_\infty\mathcal{D}$  be the hyperbolic Gauss map of  $\Phi \circ Y$ . By [Ga-Go, Lemma 1] we have

$$n(z) = \frac{Y + N}{(Y_0 + N_0)}(z),$$

where  $Y = (Y_0, Y_1, Y_2, Y_3)$  and  $N = (N_0, N_1, N_2, N_3)$  is the unitary normal field on  $Y(M)$  such that  $(Y, Y_x, Y_y, N)(z)$  is a positive basis of  $\mathbf{L}^4$ . As  $Y$  is a

conformal immersion, we have

$$\langle Y_{z\bar{z}}, Y_x \rangle = \langle Y_{z\bar{z}}, Y_y \rangle = 0, \quad \langle Y_{z\bar{z}}, Y \rangle = -\frac{\lambda^2}{2},$$

where  $\langle \cdot, \cdot \rangle$  is the pseudo-metric on  $\mathbf{L}^4$  and  $\lambda^2 = \langle Y_x, Y_x \rangle = \langle Y_y, Y_y \rangle$ . Moreover, as  $Y(M)$  has constant mean curvature 1 with respect to the normal orientation given by  $N$ , we obtain

$$\langle Y_{z\bar{z}}, N \rangle = \frac{\lambda^2}{2}.$$

We deduce

$$Y + N = \frac{2}{\lambda^2} Y_{z\bar{z}},$$

and hence

$$n(z) = \frac{Y_{z\bar{z}}}{(Y_0)_{z\bar{z}}}.$$

In terms of the functions  $A, B, C$ , and  $D$  (defined in (1.1)), we obtain locally

$$G(z) = (I_\infty \circ n)(z) = \frac{\bar{A}'C' + \bar{B}'D'}{|A'|^2 + |B'|^2}(z),$$

where the dash denotes the derivative with respect to  $z$ . Using the relation  $A'D' - B'C' = 0$ , we conclude

$$G = \frac{C'}{A'} = \frac{dC}{dA} = \frac{dD}{dB}. \quad \square$$

REMARK 1.11.

(1) Our expression for the hyperbolic Gauss map  $G$  differs slightly from that of Bryant (see [Br, page 339]) where, with our notations,  $G = [e_0 + e_3]$  and  $A = F_1, C = F_3$ . This is due to different choices of the isometries. In fact, in Bryant's paper the surface is immersed in the Minkowski model of hyperbolic space, while in Lemma 1.10, the surface is immersed in the half-space model. Of course, the expression for  $G$  depends on the choice of the immersion. For example, in  $\mathbf{H}^3$  let  $I$  be the orthogonal reflection with respect to the vertical plane  $\{v = 0\}$  composed with the reflection with respect to the unit sphere centered at 0. Note that  $I$  is an orientation preserving isometry of  $\mathbf{H}^3$  and extends to  $\partial_\infty \mathbf{H}^3 = \mathbf{C} \cup \{\infty\}$  as the map  $z \rightarrow z^{-1}$ . Then, if  $G = \frac{dC}{dA}$  is the hyperbolic Gauss map of  $\tilde{X}$ , we deduce that the hyperbolic Gauss map of the immersion  $I \circ \tilde{X}$  is  $G_I = G^{-1} = \frac{dA}{dC}$ .

(2) Let  $\tilde{X}: M \rightarrow \mathbf{H}^3$  be a constant mean curvature 1 immersion defined by a matrix  $F$ , as in Remark 1.4(2). The immersion defined by the inverse matrix  $F^{-1}$  is called the dual immersion, according to Rossman, Umehara, and Yamada (see [R-U-Y]). Let  $(g, \omega)$  be the Weierstrass representation of

the minimal surface in  $\mathbf{R}^3$  associated to  $F$ . Then, using (1.3) and letting  $(g^\#, \omega^\#)$  be the Weierstrass representation associated to  $F^{-1}$ , we get

$$g^\# = \frac{1}{G}, \quad \omega^\# = -\frac{g'}{\left(\frac{1}{G}\right)}\omega.$$

This expression differs from that of [R-U-Y] because our formula for  $G$  is different; see Remark 1.11(1) above. Consider, for instance, Example 1.8 with  $e^{i\theta} = -1$  and  $\lambda = 1/4$ . Using the formulas for the functions  $A, B, C$ , and  $D$  given in this example, we find, via Lemma 1.10, that  $G(z) = \frac{2-z}{4}$ . The Weierstrass representation for the dual surface then gives

$$g^\#(z) = \frac{4}{2-z}, \quad \omega^\# = \frac{(2-z)^2}{16} dz.$$

Making the substitution  $\zeta = \frac{4}{2-z}$ , we obtain

$$g^\#(\zeta) = \zeta, \quad \omega^\# = \frac{4}{\zeta^4} d\zeta.$$

This is the Weierstrass representation of the Enneper minimal surface in  $\mathbf{R}^3$ . To see this, let  $(x_1, x_2, x_3)$  be the coordinates in  $\mathbf{R}^3$ , and consider a rotation by an angle  $\pi$  with respect to the  $x_1$ -axis. Setting  $z = 1/\zeta$ , we obtain  $g^\#(z) = z$  and  $\omega^\# = 4 dz$ . Thus, the surface in Example 1.8 with  $e^{i\theta} = -1$  and  $\lambda = 1/4$  is an Enneper cousin dual.

**2. Asymptotic behavior and classification of embedded regular ends in  $\mathbf{H}^3$  with mean curvature 1 and finite total curvature**

DEFINITION 2.1.

(1) Set  $\mathbf{D}^* = \{z \in \mathbf{C}, 0 < |z| < 1\}$ . Let  $\tilde{X}: \mathbf{D}^* \rightarrow \mathbf{H}^3$  be a conformal and constant mean curvature 1 immersion. Suppose that, for each path  $\gamma: [0, 1[ \rightarrow \mathbf{D}^*$  such that  $\gamma(t) \rightarrow 0$  when  $t \rightarrow 1$ , the path  $\tilde{X}(\gamma)$  has infinite hyperbolic length in  $\mathbf{H}^3$ . We then say that  $\tilde{X}$  (or  $\tilde{X}(\mathbf{D}^*)$ ) is an *end*, and we will denote it by  $E$ . In [U-Y1], Umehara and Yamada called an end *regular* if the hyperbolic Gauss map  $G$  extends analytically to the puncture 0.

(2) We define a *vertical graph* in  $\mathbf{H}^3$  as any surface such that the third coordinate is a function of the two others,  $w = w(u, v)$ .

REMARK 2.2.

(1) Let us consider a regular end of  $\mathbf{H}^3$  with finite total curvature which is not part of a horosphere. According to Bryant (see [Br]) the associated Weierstrass representation  $(g, \omega)$  (see Remark 1.4(2)) has the following general form, which may be compared with the Weierstrass representation of catenoid cousins, given in Example 1.5:

$$(2.1) \quad g(z) = z^\mu \cdot f(z), \quad \omega = z^\nu \cdot h(z) dz.$$

Here  $z \in \mathbf{D} = \{z \in \mathbf{C}, |z| < 1\}$ ,  $f$  and  $h$  are holomorphic functions on  $\mathbf{D}$  with  $f(0), h(0) \neq 0$ ,  $\mu, \nu \in \mathbf{R}$ ,  $\mu > 0$ ,  $\nu \leq -1$ ,  $\nu + \mu \in \mathbf{Z}$ , and  $\nu + \mu \geq -1$ . For the sake of completeness we give a proof of this result.

