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MINIMALITY AND HARMONICITY FOR HOPF VECTOR FIELDS

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Abstract. We determine when the Hopf vector fields on orientable real hypersurfaces (M, g) in complex space forms are minimal or harmonic. Furthermore, we determine when these vector fields give rise to harmonic maps from (M, g) to the unit tangent sphere bundle (T_1M, g_S). In particular, we consider the special case of Hopf hypersurfaces and of ruled hypersurfaces. The Hopf vector fields on Hopf hypersurfaces with constant principal curvatures provide examples. The minimal ruled real hypersurfaces form another class of particular examples.

1. Introduction

Let (M, g) be a Riemannian manifold and let (T_1M, g_S) be its unit tangent sphere bundle equipped with the Sasaki metric g_S. A unit vector field ξ on (M, g) determines a map from the manifold into its unit tangent sphere bundle, and the image of this map is a submanifold of M. When M is closed and orientable, this gives rise to two functionals on the set of unit vector fields $\mathcal{X}^1(M)$: the energy of the map, called the *energy* of the vector field ξ, and the volume of the submanifold, called the *volume* of ξ. These functionals yield two critical point conditions, which may also be considered on general Riemannian manifold with non-empty $\mathcal{X}^1(M)$. A unit vector field ξ satisfying the first critical point condition is called a *harmonic vector field*, and a field ξ satisfying the second condition is said to be a *minimal vector field*. A minimal unit vector field corresponds to a minimal submanifold, but a harmonic unit vector field does not necessarily yield a harmonic map. We refer to [7], [8], [9], [10], [14], [20], [22] and [23] for a general treatment of this and related problems. Examples of minimal and harmonic vector fields have been given in [5], [6], [7], [8], [11], [12], [13] and [21].

The main purpose of this paper is to consider another natural class of manifolds equipped with a unit vector field. Let $(\overline{M}, g, J)$ be an almost Hermitian

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manifold and let (M,g) be an orientable real hypersurface with induced metric g. Furthermore, let N be a unit normal vector field of M. Then $\xi = -JN$ determines a unit tangent vector field on M, called the Hopf vector field. Here we investigate the harmonicity and minimality condition for ξ and for the case when the ambient space $(\overline{M}, g, J)$ is a complex space form. In particular, we consider this situation when (M,g) is a Hopf hypersurface, that is, when ξ is an eigenvector of the shape operator, or a ruled real hypersurface which is not of Hopf type. This again provides a series of examples, in particular for Hopf hypersurfaces with constant principal curvatures and for minimal ruled real hypersurfaces.

2. Preliminaries

In this section we recall some basic facts about minimal and harmonic vector fields, and about orientable real hypersurfaces in complex space forms.

Let (M,g) be an m-dimensional Riemannian manifold of class C^∞, ∇ its Levi Civita connection and R the Riemannian curvature tensor. Furthermore, let $\mathcal{X}^1(M)$ denote the set of all smooth unit vector fields on M which we suppose to be non-empty. A unit vector field ξ can be regarded as an immersion of M into its unit tangent sphere bundle (T_1M, g_S), where g_S denotes the Sasaki metric. Then the induced metric $\xi^* g_S$ is given by

$$(\xi^* g_S)(X,Y) = g(X,Y) + g(\nabla_X \xi, \nabla_Y \xi).$$

We define two tensor fields of type (1,1), A_ξ and L_ξ, by

$$A_\xi = -\nabla \xi, \qquad L_\xi = I + A_\xi^t A_\xi$$

and a function f by $f(\xi) = (\det L_\xi)^{1/2}$. Then, for a closed oriented manifold M, the energy $E(\xi)$ and the volume $\mathrm{Vol}(\xi)$ of ξ are defined by

$$\begin{aligned} E(\xi) &= \tfrac{1}{2}\int_M \mathrm{tr}\, L_\xi \mathrm{d}v = \tfrac{m}{2}\mathrm{vol}\,(M) + \tfrac{1}{2}\int_M |\nabla\xi|^2 \mathrm{d}v, \\ \mathrm{Vol}\,(\xi) &= \int_M f(\xi)\mathrm{d}v, \end{aligned}$$

where $\mathrm{d}v$ denotes the volume form on (M,g). Note that $E(\xi)$ is, up to constants, equal to the quantity $\int_M |\nabla\xi|^2 \mathrm{d}v$, known as the total bending of ξ [22].

