Illinois Journal of Mathematics
Volume 45, Number 2, Summer 2001, Pages 615–629
S 0019-2082

MANIFOLDS CLOSE TO THE ROUND SPHERE

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Abstract. We prove that the manifold M^n of minimal radial curvature $K_o^{\min} \geq 1$ is homeomorphic to the sphere S^n if its radius or volume is larger than half the radius or volume of the round sphere of constant curvature 1. These results are optimal and give a complete generalization of the corresponding results for manifolds of sectional curvature bounded from below.

0. Introduction and results

0.1. A standard problem in Riemannian Geometry is to find conditions which guarantee that a given Riemannian manifold is topologically or metrically close to the round sphere S^n of constant curvature 1 (so-called sphere recognition theorems; see [GM1, GM2]). Two of the best known results of this type are contained in the following theorem.

Theorem 1. *Let M^n be an n-dimensional compact Riemannian manifold without boundary of sectional curvature $K \geq 1$ and $\mathrm{rad}(M^n) > \pi - \epsilon$. Then we have:*

(1) (Grove-Shiohama [GS]) *M^n is homeomorphic to S^n if* $\mathrm{rad}(M^n) > \pi/2$.
(2) (Grove-Petersen [GP]) *The Gromov-Hausdorff distance between M^n and S^n satisfies $d_{GH}(M^n, S^n) \leq C(\epsilon)$ for some function $C(\epsilon) \to 0$ as $\epsilon \to 0$.*

We consider the class of manifolds with minimal radial curvature bounded from below by 1. This class is substantially larger than the class of manifolds in Theorem 1 with sectional curvature bounded from below by 1. (Recall that a Riemannian manifold M^n has minimal radial curvature $K_o^{\min}$ with a base point o bounded from below by k, $K_o^{\min} \geq k$, if for an arbitrary point p and every minimal geodesic $\gamma(t), 0 \leq t \leq r$, connecting o and p the sectional

Received February 15, 2000; received in final form August 22, 2000.
2000 *Mathematics Subject Classification.* 53C20, 53C21.
Supported by CNPq.

curvature of M^n is at least k in all two-dimensional directions which contain the vector $\dot{\gamma}(r)$.)[1]

In a previous paper [MM] we proved the following analog to Theorem 1 above.

Theorem A. *Let M^n be an n-dimensional compact Riemannian manifold without boundary of minimal radial curvature $K_o^{\min} \geq 1$ and $rad(M^n) > \pi - \epsilon$. Then we have:*

(1) *If ϵ is sufficiently small, then M^n is homeomorphic to S^n (i.e., M^n is a twisted sphere).*
(2) *The Gromov-Hausdorff distance between M^n and S^n satisfies $d_{GH}(M^n, S^n) \leq C(\epsilon)$ for some function $C(\epsilon) \to 0$ as $\epsilon \to 0$.* [2]

In contrast to Theorem 1 above, Theorem A has the defect that the constant ϵ in the bound $\mathrm{rad}(M^n) > \pi - \epsilon$ could be a function of the dimension n that tends to 0 as $n \to \infty$. In this paper we overcome this defect and show that, as in Theorem 1, one can take $\epsilon = \pi/2$, which is best possible.

Theorem A′. *Let M^n be an n-dimensional compact Riemannian manifold without boundary of minimal radial curvature $K_o^{\min} \geq 1$ and $rad(M^n) > \pi/2$. Then M^n is homeomorphic to S^n.*

This provides a complete generalization of Theorem 1 to manifolds of minimal radial curvature bounded from below by 1. As in Theorem 1 the projective space RP^n gives an example of a manifold with minimal radial curvature $K_o^{\min} \geq 1$, $\mathrm{rad}(M^n) = \pi/2$, which is nonhomeomorphic to S^n, and hence shows that the bound $\mathrm{rad}(M^n) > \pi/2$ is best possible.

To prove Theorem A′ we combine the arguments which had lead us earlier to conjecture Theorem A, namely a combination of the Borsuk-Ulam Theorem and the comparison angle almost nonincreasing flow introduced in [MM]. As is often the case, the simpler argument leads to a best possible result, whereas the more complicated approach of [MM] gives only a weaker result. We remark that some general problems are naturally suggested by our arguments. For instance, the following conjecture might be true.