As the end has finite total curvature, it follows from [Br, Proposition 4] that  $g$  has the form  $g(z) = z^\mu f(z)$ , where  $\mu > 0$  and  $f$  is a holomorphic function on  $\mathbf{D}$ , such that  $f(0) \neq 0$ . In the notation of Proposition 4 in [Br], we have  $\mu = 1 + \beta$ . Observe that the metric  $|\omega| (1 + |g|^2)$  is well defined on  $\mathbf{D}^*$ . Setting  $\omega = \eta(z) dz$ , we see that  $|\eta|$  is well defined on  $\mathbf{D}^*$ , although  $\eta$  may not be defined. For any  $z \in \mathbf{D}^*$  we set

$$\tilde{\eta}(z) = \lim_{t \rightarrow 2\pi} \eta(e^{ti}z), \quad 0 \leq t \leq 2\pi.$$

Note that  $\frac{\tilde{\eta}(z)}{\eta(z)}$  is locally defined and holomorphic on  $\mathbf{D}^*$ . As  $\left| \frac{\tilde{\eta}(z)}{\eta(z)} \right| \equiv 1$ , we see that this function is, in fact, defined on  $\mathbf{D}^*$  and is constant. Thus there exists  $\alpha \in \mathbf{R}$  such that  $\frac{\tilde{\eta}(z)}{\eta(z)} \equiv e^{i2\pi\alpha}$ . Hence  $\eta(z)$  has the form  $\eta(z) = z^\nu h(z)$ , where  $h$  is a holomorphic function on  $\mathbf{D}$  such that  $h(0) \neq 0$  and  $\nu \in \mathbf{R}$  satisfies  $\nu - \alpha \in \mathbf{Z}$ . As the metric  $ds^2 = [|\omega| (1 + |g|^2)]^2$  is complete at 0, we deduce that  $\nu \leq -1$ . Furthermore, by [Br, Proposition 5] we have  $\nu + \mu \in \mathbf{Z}$  (since the immersion is complete at 0 and has finite total curvature). Moreover, as the end is regular, we deduce from [Br, Proposition 6] that  $\nu + \mu \geq -1$ .

(2) We know from the work of Bryant that the end of the Enneper cousin is not regular although the total curvature is finite (see [Br, Example 1]). Conversely, consider Example 1.8 with  $e^{i\theta} = -1$  and  $\lambda = 1/4$ . The hyperbolic Gauss map in this case is  $G(z) = \frac{2-z}{4}$ . We deduce that the end of this surface is regular. Also, as this surface is translationally invariant, the end has infinite total curvature.

(3) In the sequel we shall need some results from ODE theory, which can be found, for example, in [G-S, Chapters 15.1 and 15.2]. In the context of (1), let  $A, B, C$ , and  $D$  denote the analytic (possibly multi-valued) functions associated to the immersion (or to the end), and let  $\tilde{f} = \mu f + z f'$ . Then  $A$  and  $C$  are independent solutions of

$$(2.2) \quad X'' - \frac{(z^\nu h)'}{z^\nu h} X' - h \tilde{f} z^{\nu+\mu-1} X = 0,$$

and  $B$  and  $D$  are independent solutions of

$$(2.3) \quad Y'' - \frac{(z^{\nu+2\mu} h f^2)'}{z^{\nu+2\mu} h f^2} Y' - h \tilde{f} z^{\nu+\mu-1} Y = 0,$$

satisfying  $AD - BC = 1$  and  $A'D' - B'C' = 0$ ; see Section 1. We observe that, since  $\nu + \mu \geq -1$  (because the end is regular), 0 is a regular singularity for both equations. The indicial equation associated to (2.2) and (2.3) are

$$\lambda^2 - (1 + \nu)\lambda - q = 0 \text{ and } r^2 - (1 + \nu + 2\mu)r - q = 0,$$

respectively, where

$$q = \begin{cases} 0 & \text{if } \nu + \mu \geq 0, \\ \mu f(0)h(0) & \text{if } \nu + \mu = -1. \end{cases}$$

We now state our main result, which we will prove by combining several independent lemmas with Propositions 2.7, 2.8, and 2.10.

**THEOREM 2.3.** *Let  $E$  be a regular embedded end into  $\mathbf{H}^3$  with constant mean curvature 1 and finite total curvature. Then one of the following three cases holds:*

- (i)  $E$  is asymptotic to an embedded catenoid cousin.
- (ii)  $E$  is asymptotic to a nonembedded catenoid cousin.
- (iii)  $E$  is asymptotic to a horosphere.

Furthermore, up to an isometry of  $\mathbf{H}^3$ , the end  $E$  is a vertical Euclidean graph over an exterior domain in  $\partial_\infty \mathbf{H}^3$ . In this situation,  $E$  is asymptotic to either an end of a catenoid cousin or a horosphere, regarded as vertical Euclidean graphs. More precisely, if  $w_E$  (resp.,  $w_C$ ) is the function whose graph is  $E$  (resp., the end of the catenoid cousin or the horosphere), then

$$\lim_{u^2+v^2 \rightarrow \infty} (w_E - w_C)(u, v) = 0.$$

Moreover,

$$\lim_{u^2+v^2 \rightarrow \infty} \log \left( \frac{w_E}{w_C} \right) (u, v) = 0,$$

so  $E$  is also asymptotic in the hyperbolic sense.

We remark that it suffices to assume that  $E$  is a properly embedded end since, by [C-H-R, Theorem 10], such an end must be regular and have finite total curvature.

*Proof.* The Weierstrass representation  $(g, \omega)$  associated to the end  $E$  is given by (2.1). We first treat the case where  $\nu + \mu = -1$  (see Lemmas 2.4 and 2.5 and Propositions 2.7 and 2.8), and then consider the case where  $\nu + \mu \geq 0$  (see Lemma 2.9 and Proposition 2.10). □

**LEMMA 2.4.** *Consider the Weierstrass representation  $(g, \omega)$  on  $\mathbf{D}^*$ , given by (2.1), with  $\mu > 0$ . Suppose that  $\nu + \mu = -1$ . Then  $(g, \omega)$  determines an embedding of  $\mathbf{D}^*$  (more precisely, a neighborhood of 0 in  $\mathbf{D}^*$ ) into  $\mathcal{H}^3$  if and only if*

$$f(0)h(0) = \frac{1 - \mu^2}{4\mu} \text{ and } (1 + \mu) \frac{h'}{h}(0) = (2\mu h'f + 2(1 + \mu)hf')(0).$$



Furthermore, we have  $\mu \neq 1$ , and the solutions of the indicial equations are

$$\begin{aligned} \lambda_1 &= \frac{-1 - \mu}{2}, & \lambda_2 &= \frac{1 - \mu}{2}, \\ r_1 &= \frac{\mu - 1}{2}, & r_2 &= \frac{\mu + 1}{2}. \end{aligned}$$

*Proof.* Let  $A$  and  $C$  (resp.,  $B$  and  $D$ ) be two independent solutions of equation (2.2) (resp., (2.3)) satisfying  $AD - BC \equiv 1$  and  $A'D' - B'C' \equiv 0$ . Let  $\lambda_1$  and  $\lambda_2$  (resp.,  $r_1$  and  $r_2$ ) denote the solutions of the indicial equation associated to (2.2) (resp., 2.3).

Suppose first that the functions  $u, v$ , and  $w$  in (1.2) define an embedding of a neighborhood of 0 in  $\mathbf{D}^*$  into  $\mathbf{H}^3$ . Then, by [U-Y1, Theorem 5.2], we have  $\min(|\lambda_1 - \lambda_2|, |r_1 - r_2|) = 1$ . Hence the indicial equations become

$$\lambda^2 + \mu\lambda - \mu f(0)h(0) = 0 \quad \text{and} \quad r^2 - \mu r - \mu f(0)h(0) = 0.$$

Note that both equations have discriminant  $\Delta = \mu^2 + 4\mu f(0)h(0)$ . Since we necessarily have  $\Delta = 1$ , we obtain  $f(0)h(0) = \frac{1-\mu^2}{4\mu}$ . This yields the solutions of the two indicial equations, as asserted in the lemma. Since  $f(0)h(0) \neq 0$ , we have  $\mu \neq 1$ .

A basis of solutions to the differential equation (2.2) has the form

$$X_2(z) = z^{\lambda_2} f_2 \quad \text{and} \quad X_1(z) = \lambda \log z \cdot X_2 + z^{\lambda_1} f_1,$$

where  $\lambda \in \mathbf{C}$  is a complex number and  $f_1$  and  $f_2$  are analytic functions on  $\mathbf{D}$  such that  $f_i(0) = 1$ . Substituting the formula for  $X_1$  into the equation (2.2), we get (after multiplication by  $z^{\frac{5+\mu}{2}}$ ),

$$(\mu h(0)f(0) - h\tilde{f})f_1 + (\lambda f_2 - \lambda_1 \frac{h'}{h} f_1)z + (2\lambda f_2' + f_1'' - \lambda \frac{h'}{h} f_2 - \frac{h'}{h} f_1')z^2 = 0.$$

In particular, the derivative of the function on the left side vanishes for  $z = 0$ , and so

$$(*) \quad \lambda - \left( (h\tilde{f})' + \lambda_1 \frac{h'}{h} \right) (0) = 0.$$

Now, as the immersion associated to  $(g, \omega)$  is single-valued, the hyperbolic Gauss map  $G$  also must be single-valued. On the other hand, by Lemma 1.10, we have  $G = C'/A'$ , where  $A$  and  $C$  are independent solutions of (2.2). Thus we must have  $\lambda = 0$ . By (\*) we deduce that  $\left( (h\tilde{f})' + \lambda_1 \frac{h'}{h} \right) (0) = 0$ . Using the expressions of  $\lambda_1$  and  $\tilde{f}$ , this yields

$$(1 + \mu) \frac{h'}{h} (0) = (2\mu h'f + 2(1 + \mu)hf')(0),$$

as desired.