The critical point conditions for the functionals E and Vol on $\mathcal{X}^1(M)$ have been derived in [22] and [8], respectively. (See also [7] for a unified treatment.) To state these conditions, we introduce some tensor fields. The one-forms ν_ξ and $\tilde{\nu}_\xi$ associated to the unit vector field ξ are defined by

$$\begin{aligned} \nu_\xi(X) &= \mathrm{tr}\,(Z \mapsto (\nabla_Z A_\xi^t)X), \\ \tilde{\nu}_\xi(X) &= \mathrm{tr}\,(Z \mapsto R(A_\xi Z, \xi)X). \end{aligned}$$

Then ξ is a critical point for the energy functional E if and only if ν_ξ vanishes on $\xi^\perp$. Here $\xi^\perp$ denotes the distribution determined by tangent vectors orthogonal to ξ. A unit vector field ξ on (M,g) is said to be a *harmonic vector* *field* if ν_ξ vanishes on $\xi^\perp$. A harmonic field ξ does not always give rise to a harmonic map of (M, g) into (T_1M, g_S). As was shown in [7], ξ determines a harmonic map if and only if ξ is harmonic and moreover, $\tilde{\nu}_\xi$ vanishes on the whole tangent bundle TM.

Next, we define a tensor field K_ξ and a one-form ω_ξ, associated to ξ, by

$$\begin{aligned} K_\xi &= -f(\xi)L_\xi^{-1}A_\xi^t, \\ \omega_\xi(X) &= \operatorname{tr}(Z \mapsto (\nabla_Z K_\xi)X). \end{aligned}$$

Then ξ is a critical point for the volume functional Vol if and only if ω_ξ vanishes on $\xi^\perp$. A unit vector field ξ is called a *minimal vector field* if ω_ξ vanishes on $\xi^\perp$. A field ξ is minimal if and only if the submanifold $\xi(M)$ is a minimal submanifold of (T_1M, g_S) (see [8]).

We now recall some facts about orientable real hypersurfaces in complex space forms; we refer to [2] and [19] for more details and further references.

We denote by $(\overline{M}(c), g, J)$ a complex space form of constant holomorphic sectional curvature c and with real dimension $2n$. Let M be a connected, orientable real hypersurface of $\overline{M}(c)$ and N a unit normal vector field of M. For any vector field X on M, we put

$$JX = \varphi X + \eta(X)N, \quad JN = -\xi, \tag{2.1}$$

where φ is a tensor field of type $(1, 1)$ and φX is the tangential part of JX, η is a one-form and ξ is a unit vector field on M. We also denote by g the induced Riemannian metric on M. Then (φ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y on M. The field ξ is called the *Hopf vector field* on M [2].

The Gauss and Weingarten formulas for M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + g(SX, Y)N, \qquad \overline{\nabla}_X N = -SX, \tag{2.3}$$

where $\overline{\nabla}$ and ∇ denote the Levi Civita connections of $(\bar{M}(c), g)$ and (M, g) respectively, and S is the shape operator of M. From (2.1) and (2.3) we obtain

$$(\nabla_X \varphi)Y = \eta(Y)SX - g(SX, Y)\xi, \qquad \nabla_X \xi = \varphi SX. \tag{2.4}$$

Furthermore, we have the Gauss and Codazzi equations

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\, \{&g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ &-2g(\varphi X, Y)\varphi Z\} + g(SY, Z)SX - g(SX, Z)SY\}, \end{aligned} \tag{2.5}$$

$$(\nabla_X S)Y - (\nabla_Y S)X = \frac{c}{4}\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\}, \tag{2.6}$$

where R is taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

M is called a *Hopf hypersurface* if ξ is a principal curvature vector, that is, if $S\xi = \alpha\xi$. Tubes about complex submanifolds in $\overline{M}(c)$ provide a class of nice examples. Hopf hypersurfaces have some remarkable properties. When $c \neq 0$, then α is constant, and for principal curvatures λ whose corresponding principal vectors lie in $\xi^\perp$ (as in [19], we express this by saying that λ is a principal curvature on $\xi^\perp$) we have the following result.

Proposition 2.1. *For $X \in \xi^\perp$ and $SX = \lambda X$, we have*

$$(2\lambda - \alpha)S\varphi X = (\alpha\lambda + \frac{c}{2})\varphi X.$$

Finally, let $h = \mathrm{tr}\ S$ denote the mean curvature of M. Then the following result holds.