Conjecture. *Let (Σ, d) be a space equipped with a metric d (not necessarily the length, i.e., in general, the distance between two points need not be*

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[1] As was noted by Machigashira and Shiohama, some well-known results on the geometry of Riemannian manifolds of nonnegative sectional curvature, such as Toponogov comparison theorem, estimates on diameter and radius, or the sphere theorem, can be generalized to manifolds with non-negative minimal radial curvature; see [Mch1], [Mch2] and [MS].

[2] Note that the proof of this second claim given in [MM] is by some direct geometric arguments, it does not use the result due to Gromov (see [G]) on the relations between $\mathrm{Fillrad}(M^n)$ and $\mathrm{Fillvol}(M^n)$ as in [GP].

the length of the shortest curve between these points), and let Σ *be homeomorphic to a sphere* S^n *and satisfy* $\pi/2+\epsilon \leq \mathrm{rad}(\Sigma, d) \leq \mathrm{diam}(\Sigma) \leq \pi$ *for* $\epsilon > 0$. *Then there exists a continuous map* $(\ )^* : \Sigma \to \Sigma$ *such that* $d(x, x^*) > d(\epsilon)$ *for some* $d(\epsilon) > 0$ *and* $d(\epsilon) \to \pi$ *as* $\epsilon \to \pi/2$. [3]

0.2. According to Theorem A the manifold M^n is topologically and metrically close to the round sphere S^n of constant curvature 1 if its radius is close to the radius of the sphere. The same is true for the volume.

THEOREM 2. *Let* M^n *be an* n*-dimensional compact Riemannian manifold without boundary with minimal radial curvature* $K_o^{\min} \geq 1$ *and* $\mathrm{vol}(M^n) > vol(S^n) - \epsilon$. *Then we have:*

(1) (Machigashira-Shiohama [MS]) M^n *is homeomorphic to* S^n *if* $\mathrm{vol}(M^n) > (3/4)\,\mathrm{vol}(S^n)$.
(2) ([MM]) *The Gromov-Hausdorff distance between* M^n *and* S^n *satisfies* $d_{GH}(M^n, S^n) \leq C(\epsilon)$ *for some function* $C(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Our next result is the following.

THEOREM A$''$. *Let* M^n *be an* n*-dimensional compact Riemannian manifold without boundary, of minimal radial curvature* $K_o^{\min} \geq 1$, *and satisfying* $\mathrm{vol}(M^n) > (1/2)\,\mathrm{vol}(S^n)$. *Then* M^n *is homeomorphic to* S^n.

Again, the example of the projective space RP^n shows that this result is sharp.

In previous papers, we pointed out that there is an interesting analogy between results for manifolds with minimal radial curvature bounded from below by 1 and manifolds with Ricci curvature bounded from below by $(n-1)$: the theorems of [MM] and [M] are analogous to results of [Wl] and [Cl1], [Cl3]. It is natural to conjecture that the assertions of Theorems A$'$ and A$''$ are still true for manifolds with corresponding bounds on the Ricci curvature. [4]

I would like to express my sincere gratitude to Sergio J. X. Mendonça for calling my attention to the class of manifolds with minimal radial curvature bounded from below.

1. Some basic facts and notations

Throughout this paper, M^n denotes a complete compact Riemannian manifold without boundary and of minimal radial curvature $K_o^{\min} \geq 1$, and $r_{\mathrm{inj}}(M^n)$ denotes the injectivity radius of M^n.

[3] Is there any involution $(\ )^*$ that has this property?

[4] To the author's knowledge, there are examples due to Anderson and Otsu of closed manifolds M^n with $\mathrm{Ric}(M^n) \geq (n-1)$ and diameter arbitrarily close to π, i.e., that of a round sphere (see [A1]-[A4] and [Ot]), which have $\mathrm{rad}(M^n) < \pi/2$.