Conversely, assume that  $f(0)h(0) = \frac{1-\mu^2}{4\mu}$  and  $(1 + \mu) \frac{h'}{h} (0) = (2\mu h'f + 2(1 + \mu)hf')(0)$ . We deduce that  $|\lambda_1 - \lambda_2| = |r_1 - r_2| = 1$ . Furthermore, by

(\*) we have  $\lambda = 0$ . That is, the function  $X_1$  has no logarithmic term; hence,  $A$  and  $C$  have no logarithmic term either. Since  $\frac{C'}{A'} = \frac{D'}{B'}$ , the same holds for  $B$  and  $D$ . This shows that the coordinate functions  $t(z)$ ,  $x_1(z)$ ,  $x_2(z)$ , and  $x_3(z)$  defined in (1.1) are single-valued. Hence, the immersion is also single-valued. Finally the immersion is one-to-one because  $\min(|\lambda_1 - \lambda_2|, |r_1 - r_2|) = 1$ .  $\square$

LEMMA 2.5. *Consider the Weierstrass representation  $(g, \omega)$  on  $\mathbf{D}^*$  given by (2.1) with  $\mu > 0$ . Suppose that  $\nu + \mu = -1$ ,  $f(0)h(0) = \frac{1-\mu^2}{4\mu}$ , and  $(1 + \mu)\frac{h'}{h}(0) = (2\mu h'f + 2(1 + \mu)hf')(0)$ . Let  $A$ ,  $B$ ,  $C$ , and  $D$  be the functions giving the embedding (1.2) associated to  $(g, \omega)$ . Then, up to an isometry of  $\mathbf{H}^3$ , we can choose  $A$ ,  $B$ ,  $C$ , and  $D$  as follows:*

$$\begin{aligned} A(z) &= z^{\lambda_2} f_2, & B(z) &= f(0) \frac{\mu - 1}{\mu + 1} z^{r_2} g_2, \\ C(z) &= \frac{\mu^2 - 1}{4\mu f(0)} z^{\lambda_1} f_1, & D(z) &= \frac{(1 + \mu)^2}{4\mu} z^{r_1} g_1, \end{aligned}$$

Here  $z \in \mathbf{D}^*$ , and  $f_i$  and  $g_i$  are analytic functions near 0 satisfying  $f_i(0) = 1$ , and  $g_i(0) = 1$ .

*Proof.* The argument is similar to the proof of Lemma 5.3 in [U-Y1]. By Lemma 2.4 the Weierstrass representation  $(g, \omega)$  defines a single-valued embedding of a neighborhood of 0 in  $\mathbf{D}^*$  into  $\mathbf{H}^3$ . Furthermore,  $A$  and  $C$  are independent solutions to (2.2), and  $B$  and  $D$  are independent solutions to (2.3). Then Lemma 2.4 and the theory of ordinary differential equations show that  $A$  and  $C$  are linear combinations of

$$X_1(z) = z^{\lambda_1} f_1(z), \quad X_2(z) = z^{\lambda_2} f_2(z),$$

and  $B$  and  $D$  are linear combinations of

$$Y_1 = z^{r_1} \rho(z), \quad Y_2(z) = z^{r_2} g_2(z),$$

where  $f_1(z)$ ,  $f_2(z)$ ,  $g_2(z)$ , and  $\rho(z)$  are analytic functions near 0 such that  $f_1(0) = 1$ ,  $f_2(0) = 1$ ,  $g_2(0) = 1$ , and  $\rho(0) = 1$ . Hence

$$\begin{aligned} A(z) &= a_1 z^{\lambda_1} f_1 + a_2 z^{\lambda_2} f_2, & B(z) &= b_1 z^{r_1} \rho + b_2 z^{r_2} g_2, \\ C(z) &= c_1 z^{\lambda_1} f_1 + c_2 z^{\lambda_2} f_2, & D(z) &= d_1 z^{r_1} \rho + d_2 z^{r_2} g_2. \end{aligned}$$

Consider any matrix  $P \in \mathbf{SL}(2, \mathbf{C})$ , define

$$F(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (z),$$

and set  $F_1(z) = (P \cdot F)(z)$ . Since  $F^{-1} \cdot dF = F_1^{-1} \cdot dF_1$ , the two matrices  $F$  and  $F_1$  define the same Weierstrass representations; see Remark 1.4(2). This means that  $F_1$  and  $F$  define the same immersion into  $\mathbf{H}^3$ , up to a global isometry.

Now let  $\alpha, \beta, \gamma,$  and  $\delta$  denote the complex entries of  $P,$  with  $\alpha\delta - \beta\gamma = 1.$  We have

$$F_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

with

$$\begin{aligned} \tilde{A} &= (\alpha a_1 + \beta c_1)z^{\lambda_1} f_1 + (\alpha a_2 + \beta c_2)z^{\lambda_2} f_2, \\ \tilde{B} &= (\alpha b_1 + \beta d_1)z^{r_1} \rho + (\alpha b_2 + \beta d_2)z^{r_2} g_2, \\ \tilde{C} &= (\gamma a_1 + \delta c_1)z^{\lambda_1} f_1 + (\gamma a_2 + \delta c_2)z^{\lambda_2} f_2, \\ \tilde{D} &= (\gamma b_1 + \delta d_1)z^{r_1} \rho + (\gamma b_2 + \delta d_2)z^{r_2} g_2. \end{aligned}$$

Since  $AD - BC \equiv 1,$  we have  $a_1 d_1 - b_1 c_1 = 0.$  Also,  $a_1 c_2 - a_2 c_1 \neq 0$  since  $A$  and  $C$  are two independent functions. For the same reason,  $b_1 d_2 - b_2 d_1 \neq 0.$  Therefore we can find complex numbers  $\alpha, \beta, \gamma,$  and  $\delta$  such that  $\alpha a_1 + \beta c_1 = \alpha b_1 + \beta d_1 = \gamma a_2 + \delta c_2 = 0$  and  $\alpha\delta - \beta\gamma = 1.$  This shows that, up to an isometry in  $\mathbf{H}^3,$  we can assume

$$\begin{aligned} A(z) &= az^{\lambda_2} f_2, & B(z) &= bz^{r_2} g_2, \\ C(z) &= cz^{\lambda_1} f_1, & D(z) &= dz^{r_1} g_1, \end{aligned}$$

where  $a, b, c, d \in \mathbf{C}^*, ad - bc = 1,$  and  $g_1$  is a linear combination of  $\rho$  and  $z^{r_2-r_1} g_2$  with  $g_1(0) = 1.$  However, by (1.3) we have  $g \cdot z^\nu \cdot h(z) = A'D - BC' (= g\eta)$  and  $z^\nu \cdot h(z) = AC' - A'C (= \eta).$  From the first equality we get

$$ad(\lambda_2 g_1 f_2 + z g_1 f_2') - bc(\lambda_1 g_2 f_1 + z g_2 f_1') = h(z)f(z).$$

Setting  $z = 0,$  we obtain  $\lambda_2 ad - \lambda_1 bc = h(0)f(0).$  Substituting the values of  $h(0)f(0), \lambda_1,$  and  $\lambda_2$  in the function of  $\mu$  and using  $ad - bc = 1,$  we get  $d = \frac{(1+\mu)^2}{4\mu a}.$  Similarly, using the other equality we get  $c = \frac{\mu^2-1}{4\mu a f(0)}.$  Also, since  $ad - bc = 1,$  we have  $b = af(0)\frac{\mu-1}{\mu+1}.$