Proposition 2.2. *Principal curvatures on $\xi^\perp$ are constant along the integral curves of ξ. In particular, $\xi h = 0$ for complex space forms of non-zero curvature.*

Proof. Let E be a local unit vector field on $\xi^\perp$ satisfying $SE = \lambda E$. (As usual, we restrict to the dense open subset of M on which the multiplicities of the eigenvalues of S are locally constant, if necessary.) Then, by (2.6), we have

$$\begin{aligned} 0 &= g((\nabla_\xi S)E, E) - g((\nabla_E S)\xi, E) \\ &= \xi\lambda + (\lambda - \alpha)g(\nabla_E \xi, E) \\ &= \xi\lambda + (\lambda - \alpha)\lambda g(\varphi E, E) \\ &= \xi\lambda. \end{aligned}$$

Since α is constant for complex space forms with $c \neq 0$, it is obvious that $\xi h = 0$. □

3. Harmonic Hopf vector fields

We start this section by deriving a useful criterion for the harmonicity of a Hopf vector field.

Theorem 3.1. *Let M be a (connected) orientable real hypersurface of a complex space form $\overline{M}(c)$. Then the Hopf vector field ξ is harmonic if and only if*

$$Xh - g(\varphi S^2\xi, X) = 0 \tag{3.1}$$

for all $X \in \xi^\perp$, where h denotes the mean curvature of M.

Proof. Since $A_\xi = -\nabla\xi = -\varphi S$, we have $A_\xi^t = S\varphi$. Further, let Y be an arbitrary vector field of $\xi^\perp$ and $\{E_1, \cdots, E_{2n-1}\}$ a local orthonormal frame field. Then

$$\begin{aligned}\nu_\xi(Y) &= \sum_{i=1}^{2n-1} g((\nabla_{E_i} A^t_\xi)Y, E_i) = \sum_{i=1}^{2n-1} g((\nabla_{E_i}(S\varphi))Y, E_i)\\ &= \sum_{i=1}^{2n-1} g((\nabla_{E_i} S)\varphi Y, E_i) + \sum_{i=1}^{2n-1} g(S(\nabla_{E_i}\varphi)Y, E_i).\end{aligned}$$

For the first term we use (2.6) to obtain

$$\sum_{i=1}^{2n-1} g((\nabla_{E_i} S)\varphi Y, E_i) = \sum_{i=1}^{2n-1} g((\nabla_{\varphi Y} S)E_i, E_i) = (\varphi Y)h.$$

For the second term we obtain, using (2.4),

$$\sum_{i=1}^{2n-1} g(S(\nabla_{E_i}\varphi)Y, E_i) = \sum_{i=1}^{2n-1} g((\nabla_{E_i}\varphi)Y, SE_i) = -g(S^2\xi, Y).$$

Therefore we have

$$\nu_\xi(Y) = (\varphi Y)h - g(S^2\xi, Y),$$

and the required result follows by setting $Y = \varphi X$. □

From this we get immediately, using Proposition 2.2, the following corollary.

Corollary 3.2. *Let (M, g) be a Hopf hypersurface in a complex space form $\bar{M}(c), c \neq 0$. Then ξ is harmonic if and only if the mean curvature is constant.*

Next, by using (2.5), we obtain

$$\begin{aligned}\tilde{\nu}_\xi(X) &= \sum_{i=1}^{2n-1} g(R(A_\xi E_i, \xi)X, E_i)\\ &= -\sum_{i=1}^{2n-1} g(R(\varphi SE_i, \xi)X, E_i) \qquad (3.2)\\ &= \frac{c}{4} g(\varphi S\xi, X) + g(S\varphi S^2\xi, X).\end{aligned}$$

This yields the following result.