1.1. Gromov-Hausdorff distance. Recall that the radius $\mathrm{rad}(X)$ of a compact metric space X is the radius of the smallest ball which contains X, that is

$$\mathrm{rad}(X) = \min_{x\in X} \max_{y\in X} |xy|, \tag{1.1}$$

where $|xy|$ denotes the distance between x and y. Obviously, $\mathrm{diam}(X)/2 \leq \mathrm{rad}(X) \leq \mathrm{diam}(X)$, where $\mathrm{diam}(X)$ is the diameter of X, and for an arbitrary point x there exists a (generally nonunique) point x^* such that $\mathrm{dist}(x, x^*) \geq \mathrm{rad}(X)$. We will denote by x^* the point in X having maximal distance to x, i.e., $\mathrm{dist}(x, x^*) = \sup\{\mathrm{dist}(x, y) | y \in X\}$.

A metric space X is called a *length space* if the distance between any two points equals the infimum of lengths of continuous curves between them. If X is complete, connected and locally compact, standard arguments prove the existence of a minimal geodesic connecting any two given points, i.e., a continuous curve γ_{xy} connecting x and y, whose length equals the distance $|xy|$; see [Pl].

Recall that for given compact metric spaces X and Y, the Gromov-Hausdorff distance $d_{GH}(X, Y)$ between X and Y is the infimum of values of $\epsilon > 0$, such that, for some metric in the disjoint union $X \uplus Y$, the ϵ-neighborhood $B(X, \epsilon)$ of X contains Y (we say in this case that Y is ϵ-close to X) and the ϵ-neighborhood $B(Y, \epsilon)$ of Y contains X, or X and Y are ϵ-close to each other. A map $f : X \to Y$ is said to be an ϵ-approximation if the image $f(X)$ is ϵ-dense in Y and, for any $x, y \in X$, $|\mathrm{dist}(f(x), f(y)) - \mathrm{dist}(x, y)| < \epsilon$. The distance $d_{GH}(X, Y)$ can be defined in an equivalent way as the infimum of the values of $\epsilon > 0$ such that there exist ϵ-approximations $f : X \to Y$ and $g : Y \to X$. Another equivalent definition is that if there exist ϵ-dense nets $\{p_i\}_{i=1,\ldots,N} \subset X$ and $\{q_j\}_{j=1,\ldots,N} \subset Y$ such that, if for all i, j we have $|\mathrm{dist}(p_i, p_j) - \mathrm{dist}(q_i, q_j)| < \epsilon$, then $d_{GH}(X, Y) < 3\epsilon$. For details see [GLP]; for a recent exposition see [Pt].

1.2. Toponogov and Bishop-Gromov comparison theorems for manifolds with minimal radial curvature bounded from below. For a triangle $\triangle_{pqr}$ in M^n let $\triangle\bar{p}\bar{q}\bar{r}$ be a triangle in the space form S^2 of constant curvature 1 with the same lengths, called the *comparison triangle* of $\triangle_{pqr}$ (if such a triangle exists). We denote by $\measuredangle qpr$, or simply $\measuredangle p$, the angle at p of the triangle $\triangle_{pqr}$. The corresponding angle in $\triangle\bar{p}\bar{q}\bar{r}$ is denoted by $\widetilde{\measuredangle} qpr$, or simply $\widetilde{\measuredangle} p$. We let γ_{pq} denote a geodesic in $\triangle_{pqr}$ joining p and q. All geodesics are assumed to be parameterized by the arc-length unless stated otherwise. Machigashira (see [Mch1] and [Mch2]) obtained the following two results, which extend the Toponogov Comparison Theorem and the Toponogov-Alexandrov monotonicity property to the case of manifolds with minimal radial curvature bounded from below and for triangles which have the base point o as one of their vertices.

Proposition 1. *Let M^n be a complete manifold with $K_o^{\min} \geq 1$ and let $\triangle_{opq}$ be a triangle of minimal geodesic segments in M^n. Then there exists a comparison triangle $\triangle\bar{p}\bar{q}\bar{r}$ in a space form S^2 of constant curvature* 1, *and we have $\measuredangle o \geq \widetilde{\measuredangle} o$, $\measuredangle p \geq \widetilde{\measuredangle} p$ and $\measuredangle q \geq \widetilde{\measuredangle} q$.*

Proposition 2. *Let M^n be a complete manifold with $K_o^{\min} \geq 1$. Consider points $p, q, r, s \in M^n$ such that γ_{op} and γ_{oq} are minimal geodesics and r and s belong to γ_{op} and γ_{oq}, respectively. Then we have $\widetilde{\measuredangle} ros \geq \widetilde{\measuredangle} poq$.*

In a standard manner, we obtain as simple consequences of Proposition 1 that $\operatorname{diam}(M^n) \leq \pi$ and $\operatorname{dist}(o,p) + \operatorname{dist}(o,q) + \operatorname{dist}(p,q) \leq 2\pi$ for any points o, p, q of M^n. In [MS] it was shown that if equality holds in any of these inequalities, then M^n is isometric to S^n.