Finally, by considering the matrix  $F_1 = P \cdot F,$  where

$$P = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

we can assume that  $a = 1.$  □

REMARK 2.6. A calculation shows that under the conditions of Lemma 2.5 (and using equations (1.2)) the embedding into  $\mathbf{H}^3$  has the following coordinates:

$$\begin{aligned} (u + iv)(z) &= \frac{1}{z} \frac{\mu^2 - 1}{4\mu f(0)} \frac{f_1 \bar{f}_2 + g_1 \bar{g}_2 |z|^{2\mu} |f(0)|^2}{|f_2|^2 + |f(0)|^2 \left(\frac{\mu-1}{\mu+1}\right)^2 |g_2|^2 |z|^{2\mu}}, \\ w(z) &= |z|^{\mu-1} \frac{1}{|f_2|^2 + |f(0)|^2 \left(\frac{\mu-1}{\mu+1}\right)^2 |g_2|^2 |z|^{2\mu}}. \end{aligned}$$

PROPOSITION 2.7. *Consider the Weierstrass representation  $(g, \omega)$  on  $\mathbf{D}^*$  given by (2.1). Suppose that  $\nu + \mu = -1$ ,  $f(0)h(0) = \frac{1-\mu^2}{4\mu}$ ,  $(1 + \mu)\frac{h'}{h}(0) = (2\mu h'f + 2(1 + \mu)hf')(0)$ , and  $0 < \mu < 1$ . Then, up to an isometry of  $\mathbf{H}^3$ , the asymptotic boundary of the end, given by (1.2), is the point  $\infty$ , and the end  $E$  is asymptotic (in both the Euclidean and the hyperbolic sense) to an end  $C$  of an embedded catenoid cousin. More precisely,  $E$  is a vertical graph over the complement of a disk in the plane  $\{w = 0\}$ . The function whose graph is  $E$  is given by*

$$w(u + iv) = \gamma_1 r^{1-\mu} + \gamma_3 r^{1-3\mu} + \dots + \gamma_{2k+1} r^{1-(1+2k)\mu} + o(1),$$

where  $\gamma_1 > 0$ ,  $\gamma_{2i+1} \in \mathbf{R}$  ( $i = 0, \dots, k$ ) depend only on  $\mu$ ,  $k$  is the greatest integer such that  $2k\mu < 1$ ,  $r = |u + iv|$ , and  $o(1)$  is a real smooth function of  $u$  and  $v$  such that  $o(1) \rightarrow 0$  when  $|u + iv| \rightarrow \infty$ . Furthermore, if  $w_C$  is the function whose graph is  $C$ , we have

$$\lim_{u^2+v^2 \rightarrow \infty} (w - w_C)(u, v) = 0 \quad \text{and} \quad \lim_{u^2+v^2 \rightarrow \infty} \log \left( \frac{w}{w_C} \right) (u, v) = 0.$$

*Proof.* Since  $0 < \mu < 1$ , the formula for  $(u + iv)(z)$  given in Remark 2.6 gives

$$\begin{aligned} (u + iv)(z) &= \frac{1}{z} \cdot \frac{\mu^2 - 1}{4\mu f(0)} \cdot \frac{1 + f_1 \bar{f}_2 - 1 + g_1 \bar{g}_2 |z|^{2\mu} |f(0)|^2}{1 + |f_2|^2 - 1 + |f(0)|^2 \left(\frac{\mu-1}{\mu+1}\right)^2 |g_2|^2 |z|^{2\mu}} \\ &= \frac{1}{z} \cdot \frac{\mu^2 - 1}{4\mu f(0)} \cdot \frac{1 + |f(0)|^2 |z|^{2\mu} + O(z)}{1 + |f(0)|^2 \left(\frac{\mu-1}{\mu+1}\right)^2 |z|^{2\mu} + O(z)} \\ &= \frac{1}{z} \cdot \frac{\mu^2 - 1}{4\mu f(0)} \cdot (1 + |f(0)|^2 |z|^{2\mu} + O(z)) \\ &\quad \times \left( \sum_{p=0}^k (-1)^p |f(0)|^2 \left(\frac{\mu-1}{\mu+1}\right)^2 |z|^{2\mu p} + O(z) \right), \end{aligned}$$

and hence

$$(*) \quad (u + iv)(z) = \frac{1}{z} \cdot \frac{\mu^2 - 1}{4\mu f(0)} \cdot (1 + \beta_1 |z|^{2\mu} + \dots + \beta_k |z|^{k \cdot 2\mu} + O(z)),$$

where the coefficients  $\beta_1, \dots, \beta_k \in \mathbf{R}$  depend only on  $f(0)$  and  $\mu$ , and  $k$  is the greatest integer such that  $k \cdot 2\mu < 1$ .

Set  $r = |u + iv|$ . We claim that

$$(**) \quad |z| = \frac{1}{r} \cdot \frac{1 - \mu^2}{4\mu |f(0)|} \left( 1 + b_1 r^{-2\mu} + \dots + b_k r^{-k \cdot 2\mu} + O\left(\frac{1}{r}\right) \right),$$

where  $b_1, \dots, b_k \in \mathbf{R}$  depend only on  $f(0)$  and  $\mu$ , and  $O(1/r)$  is a smooth real function such that  $r \cdot O(1/r)$  is bounded when  $r = |u + iv| \rightarrow +\infty$ .

Indeed, by (\*) we have  $z \rightarrow 0$  if and only if  $r \rightarrow +\infty$ . Now, suppose that, for some integer  $p$  such that  $0 \leq p < k$ , we have

$$(***) \quad |z| = \frac{1}{r} \cdot \frac{1 - \mu^2}{4\mu|f(0)|} \left( 1 + c_1 r^{-2\mu} + \dots + c_p r^{-p \cdot 2\mu} + O\left(r^{-2(p+1)\mu}\right) \right),$$

where  $c_1, \dots, c_p \in \mathbf{R}$  depend only on  $f(0)$  and  $\mu$ . Then, substituting this expression of  $|z|$  in (\*), we get

$$\begin{aligned} |z| &= \frac{1}{r} \cdot \frac{1 - \mu^2}{4\mu|f(0)|} \left( 1 + \sum_{q=1}^k \beta_q \left[ \frac{\alpha}{r} \left( 1 + c_1 r^{-2\mu} + \dots \right. \right. \right. \\ &\qquad \qquad \qquad \left. \left. \left. + c_p r^{-p \cdot 2\mu} + O\left(r^{-2(p+1)\mu}\right) \right) \right]^{2q\mu} + O\left(\frac{1}{r}\right) \right) \\ &= \frac{1}{r} \cdot \frac{1 - \mu^2}{4\mu|f(0)|} \left( 1 + \tilde{c}_1 r^{-2\mu} + \dots \right. \\ &\qquad \qquad \qquad \left. + \tilde{c}_{p+1} r^{-2(p+1)\mu} + O\left(r^{-2(p+2)\mu}\right) + O\left(\frac{1}{r}\right) \right), \end{aligned}$$

where  $\alpha = \frac{1 - \mu^2}{4\mu|f(0)|}$  and  $\tilde{c}_1, \dots, \tilde{c}_{p+1} \in \mathbf{R}$  depend on only  $f(0)$  and  $\mu$ . If  $p + 1 = k$ , then  $r \cdot O\left(r^{-2(p+2)\mu}\right) = r^{1-2(k+1)\mu} \cdot r^{2(k+1)\mu} O\left(r^{-2(k+1)\mu}\right)$  is a bounded function as  $r \rightarrow +\infty$  (because  $2(k + 1)\mu \geq 1$ ). Hence (\*\*) follows in this case. If  $p + 1 < k$ , then  $r^{2(p+2)\mu} O(1/r) = r^{-1+2(p+2)\mu} \cdot r O(1/r)$  is a bounded function when  $r \rightarrow +\infty$ , so in this case we have

$$|z| = \frac{1}{r} \cdot \frac{1 - \mu^2}{4\mu|f(0)|} \left( 1 + \tilde{c}_1 r^{-2\mu} + \dots + \tilde{c}_{p+1} r^{-2(p+1)\mu} + O\left(r^{-2(p+2)\mu}\right) \right).$$

Therefore, by induction on  $p$ , we see that to prove (\*\*) for  $0 \leq p < k$ , it suffices to show (\*\*\*) for some  $p$  with  $0 \leq p < k$ . Finally, using (\*), it is easily seen that (\*\*\*) is true for  $p = 0$ . This proves the claim.