Proposition 3.3. *Let (M, g) be a (connected) orientable real hypersurface of $\overline{M}(c)$, $c \neq 0$, with constant mean curvature. Then ξ determines a harmonic map of (M, g) into (T_1M, g_S) if and only if (M, g) is a Hopf hypersurface.*

Proof. For a Hopf hypersurface the result follows at once from Corollary 3.2 and (3.2). Conversely, let ξ determine a harmonic map. Then (3.1) yields $\varphi S^2\xi = 0$ and since $\tilde{\nu}_\xi(X) = 0$ for all X, we then obtain $\varphi S\xi = 0$ or, equivalently, $S\xi = \alpha\xi$. Thus, (M, g) is a Hopf hypersurface. □

Corollary 3.4. *The Hopf vector field on a Hopf hypersurface with constant principal curvatures determines a harmonic map.*

Note that Hopf hypersurfaces with constant principal curvatures have been classified in [3] and [15].

Next, we turn to another interesting class of hypersurfaces which are not of Hopf type. Let M be an orientable real hypersurface of $\overline{M}(c), c \neq 0$. If the distribution $\xi^{\perp}$ is integrable and each integral submanifold of $\xi^{\perp}$ is a totally geodesic submanifold in $\overline{M}(c)$, then M is called a *ruled real hypersurface* (see [16], [17]). For such a hypersurface it is easily seen [17] that the shape operator satisfies

$$\begin{aligned} S\xi &= \mu\xi + \nu U, \qquad \nu \neq 0, \\ SU &= \nu\xi, \\ SX &= 0 \quad \text{for any} \quad X \perp \xi, U, \end{aligned} \tag{3.3}$$

where U is a unit vector field of $\xi^{\perp}$ and μ and ν are differentiable functions on M. Then the mean curvature h is equal to μ and the Hopf vector field ξ is not a principal curvature vector. From (2.4) we get $\nabla_X \xi = 0$ for $X \in \xi^{\perp}$. Furthermore, from (2.6) we obtain

$$X\mu - \mu\nu g(X, \varphi U) - \nu g(X, \nabla_\xi U) = 0, \quad U\mu = \xi\nu, \tag{3.4}$$

for X orthogonal to the two-plane field determined by ξ and U. Now, applying Theorem 3.1, yields the following result.

Proposition 3.5. *The Hopf vector field ξ on a ruled real hypersurface is harmonic if and only if*

$$(\varphi U)\mu - \mu\nu = 0, \qquad X\mu = 0, \tag{3.5}$$

for all X orthogonal to ξ and φU.

Corollary 3.6. *The Hopf vector field on a minimal ruled real hypersurface is always harmonic.*

Examples of minimal ruled real hypersurfaces have been given in [1], [4], [16], and [18].

Note that, since $\tilde{\nu}_\xi(X) = \frac{c}{4}\nu g(\varphi U, X)$, ξ never determines a harmonic map.

4. Minimal Hopf vector fields on Hopf hypersurfaces

In this section we concentrate on the minimality condition for the Hopf vector field on a Hopf hypersurface M in a complex space form $\overline{M}(c)$. Let $\lambda_i, i = 1, \cdots, 2(n-1)$, be the principal curvatures corresponding to principal vectors in $\xi^{\perp}$, and let $\mathcal{U}$ be the dense open subset of M where the multiplicities of these principal curvatures are locally constant. Furthermore, let $\tilde{h}$ be the *modified* mean curvature function defined by

$$\tilde{h} = \sum_{i=1}^{2(n-1)} \text{arc cot } \lambda_i$$

where $0 < \text{arc cot } \lambda_i < \pi$. Then $\tilde{h}$ is differentiable on $\mathcal{U}$ and we have the following general result.

THEOREM 4.1. *On $\mathcal{U}$ we have*

$$\omega_\xi(X) = f(\xi) d\tilde{h}(L_\xi^{-1} \varphi X)$$

for $X \in \xi^\perp$.

Proof. On $\mathcal{U}$ we choose a local orthonormal frame field $\{E_1, \cdots, E_{2(n-1)}\}$ of $\xi^\perp$ which satisfies $SE_i = \lambda_i E_i$ for $i = 1, \cdots, 2(n-1)$. Since $A_\xi = -\nabla\xi = -\varphi S$, we have

$$L_\xi \xi = \xi, \qquad L_\xi E_i = (1 + \lambda_i^2) E_i$$

and hence

$$f(\xi) = (\det L_\xi)^{1/2} = \left(\prod_{i=1}^{2(n-1)} (1 + \lambda_i^2) \right)^{1/2},$$

$$L_\xi^{-1} \xi = \xi \quad , \quad L_\xi^{-1} E_i = (1 + \lambda_i^2)^{-1} E_i \ .$$