It is also easy to see that manifolds of minimal radial curvature $K_o^{\min} \geq 1$ satisfy the Bishop-Gromov volume comparison theorem for balls with a center at the base point o; see [MS].

Proposition 3. *Let M^n be a manifold of minimal radial curvature $K_o^{\min} \geq 1$, and let $B(o,r)$ be a metric ball in the manifold M^n with center o and radius r. Let $B(\bar{o},r)$ be the corresponding r-ball in the sphere S^n of constant curvature* 1. *Then the function* $\operatorname{vol}(B(o,r))/\operatorname{vol}(B(\bar{o},r))$ *is monotone nonincreasing.*

We will exploit a "comparison angle almost nonincreasing" flow that was constructed in our earlier papers [MM] and [M] and which could be regarded as the parameterized version of Proposition 2; see Proposition 5 below.

1.3. Distance functions and their critical points. We will consider distance functions to some sets along integral trajectories of some smooth vectors fields. It is known that for every smooth vector field X the distance from a fixed point q to the trajectory $x(s)$ of X is a function which has one-sided derivatives at all points s, and for almost all s the left and right derivatives coincide. The left derivative of $\operatorname{dist}(o, x(s))$ at $s = \tau$ is the maximum of the scalar products between $-X(x(\tau))$ and the directions of minimal geodesics from $x(\tau)$ to q:

$$\begin{aligned} \operatorname{dist}'_-(q, x(s))\big|_{s=\tau} = \max\{&(-X(x(\tau)), \dot\gamma(s)) : \\ &\gamma(s) \text{ is a minimal geodesic from } x(s) \text{ to } q\}. \end{aligned} \tag{1.2}$$

This fact is well-known, and its proof is left to the reader. It is also easy to see that, for any set C in M^n, we have

$$\begin{aligned} \operatorname{dist}'_-(C, x(s))\big|_{s=\tau} = \min\{&\operatorname{dist}'_-(q, x(s))\big|_{s=\tau} : \\ &q \in C, \quad \operatorname{dist}(q, x(\tau)) = \operatorname{dist}(C, x(\tau))\}. \end{aligned} \tag{1.3}$$

In what follows, C is assumed to be compact.

The important idea to use general nonsmooth distance functions as Morse functions on Riemannian manifolds is due to Grove and Shiohama; see [GS], where the authors proved, among other results, the first assertion of Theorem 1.

Definition 1. We say that the point x is the critical point of the distance function $\operatorname{dist}(\ ,C)$ if, for an arbitrary direction v at this point and any curve $x(s)$ starting at this point in direction v, we have $\operatorname{dist}'(x(s),C)_{s=0} \leq 0$.

By (1.3) the point x is a critical point of $\operatorname{dist}(\ ,C)$ if, for an arbitrary vector v at this point, there exists a minimal geodesic $\gamma(s)$ from x to C having angle at most $\pi/2$ with v. The importance of the notion of critical points is clear from the following *Isotopy Lemma* (see Lemma 1.4 in [C]).

Proposition 4. *Denote by* $C(r) = \{x| \ \operatorname{dist}(x,C) < r\}$ *the* r*-sublevel set of the distance function. If the set* $A_{r,R} = \bar{C}(R)\backslash C(r)$ *contains no critical points of the function* $\operatorname{dist}(\ ,C)$*, then there exists a continuous family of retractions* $\Phi_s : C(R) \to C(s)$*,* $r \leq s \leq R$*.*

The retractions can be constructed as follows:

Given a point z from $A_{r,R}$ choose a vector $X(z)$ that has angle strictly less than $\pi/2$ with every minimal geodesic $\gamma_{zz'}$ from z to some $z' \in C$ such that $\gamma_{zz'}$ realizes the distance to C, i.e., $\operatorname{dist}(z,z') = \operatorname{dist}(z,C)$. Since $A_{r,R}$ does not contain critical points, such a vector exists, by our definition.