By the same argument as in the proof of (\*), Remark 2.6 implies

$$w(z) = |z|^{-1+\mu} \left( 1 + \delta_1 |z|^{2\mu} + \dots + \delta_k |z|^{k \cdot 2\mu} + O(z) \right),$$

where the coefficients  $\delta_1, \dots, \delta_k \in \mathbf{R}$  depend on only  $f(0)$  and  $\mu$ . Now observe that, up to a change of coordinates in  $\mathbf{D}^*$ , we may assume that  $f \equiv 1$ , without changing the values of  $\mu$  and  $\nu$ . Hence in (\*\*) and the last equality we can assume that the constants  $b_j$  and  $\delta_j$  depend only on  $\mu$ . Substituting for  $|z|$  the expression given by (\*\*), we obtain from the last equality  $w$  as a function of  $u$  and  $v$ , as asserted, with  $\gamma_1 = (|\mu^2 - 1| / (4\mu))^{-1+\mu}$ .

On the other hand, consider the Weierstrass representation of catenoid cousins given in Example 1.5. The above argument shows that a neighborhood of 0 is an end and is a vertical graph whose expression,  $w_C$ , is completely determined by  $\mu$ , up to an additive function vanishing at infinity. This shows that the graphs of the two ends (i.e., the general end  $E$  and the catenoid

cousin end) are equal, up to an additive smooth real function vanishing at  $\infty$ . Hence those two ends are asymptotic (as vertical graphs) in the Euclidean sense. Furthermore, since

$$w_C(u + iv) = \gamma_1 r^{1-\mu} + \gamma_3 r^{1-3\mu} + \dots + \gamma_{2k+1} r^{1-(1+2k)\mu} + o(1),$$

we get

$$\lim_{u^2+v^2 \rightarrow \infty} \log \left( \frac{w}{w_C} \right) (u, v) = 0,$$

so the two ends are also asymptotic in the hyperbolic sense. □

**PROPOSITION 2.8.** *Consider the Weierstrass representation  $(g, \omega)$  on  $\mathbf{D}^*$  given by (2.1). Suppose that  $\nu + \mu = -1$ ,  $f(0)h(0) = \frac{1-\mu^2}{4\mu}$ ,  $(1 + \mu)\frac{h'}{h}(0) = (2\mu h'f + 2(1 + \mu)hf')(0)$ , and  $\mu > 1$ . Then the end  $E$  is asymptotic (in the Euclidean and the hyperbolic sense) to the end  $C$  of a nonembedded catenoid cousin. More precisely, up to an isometry in  $\mathbf{H}^3$ , the end is a vertical graph over the complement of a disk in the plane  $\{w = 0\}$  and is asymptotic (as a vertical graph) to a nonembedded catenoid cousin. Moreover, the function whose graph is  $E$  is given by*

$$w(u + iv) = r^{1-\mu} \cdot \left( \frac{\mu^2 - 1}{4\mu} \right)^{\mu-1} \left( 1 + O\left(\frac{1}{r}\right) \right),$$

where  $r = |u + iv|$  and  $O\left(\frac{1}{r}\right)$  is a smooth real function such that  $r \cdot O\left(\frac{1}{r}\right)$  is bounded when  $r = |u + iv| \rightarrow +\infty$ . In particular,  $w \rightarrow 0$  as  $|u + iv| \rightarrow +\infty$ . Furthermore, if  $w_C$  is the function whose graph is  $C$ , we have

$$\lim_{u^2+v^2 \rightarrow \infty} (w - w_C)(u, v) = 0 \quad \text{and} \quad \lim_{u^2+v^2 \rightarrow \infty} \log \left( \frac{w}{w_C} \right) (u, v) = 0.$$

*Proof.* The proof is analogous to that of Proposition 2.7, so we give only an outline.

Since  $\mu > 1$ , it follows from Remark 2.6 that

$$(u + iv)(z) = \frac{1}{z} \cdot \frac{\mu^2 - 1}{4\mu f(0)} (1 + O(z)),$$

$$w(z) = |z|^{\mu-1} (1 + O(z)).$$

This implies

$$|z| = \frac{1}{r} \frac{\mu^2 - 1}{4\mu |f(0)|} \left( 1 + O\left(\frac{1}{r}\right) \right),$$

with  $r = |u + iv|$ . It follows that

$$w(u + iv) = r^{1-\mu} \cdot \left( \frac{\mu^2 - 1}{4\mu |f(0)|} \right)^{\mu-1} \left( 1 + O\left(\frac{1}{r}\right) \right).$$

Note that, up to a change of coordinates in  $\mathbf{D}^*$ , we can assume that  $f \equiv 1$ . It follows that

$$w(u + iv) = r^{1-\mu} \cdot \left(\frac{\mu^2 - 1}{4\mu}\right)^{\mu-1} \left(1 + O\left(\frac{1}{r}\right)\right).$$

To complete the proof, it suffices to compare this representation with that of the catenoid cousin given in Example 1.5, for the same value of  $\mu > 1$ .  $\square$

We now turn to the case when  $\nu + \mu \geq 0$ .

LEMMA 2.9. *Consider the Weierstrass representation  $(g, \omega)$  on  $\mathbf{D}^*$  given by (2.1) with  $\mu > 0$ ,  $\nu \leq -1$ , and  $\mu + \nu \in \mathbf{Z}$ . Suppose  $\nu + \mu \geq 0$ . Then  $(g, \omega)$  defines a single-valued embedding if and only if  $\nu = -2$ ,  $\mu \in \mathbf{N}$ ,  $\mu \geq 2$ , and*

- (i)  $h'(0) = 2h^2(0)f(0)$  if  $\mu = 2$ ,
- (ii)  $h'(0) = 0$  if  $\mu \geq 3$ .

Furthermore, the solutions of the indicial equations are

$$\begin{aligned} \lambda_1 &= -1, & \lambda_2 &= 0, \\ r_1 &= 0, & r_2 &= 2\mu - 1. \end{aligned}$$

*Proof.* The proof is analogous to the proof of Lemma 2.4. Let  $A$  and  $C$  (resp.,  $B$  and  $D$ ) be two independent solutions of equation (2.2) (resp., (2.3)) satisfying  $AD - BC \equiv 1$  and  $A'D' - B'C' \equiv 0$ . The indicial equations are  $\lambda^2 - (1 + \nu)\lambda = 0$  and  $r^2 - (1 + \nu + 2\mu)r = 0$ , whose solutions are, respectively,  $\lambda_1 = 1 + \nu, \lambda_2 = 0$  and  $r_1 = 0, r_2 = 1 + \nu + 2\mu$ . Observe that  $\lambda_1 \leq \lambda_2$  and  $r_1 \leq r_2$ .

Suppose first that the functions  $u, v$ , and  $w$  in (1.2) define an embedding of a neighborhood of 0 in  $\mathbf{D}^*$  into  $\mathbf{H}^3$ . By [U-Y1, Theorem 5.2] we have  $\min(|\lambda_1 - \lambda_2|, |r_1 - r_2|) = 1$ . Since  $\nu + \mu \geq 0$  and  $\mu > 0$ , we have  $\nu = -2$ , and hence  $\mu \in \mathbf{Z}$  and  $\mu \geq 2$ . Therefore, we get  $\lambda_1 = -1, \lambda_2 = 0, r_1 = 0$ , and  $r_2 = 2\mu - 1$ .

A basis of solutions to the differential equation (2.2) is

$$X_2(z) = f_2(z) \quad \text{and} \quad X_1(z) = \lambda \log z \cdot X_2 + \frac{f_1(z)}{z},$$

where  $\lambda \in \mathbf{C}$  and  $f_i$  are analytic functions on  $\mathbf{D}$  such that  $f_i(0) = 1$ . Substituting the function  $X_1$  in the equation (2.2) gives

$$\lambda f_2 + \frac{h'}{h} f_1 + \left(2\lambda f_2' + f_1'' - \lambda \frac{h'}{h} f_2 - \frac{h'}{h} f_1'\right) z - h \tilde{f} f_1 z^{\mu-2} = 0.$$

In particular, setting  $z = 0$  in the last relation we get

- (i)  $\lambda + \frac{h'}{h}(0) - 2(hf)(0)$  if  $\mu = 2$ ,
- (ii)  $\lambda + \frac{h'}{h}(0)$  if  $\mu \geq 3$ .