Since $K_\xi = -f(\xi) L_\xi^{-1} A_\xi^t = -f(\xi) L_\xi^{-1} S\varphi$, we get $K_\xi \xi = 0, K_\xi(\xi^\perp) \subset \xi^\perp$. In particular, for $X \in \xi^\perp$ we have

$$g((\nabla_\xi K_\xi) X, \xi) = 0.$$

Note that $\nabla_\xi \xi = 0$. Therefore we have

$$\omega_\xi(X) = \sum_{i=1}^{2(n-1)} g((\nabla_{E_i} K_\xi) X, E_i).$$

Now, put $X = \varphi E_j$ in this formula to get

$$\omega_\xi(\varphi E_j) = \sum_{i=1}^{2(n-1)} g(\nabla_{E_i}(K_\xi \varphi E_j), E_i) - \sum_{i=1}^{2(n-1)} g(K_\xi \nabla_{E_i}(\varphi E_j), E_i). \tag{4.1}$$

We evaluate the two terms on the right-hand side of this relation.

For the first term we have

$$\sum_{i=1}^{2(n-1)} g(\nabla_{E_i}(K_\xi \varphi E_j), E_i) = \sum_{i=1}^{2(n-1)} g(\nabla_{E_i}(f(\xi)\frac{\lambda_j}{1+\lambda_j^2}E_j), E_i)$$

$$= E_j(f(\xi)\frac{\lambda_j}{1+\lambda_j^2}) + f(\xi)\frac{\lambda_j}{1+\lambda_j^2}\sum_{i=1}^{2(n-1)} g(\nabla_{E_i}E_j, E_i).$$

Furthermore,

$$\begin{aligned} E_j f(\xi) &= E_j(\det\, L_\xi)^{1/2} = \frac{1}{2}f(\xi)E_j(\log\det L_\xi) \\ &= \frac{1}{2}f(\xi)E_j(\log\prod_{i=1}^{2(n-1)}(1+\lambda_i^2)) = f(\xi)\sum_{i=1}^{2(n-1)}\frac{\lambda_i}{1+\lambda_i^2}E_j(\lambda_i). \end{aligned}$$

Hence, we get

$$E_j(f(\xi)\frac{\lambda_j}{1+\lambda_j^2}) = f(\xi)\left\{\frac{1}{(1+\lambda_j^2)^2}E_j(\lambda_j) + \frac{\lambda_j}{1+\lambda_j^2}\sum_{i\neq j}\frac{\lambda_i}{1+\lambda_i^2}E_j(\lambda_i)\right\}.$$

Next, we compute the second term in (4.1). We have

$$\begin{aligned} \sum_{i=1}^{2(n-1)} g(K_\xi \nabla_{E_i}(\varphi E_j), E_i) &= f(\xi)\sum_{i=1}^{2(n-1)} g(\nabla_{E_i}(\varphi E_j), \varphi S L_\xi^{-1}E_i) \\ &= f(\xi)\sum_{i=1}^{2(n-1)}\frac{\lambda_i}{1+\lambda_i^2}g(\nabla_{E_i}(\varphi E_j), \varphi E_i) \\ &= f(\xi)\sum_{i=1}^{2(n-1)}\frac{\lambda_i}{1+\lambda_i^2}g(\nabla_{E_i}E_j, E_i). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \omega_\xi(\varphi E_j) = f(\xi)\Bigg\{&\frac{1}{(1+\lambda_j^2)^2}E_j(\lambda_j) + \frac{\lambda_j}{1+\lambda_j^2}\sum_{i\neq j}\frac{\lambda_i}{1+\lambda_i^2}E_j(\lambda_i) \\ &+\sum_{i\neq j}\frac{1-\lambda_i\lambda_j}{(1+\lambda_j^2)(1+\lambda_i^2)}(\lambda_j-\lambda_i)g(\nabla_{E_i}E_j, E_i)\Bigg\}\ . \end{aligned}$$

Now, by (2.6) we have

$$E_j(\lambda_i) = (\lambda_j - \lambda_i)g(\nabla_{E_i}E_j, E_i) \quad \text{for } i \neq j.$$

Hence we obtain

$$\begin{aligned}\omega_\xi(\varphi E_j) &= f(\xi)\frac{1}{1+\lambda_j^2}\sum_{i=1}^{2(n-1)}\frac{1}{1+\lambda_i^2}E_j(\lambda_i)\\ &= -f(\xi)\frac{1}{1+\lambda_j^2}\sum_{i=1}^{2(n-1)}E_j(\text{arc cot }\lambda_i)\\ &= -f(\xi)\frac{1}{1+\lambda_j^2}E_j(\tilde{h}).\end{aligned}$$

On the other hand, we have

$$f(\xi)d\tilde{h}(L_\xi^{-1}\varphi^2E_j) = -f(\xi)\frac{1}{1+\lambda_j^2}E_j(\tilde{h}).$$

The required result now follows. □

Using again Proposition 2.2, we obtain the following corollary.