Now, recall that  $A$  and  $C$  are independent linear combinations of  $X_1$  and  $X_2$ . Furthermore, the hyperbolic Gauss map  $G(z) = \frac{C'}{A'}$  is well defined. This forces  $\lambda = 0$ . Using (i) and (ii) we obtain  $h'(0) = 2h^2(0)f(0)$  if  $\mu = 2$ , and  $h'(0) = 0$  if  $\mu \geq 3$ .

Conversely, suppose that  $\nu = -2$  and that  $h'(0) = 2h^2(0)f(0)$  if  $\mu = 2$  and  $h'(0) = 0$  if  $\mu \geq 3$ . Then, as in the proof of Lemma 2.4, we see that the immersion (1.2) defined by  $(g, \omega)$  is an embedding.  $\square$

**PROPOSITION 2.10.** *Consider the Weierstrass representation  $(g, \omega)$  on  $\mathbf{D}^*$  given by (2.1). Suppose  $\nu = -2$ ,  $\mu \geq 2$ ,  $\mu \in \mathbf{N}$ , and*

- (i)  $h'(0) = 2h^2(0)f(0)$  if  $\mu = 2$ ,
- (ii)  $h'(0) = 0$  if  $\mu \geq 3$  (see Lemma 2.9).

*Then the constant mean curvature 1 end  $E$  defined by  $(g, \omega)$  in  $\mathbf{H}^3$  is asymptotic to a horosphere in both the Euclidean and the hyperbolic sense. More precisely, up to an isometry in  $\mathbf{H}^3$ ,  $E$  is a vertical graph over the complement of a disk in the plane  $\{w = 0\}$ , and the function whose graph is  $E$  has the form*

$$w(u + iv) = 1 + O(r^{1-\mu}).$$

where  $r = |u + iv|$ .

*Proof.* As  $A$  and  $C$  are solutions to (2.2), these functions are linear combinations of solutions  $X_1(z) = z^{-1}f_1$  and  $X_2 = f_2(z)$ , where the functions  $f_i$  are analytic near 0 and satisfy  $f_i(0) = 1$  (see Lemma 2.9). Similarly, since  $B$  and  $D$  are solutions to (2.3), these functions are linear combinations of  $Y_1(z) = \rho(z)$  and  $Y_2(z) = z^{2\mu-1}g_2(z)$ , where  $\rho$  and  $g_2$  are analytic near 0 and satisfy  $\rho(0) = g_2(0) = 1$ . As in Lemma 2.5, we can choose  $A, B, C$ , and  $D$  by

$$\begin{aligned} A(z) &= f_2(z), & B(z) &= bz^{2\mu-1}g_2(z), \\ C(z) &= cz^{-1}f_1(z), & D(z) &= g_1(z), \end{aligned}$$

where  $b, c \in \mathbf{C}^*$ , and  $g_1$  is a linear combination of  $\rho$  and  $z^{2\mu-1}g_2$  such that  $g_1(0) = 1$ . By the formulas for the coordinate functions given in (1.2), we deduce

$$\begin{aligned} (u + iv)(z) &= \frac{c}{z} \cdot (1 + O(z)), \\ w(z) &= 1 + \sum_{n \geq 1} \left(1 - |f_2|^2 + O(|z|^{4\mu-2})\right)^n. \end{aligned}$$

Now  $f_2$  has the form  $f_2(z) = 1 + a_p z^p + O(z^{p+1})$ , where  $a_p \in \mathbf{C}^*$  and  $p \in \mathbf{N}^*$ . But since  $f_2$  is a solution to (2.2), we have

$$p(p + 1)a_p z^{p-2} - h\tilde{f}z^{\mu-3} + O(z^{p-1}) = 0.$$



Hence  $p - 2 = \mu - 3$  and  $p(p + 1)a_p = (h\tilde{f})(0)$ ; that is,  $p = \mu - 1$  and  $a_p = \frac{h(0)f(0)}{\mu - 1}$ . So we have

$$w(z) = 1 - 2 \operatorname{Re} \left( \frac{h(0)f(0)}{\mu - 1} z^{\mu-1} \right) + O(|z|^\mu).$$

This implies that the end is a vertical graph, as claimed. This completes the proof of Proposition 2.10 as well as that of Theorem 2.3.  $\square$

REMARK 2.11. The proof of Proposition 2.10 shows that the end crosses the horosphere  $\{w = 1\}$  exactly  $2(\mu - 1)$  times.

DEFINITION 2.12. The power  $(1 - \mu)$  of  $|u + iv|$ , which appears in the graph of an end  $E$  (that is embedded, regular, and with finite total curvature) is called the *growth* rate of the end.

We easily obtain the following corollary:

COROLLARY 2.13. *Let  $E_1$  and  $E_2$  be two regular embedded ends in  $\mathbf{H}^3$  with constant mean curvature 1 and finite total curvature whose asymptotic boundary is the point  $\infty$ . Suppose that  $E_1$  and  $E_2$  have the same growth. Then, up to a Euclidean homothety (which is a hyperbolic isometry),  $E_1$  is asymptotic to  $E_2$ , as vertical graphs, in both the Euclidean and the hyperbolic sense.*

### 3. Symmetries of constant mean curvature 1 surfaces in $\mathbf{H}^3$

In this section, we relate the symmetries of a simply connected minimal surface in  $\mathbf{R}^3$  to those of the associated constant mean curvature 1 surface in  $\mathbf{H}^3$ . We will need the following basic result.

LEMMA 3.1. *Let  $M \subset \mathbf{H}^3$  be a surface in  $\mathbf{H}^3$ . Let  $\alpha: ] - 1, 1[ \rightarrow M$  be a geodesic curve of  $M$ , which is a curvature line. Then  $\alpha$  is a planar curve; that is,  $\alpha$  stays on a geodesic plane in  $\mathbf{H}^3$ . Furthermore,  $M$  is orthogonal to this geodesic plane along  $\alpha$ .*

*Proof.* We suppose that  $\alpha$  is parametrized by the arc length. Let  $n(t)$  be a unit co-normal field along  $\alpha$ , and let  $N$  be a unit normal field on  $M$ . We make the following assumptions (without loss of generality):

$$\alpha(0) = (0, 0, 1); \quad \alpha'(0) = (1, 0, 0) \quad \text{and} \quad n(0) = (0, 1, 0).$$

We want to prove that  $\alpha$  stays on the geodesic plane  $\{x_2 = 0\}$ .

To show this, we consider the vector fields  $e_1 = (1, 0, 0)$ ,  $e_2 = (0, 1, 0)$ , and  $e_3 = (0, 0, 1)$ , and we denote by  $\bar{\nabla}$  the Riemannian connection in  $\mathbf{H}^3$ . As  $n(t)$  is a unit vector field along  $\alpha$ , we have  $\langle \bar{\nabla}_{\alpha'} n, n \rangle = 0$ , where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbf{H}^3$ . We also have

$$\langle \bar{\nabla}_{\alpha'} n, \alpha' \rangle = -\langle n, \bar{\nabla}_{\alpha'} \alpha' \rangle = 0.$$

The last equality holds since  $\alpha$  is a geodesic on  $M$  and  $n$  is a tangent field on  $M$ . Furthermore,

$$\langle \bar{\nabla}_{\alpha'} n, N \rangle = -\langle n, \bar{\nabla}_{\alpha'} N \rangle = 0$$

since  $\alpha$  is a curvature line on  $M$ . From this we obtain  $\bar{\nabla}_{\alpha'} n = 0$ . On the other hand, setting  $n = (n_1, n_2, n_3)$ , a computation gives

$$\begin{aligned} \bar{\nabla}_{\alpha'} n &= n'(t) + n_1(t)\bar{\nabla}_{\alpha'} e_1 + n_2(t)\bar{\nabla}_{\alpha'} e_2 + n_3(t)\bar{\nabla}_{\alpha'} e_3 \\ &= n'(t) + \frac{1}{\alpha_3(t)}(-\alpha'_3 n_1 - \alpha'_1 n_3)e_1 + \frac{1}{\alpha_3(t)}(-\alpha'_3 n_2 - \alpha'_2 n_3)e_2 \\ &\quad + \frac{1}{\alpha_3(t)}(\alpha'_1 n_1 + \alpha'_2 n_2 - \alpha'_3 n_3)e_3. \end{aligned}$$