Corollary 4.2. *Let M be a Hopf hypersurface in a complex space form $\overline{M}(c)$. Then the Hopf vector field ξ is minimal if and only if $\tilde{h}$ is constant.*

This corollary implies that the Hopf vector fields on Hopf hypersurfaces with constant principal curvatures are always minimal vector fields. When the holomorphic sectional curvature equals 4, we have a remarkable stronger result.

Corollary 4.3. *Let M be a Hopf hypersurface in $\overline{M}(4)$. Then the Hopf vector field is always minimal.*

Proof. Put

$$f(\lambda) = \text{arc cot }\lambda + \text{arc cot }\frac{\alpha\lambda+2}{2\lambda-\alpha}.$$

The function f is discontinuous at $\lambda = \alpha/2$. Furthermore, we have $f'(\lambda) = 0$ at $\lambda \neq \alpha/2$ and

$$\lim_{\lambda\to+\infty} f(\lambda) = \text{arc cot }\frac{\alpha}{2}, \qquad \lim_{\lambda\to-\infty} f(\lambda) = \text{arc cot }\frac{\alpha}{2} + \pi.$$

Hence, $f(\lambda) = \text{arc cot }(\alpha/2)$ when $\lambda > \alpha/2$ and $f(\lambda) = \text{arc cot }(\alpha/2) + \pi$ when $\lambda < \alpha/2$.

Denote by m_+ (respectively m_-) the number of principal curvatures which are larger (respectively smaller) than $\alpha/2$. The numbers m_+ and m_- are both even and locally constant, and since M is connected, they are constant on M. Therefore we have

$$\begin{aligned}\tilde{h} &= \frac{m_+}{2}\text{arc cot }\frac{\alpha}{2} + \frac{m_-}{2}(\text{arc cot }\frac{\alpha}{2} + \pi)\\ &= (n-1)\text{arc cot }\frac{\alpha}{2} + \frac{m_-}{2}\pi.\end{aligned}$$

Hence, $\tilde{h}$ is constant, and the result follows from Corollary 4.2. □

To conclude this paper, we consider again ruled real hypersurfaces and derive a criterion for the minimality of the Hopf vector field ξ. A straightforward computation, using the formulas given in Section 3, yields

$$K_\xi \xi = 0, \quad K_\xi(\varphi U) = \frac{\nu}{(1+\nu^2)^{1/2}}\xi, \quad K_\xi X = 0 \quad \text{for } X \perp \xi, \varphi U.$$

Now, let $\{E_0 = \xi, E_1, \cdots, E_{2(n-1)}\}$ be a local orthonormal frame field. Then, for $Y \in \xi^\perp$ we have

$$\begin{aligned} \omega_\xi(Y) &= \sum_{i=0}^{2(n-1)} \{g(\nabla_{E_i}(K_\xi Y), E_i) - g(\nabla_{E_i} Y, K_\xi^t E_i)\} \\ &= g(\nabla_\xi(K_\xi Y), \xi) - \frac{\nu}{(1+\nu^2)^{1/2}} g(\nabla_\xi Y, \varphi U). \end{aligned}$$

Therefore, ξ is minimal if and only if the following conditions are satisfied:

$$g(\nabla_\xi U, X) = 0 \text{ for all } X \perp \xi, U; \quad \xi\nu = 0. \tag{4.2}$$

Using (3.4), it follows now that (4.2) holds if and only if (3.5) holds. Thus we have the following result.

Proposition 4.4. *The Hopf vector field on a ruled real hypersurface is minimal if and only if it is harmonic.*

From this result and Corollary 3.6 we deduce the following corollary.

Corollary 4.5. *The Hopf vector field on a minimal ruled real hypersurface is always minimal.*

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