Since  $\alpha'$  and  $n$  are orthogonal fields along  $\alpha$  and  $\bar{\nabla}_{\alpha'} n = 0$  (as we showed above), we obtain the differential system

$$n'_1 = \frac{\alpha'_1 n_3 + \alpha'_3 n_1}{\alpha_3}, \quad n'_2 = \frac{\alpha'_2 n_3 + \alpha'_3 n_2}{\alpha_3}, \quad n'_3 = 2\frac{\alpha'_3 n_3}{\alpha_3},$$

with initial conditions  $n_1(0) = n_3(0) = 0$  and  $n_2(0) = 1$ . It is easy to see that this system has a unique solution given by  $n_1 = n_3 = 0$  and  $n_2 = \alpha_3$ , that is,  $n(t) = \alpha_3(t)e_2$ . Since  $\langle n(t), \alpha'(t) \rangle = 0$ , we deduce that  $\alpha'_2 = 0$ . Hence  $\alpha$  stays on the geodesic plane  $\{x_2 = 0\}$ . Furthermore,  $M$  is orthogonal to this plane along  $\alpha$  because  $n(t)$  is a tangent field of  $M$  along  $\alpha$ , which is orthogonal to this plane. □

**PROPOSITION 3.2.** *Let  $X: U \rightarrow \mathbf{R}^3$  be a nonplanar minimal immersion, where  $U$  is a simply connected planar domain. Let  $R: \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be an orthogonal symmetry with respect to a plane such that  $R(X(U)) = X(U)$ . Then the associated constant mean curvature 1 surface in  $\mathbf{H}^3$  is invariant under an orthogonal symmetry with respect to a geodesic plane of  $\mathbf{H}^3$ .*

*Proof.* Let  $\tilde{X}: U \rightarrow \mathbf{H}^3$  be the associated immersion and  $S: U \rightarrow U$  the intrinsic isometry induced by  $R$ .

It is known that the intersection between the plane of symmetry of  $R$  and  $X(U)$  is a geodesic and a curvature line of  $X(U)$ . Let  $\gamma \subset U$  be the curve whose image under  $R$  is this geodesic. As  $X(U)$  and  $\tilde{X}(U)$  are isometric surfaces, we deduce that  $\tilde{X}(\gamma)$  is also a geodesic of  $\tilde{X}(U)$ . Moreover, since  $X(U)$  is a minimal surface in  $\mathbf{R}^3$  and  $\tilde{X}(U)$  is a constant mean curvature 1 surface in  $\mathbf{H}^3$ , it follows that  $\tilde{X}(\gamma)$  is also a curvature line of  $\tilde{X}(U)$ . By Lemma 3.1,  $\tilde{X}(\gamma)$  stays on a geodesic plane of  $\mathbf{H}^3$  and the surface  $\tilde{X}(U)$  is orthogonal to this plane. Using this and the fact that  $\tilde{X}(U)$  is an analytic surface, we conclude that  $\tilde{X}(U)$  is symmetric with respect to this geodesic plane. □

The following result holds only for surfaces which are associated to minimal surfaces containing a straight line, as the statement shows.

**PROPOSITION 3.3.** *Let  $X: U \rightarrow \mathbf{R}^3$  be a nonplanar minimal immersion, where  $U$  is a simply connected planar domain. Suppose that  $X(U)$  contains a piece of a straight line  $L$ , and let  $R$  be the rotation by the angle  $\pi$  about  $L$  (i.e., the reflection about  $L$ ) in  $\mathbf{R}^3$ . (Note that  $X(U)$  is invariant under  $R$ .) Also, let  $S: U \rightarrow U$  be the intrinsic isometry induced by  $R$ . Then the associated constant mean curvature 1 surface in  $\mathbf{H}^3$  does not admit an isometry extending  $S$  (that is, the Euclidean isometry  $R$  in  $\mathbf{R}^3$  does not extend to a hyperbolic isometry in  $\mathbf{H}^3$ ).*

*Proof.* Let  $\alpha \subset U$  be the curve on  $U$  such that  $X(\alpha) = L$ , and let  $\tilde{X}: U \rightarrow \mathbf{H}^3$  be the constant mean curvature 1 immersion associated to  $X$ . Suppose there exists an isometry  $\tilde{R}$  of  $\mathbf{H}^3$  such that  $\tilde{R}(\tilde{X}(U)) = \tilde{X}(U)$ , that is,  $\tilde{R} \circ \tilde{X} = \tilde{X} \circ S$ . Then, as each point of  $\alpha$  is fixed by  $S$ , we deduce that each point of  $\gamma = \tilde{X}(\alpha)$  is fixed under  $\tilde{R}$ . If there is no geodesic plane of  $\mathbf{H}^3$  containing  $\gamma$ , then  $\tilde{R}$  has a geodesic plane of fixed points and a fixed point outside this geodesic plane. Thus, we easily obtain that  $\tilde{R}$  is the identity map. Hence  $S$  is the identity map of  $U$ , which leads to a contradiction. We conclude that  $\gamma$  stays in a geodesic plane  $\Pi$ . If  $\gamma \subset \Pi$  is not a geodesic line (of  $\mathbf{H}^3$ ), we conclude that each point of  $\Pi$  is fixed under  $\tilde{R}$ . Therefore,  $\tilde{R}$  is the reflection about  $\Pi$ , and  $\gamma$  is a curvature line of the surface. This implies that  $L = X(\alpha)$  is a curvature line of  $X(U)$  and that the Gauss curvature vanishes along  $L$ . However, this is impossible, since  $X(U)$  is a nonplanar minimal surface in  $\mathbf{R}^3$ . Hence,  $\gamma$  must be a piece of a geodesic line of  $\mathbf{H}^3$ . Now consider any point  $p \in \gamma$  and denote by  $P$  the geodesic plane through  $p$  orthogonal to  $\gamma$ . The plane  $P$  is globally fixed under  $\tilde{R}$  and  $\tilde{R}(p) = p$ .

Suppose that the restriction of  $\tilde{R}$  to  $P$  preserves the orientation. Then  $\tilde{R}$  is a rotation about the geodesic line  $\gamma$ . As the surface is globally fixed under this rotation, the angle of rotation is  $\pi$ . This implies that the geodesic curvature at  $p$  of the curve  $\tilde{X}(U) \cap P$  is 0, but this is a normal curvature of  $\tilde{X}(U)$  at  $p$ . As the orthogonal direction of this curve at  $p$  is given by  $\gamma$ , the mean curvature of  $\tilde{X}(U)$  at  $p$  is 0, which, however, is impossible. We conclude therefore that the restriction of  $\tilde{R}$  to  $P$  does not preserve the orientation. Hence this restriction must be a reflection of a geodesic  $\beta \subset P$  orthogonal to the curve  $\tilde{X}(U) \cap P$ . Therefore,  $\tilde{R}$  is the reflection about the geodesic plane generated by  $\gamma$  and  $\beta$ . This plane must be orthogonal to  $\tilde{X}(U)$  along  $\gamma$ . This implies, as before, that  $\gamma$  is a curvature line of  $\tilde{X}(U)$ . This, however, is impossible, as we have shown above. We conclude that there exists no isometry  $\tilde{R}$  of  $\mathbf{H}^3$  such that  $\tilde{X} \circ S = \tilde{R} \circ \tilde{X}$ . □

## REMARK 3.4.

(1) In connection with Proposition 3.3, we have the following result:

*Let  $S \subset \mathbf{H}^3$  be a constant mean curvature  $H$  surface in  $\mathbf{H}^3$ , bounded by a piece of a geodesic line  $L$  of  $\mathbf{H}^3$ . Then  $S$  can be extended to a constant mean curvature surface that is symmetric with respect to  $L$  if and only if  $H = 0$ , that is, if and only if  $S$  is a minimal surface of  $\mathbf{H}^3$ .*

Note that this result shows that the equivalent property for minimal surfaces in  $\mathbf{R}^3$  does not extend to constant mean curvature 1 surfaces in  $\mathbf{H}^3$ , but only to minimal surfaces in  $\mathbf{H}^3$ . This contradicts the assertion that most properties of minimal surfaces of  $\mathbf{R}^3$  extend to constant mean curvature 1 surfaces in  $\mathbf{H}^3$ .

To see this, we note first that, by a result of B. Lawson (see [La]), a minimal surface  $S$  can be symmetrized with respect to  $L$ . That is, if  $R$  is the hyperbolic rotation by angle  $\pi$  about  $L$ , then  $S \cup R(S)$  is a smooth minimal surface in  $\mathbf{H}^3$ . Conversely, let  $p \in L$  be a point on  $L$ , and let  $P$  be the geodesic plane through  $p$  orthogonal to  $L$ . Let  $\gamma = S \cap P$  be the intersection curve between  $S$  and  $P$ . On  $P$  consider the curve  $\tilde{\gamma}$  obtained by symmetrizing  $\gamma$  with respect to  $p$ ; that is,  $\tilde{\gamma} = \gamma \cup R(\gamma)$ . Note that  $\tilde{\gamma}$  is a  $C^2$  curve whose curvature at  $p$  is 0. Varying  $p$  on  $L$ , we obtain a  $C^2$  surface with constant mean curvature 0 along  $L$ . This shows that  $H = 0$ . We conclude that the Schwarz reflection principle does not hold for constant mean curvature 1 surfaces in  $\mathbf{H}^3$  (but it holds for minimal surfaces).

(2) We observe that the helicoidal surfaces given in Example 1.8 with  $\operatorname{Re}(\tilde{\gamma}(1 + \gamma)) = 0$  and  $\gamma_2 \neq 0$  contain the geodesic line  $L = \{u = v = 0\}$  (the axis of the helicoidal motion). It can easily be verified from the formula given in Example 1.8 that those surfaces are not symmetrical with respect to  $L$ .

We will need the following result, which we state for immersions in  $\mathbf{H}^3$ , although it also holds for immersions in  $\mathbf{R}^3$ ; see [Sp] or [Ke].

LEMMA 3.5. *Let  $U \subset \mathbf{R}^2$  be a simply connected planar domain and let  $X_1, X_2: U \rightarrow \mathbf{H}^3$  be two immersions with the following properties:*

- (a) *There exists  $p \in U$  with  $X_1(p) = X_2(p)$ ,  $\frac{\partial X_1}{\partial x}(p) = \frac{\partial X_2}{\partial x}(p)$  and  $\frac{\partial X_1}{\partial y}(p) = \frac{\partial X_2}{\partial y}(p)$ , where  $(x, y)$  are the coordinates on  $U$ .*
- (b) *Let  $N_i$  be a unit normal vector field on  $X_i(U)$ ,  $i = 1, 2$ , such that  $N_1(p) = N_2(p)$ . Let  $ds_i^2$  be the first fundamental form and let  $\Pi_i$  be the second fundamental form with respect to the normal field  $N_i$  of  $X_i$  ( $i = 1, 2$ ), and assume that  $ds_1^2 = ds_2^2$  and  $\Pi_1 = \Pi_2$ .*

*Then the two immersions  $X_1$  and  $X_2$  are the same, i.e.,  $X_1 = X_2$ .*

PROPOSITION 3.6. *Let  $X: U \rightarrow \mathbf{R}^3$  be a conformal and nonplanar minimal immersion, where  $U$  is a simply connected planar domain. Let  $R_k: \mathbf{R}^3 \rightarrow$*

$\mathbf{R}^3$  be a rotation with respect to a straight line  $L$ , whose argument is  $\frac{2\pi}{k}$ ;  $k \in \mathbf{N}$ ,  $k \neq 0$ . Suppose that  $L$  intersects transversally  $X(U)$  and that  $X(U)$  is invariant by  $R_k$ . Then the associated surface in  $\mathbf{H}^3$  is also invariant by a rotation with respect to a geodesic whose argument is exactly  $\frac{2\pi}{k}$ .

*Proof.* Suppose first that  $X$  is one-to-one. Then  $S$ , the induced isometry of  $U$ , has a unique fixed point  $p \in U$ . Since  $X$  is conformal and  $X \circ S = R \circ X$ ,  $D_p S$  is a rotation whose argument is  $\frac{2\pi}{k}$ . Let  $\tilde{X}: U \rightarrow \mathbf{H}^3$  be the conformal and constant mean curvature 1 immersion associated to  $X$ . Let  $\tilde{R}_k$  be the rotation in  $\mathbf{H}^3$  with respect to the geodesic line through  $\tilde{X}(p)$  that is orthogonal to  $\tilde{X}(U)$  and whose argument is  $\frac{2\pi}{k}$ . By construction we have

$$\begin{aligned} (\tilde{X} \circ S)(p) &= (\tilde{R} \circ \tilde{X})(p), \\ \frac{\partial \tilde{X} \circ S}{\partial x}(p) &= \frac{\partial \tilde{R} \circ \tilde{X}}{\partial x}(p), \\ \frac{\partial \tilde{X} \circ S}{\partial y}(p) &= \frac{\partial \tilde{R} \circ \tilde{X}}{\partial y}(p). \end{aligned}$$

The conditions of Lemma 3.5 are therefore satisfied, and we conclude that  $\tilde{X} \circ S = \tilde{R} \circ \tilde{X}$ .

If  $X$  is not one-to-one, we apply the above argument to a neighborhood of a fixed point of  $S$  in  $U$ . Then a piece of  $\tilde{X}(U)$  is invariant under the action of a rotation with respect of a geodesic line whose argument is  $\frac{2\pi}{k}$ . Since  $\tilde{X}(U)$  is an analytic surface, it follows that the whole surface is invariant under the rotation. □

**REMARK 3.7.** Proposition 3.6 fails to hold if  $U$  is not simply connected and if the axis of the rotation in  $\mathbf{R}^3$  does not intersect  $X(U)$ . Indeed, we saw in Example 1.8 that the surface in  $\mathbf{H}^3$  that is locally associated to the catenoid  $C_{-1/4}$  of  $\mathbf{R}^3$  (defined by  $g(u) = u$  and  $\omega = (-1/4)u^{-2} du$ ,  $u \in \mathbf{C}^*$ ) is not a rotational surface in  $\mathbf{H}^3$ , but a surface that is invariant under some horizontal translations.

**Added in proof.** Kenmotsu (Math. Ann **245** (1979), 89-99) showed that any  $C^2$  solution  $E$  on a simply-connected domain  $U$  of the equation

$$E_{z\bar{z}} = 2 \frac{\overline{E}}{1 + E\overline{E}} E_z E_{\bar{z}}$$

produces a conformal immersion  $X : U^* \rightarrow \mathbf{R}^3$  of constant (non-zero) mean curvature, where  $U^* := U \setminus \{z, E_{\bar{z}} = 0\}$ . He also proved a similar result for prescribed mean curvature. As far as we know, no explicit (non-trivial) solutions of this equation are known, and the equation is still unsolved.

Recently, we derived a similar equation, namely

$$(*) \quad E_{z\bar{z}} = \frac{\bar{E}}{1 + E\bar{E}} E_z E_{\bar{z}},$$

for which every solution gives rise to a mean curvature one conformal immersion  $X : U^* \rightarrow \mathbf{H}^3$  into hyperbolic space. In contrast to the above equation, we can give a complete description of the  $C^2$  solutions of (\*). Indeed, any solution of (\*) can be expressed in terms of meromorphic data  $(h, T)$ . Conversely, given any non constant meromorphic data  $(h, T)$  with  $h \neq 1/(\alpha T + \beta)$ ,  $\alpha, \beta \in \mathbb{C}$ , there is a natural way to describe explicitly a conformal parametrization of a piece of a surface with mean curvature one into hyperbolic space, which involves just one integration,  $\int h^2 T_z dz$ . In particular, we obtain a deformation of a surface discovered by Poleni in 1729. This work will appear in a forthcoming paper, *Meromorphic data for mean curvature one surfaces in hyperbolic space*.

Any conformal immersion in the half-space model of hyperbolic space can be expressed in terms of the standard Euclidean Gauss  $E$  map and the hyperbolic Gauss map  $G$ . In another forthcoming paper, *Meromorphic data for mean curvature one surfaces in hyperbolic space, II*, we make this more precise by showing how to control the branch points. The branch points, if there are any, are isolated; in any case, it is possible to handle the branch points and obtain many complete such surfaces. Thus, it is possible to compute the coordinate maps of the immersion and its geometric quantities by the means of  $E$  and  $G$ . We emphasize that for mean curvature one immersion any geometric quantity can be expressed in terms of the Euclidean Gauss map  $E$  alone.

Finally, we have obtained a Weierstrass-Kenmotsu type theorem for prescribed mean curvature surfaces in hyperbolic space; see our paper in *Séminaire de théorie spectrale et géométrie de Grenoble*, **19** (2001), 9–23, <http://www-fourier.ujf-grenoble.fr/>.

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