Next, for any $\lambda < r_{\text{inj}}(M^n)$ define a vector field $X_{z,\lambda}$ on a λ-neighborhood $B(z,\lambda)$ of the point z as follows: for $x \in B(z,\lambda)$ the vector $X_{z,\lambda}(x)$ at the point x is the parallel transport of $X(z)$ along the unique minimal geodesic from z to x. Here the parameter λ may depend on z. Because $A_{r,R}$ is compact, the covering $A_{r,R} \subset \cup B(z,\lambda(z))$ contains a finite subcovering $A_{r,R} \subset \cup B(z_i,\lambda(z_i))$, and choosing a partition of unity $f_i : B(z_i,\lambda) \to R$, $f_i > 0$, $\sum f_i(x) \equiv 1$, associated with this subcovering, we define the smooth vector field $X(x,\lambda) = \sum f_i(x) X_{z_i,\lambda(z_i)}$, where $\lambda = \max\{\lambda(z_i)\}$. Since $A_{r,R}$ does not contain critical points, for sufficiently small λ the vector field $X(\lambda)$ is nowhere zero.

Note also that because of the compactness of $A_{r,R}$ there exists $\alpha > 0$ such that for all z of $A_{r,R}$ we have

$$\measuredangle(\gamma_{zz'}, X_\lambda(z)) < \pi/2 - \alpha. \tag{1.4}$$

By (1.3) and standard arguments there exists $\lambda_0 > 0$ depending on $A_{r,R}$ such that, if $0 < \lambda < \lambda_0$, then

$$\|X(x,\lambda)\| \geq \sin(\alpha/2) \quad \text{and} \quad \operatorname{dist}'(x(s),C)) \leq -\alpha/2, \tag{1.5}$$

where $x(s)$ denotes the integral trajectory of the smooth vector field $X(\lambda)$. Now, our retraction is a shift along these trajectories, and it provides an h-cobordism between two components $\partial C(R)$ and $\partial C(r)$ of the boundary of

$A_{r,R}$. Obviously, the same method can be applied to any set without critical points. The following is an example of such an application to a manifold M^n with minimal radial curvature $K_o^{\min} \geq 1$ and $\mathrm{rad}(M^n) > \pi/2$ (see [MS]).

Consider the distance to the base point o. Denote by o^* a point of M^n having maximal distance to o. Since $\mathrm{dist}(o, o^*) \geq \mathrm{rad}(M^n)$, we have $\mathrm{dist}(o, o^*) > \pi/2$. Take an arbitrary point x and consider a triangle $\triangle_{oo^*x}$. Since $\mathrm{dist}(o, o^*) > \pi/2$ and $\mathrm{dist}(o, x) \leq \mathrm{dist}(o, o^*)$, we see that in the comparison triangle $\triangle_{\bar{o}\bar{o}^*\bar{x}}$ in the round sphere S^2 of constant curvature 1 the angle $\measuredangle(\bar{o}\bar{x}\bar{o}^*)$ is always strictly larger than $\pi/2$, provided only that $\mathrm{dist}(o^*, x) \leq \mathrm{dist}(o, o^*)$ and $\mathrm{dist}(o, o^*) > \pi/2$. Hence, by Proposition 1 the angle between any minimal geodesic γ_{xo} between x and o and any minimal geodesic γ_{xo^*} connecting x with o^* is strictly larger than $\pi/2$. Thus, x is not critical. Because o^* is critical (since $\mathrm{dist}(o, \ )$ attains its maximum at o^*), for any minimal geodesic γ_{o^*x} connecting o^* with x there exists a minimal geodesic γ_{o^*o} between o^* and o whose angle with γ_{o^*x} at o^* is at most $\pi/2$. In particular, this implies that the point o^* of maximal distance to the base point o is unique (since, by Proposition 1, any other point o_1^* with $\mathrm{dist}(o, o_1^*) = \mathrm{dist}(o, o^*) > \pi/2$ would satisfy $\measuredangle(oo^*o_1^*) > \pi/2$ because $\mathrm{dist}(o, o^*) > \pi/2$). Thus, we see that if $R^* = \mathrm{dist}(o, o^*)$, then the closed metric ball $B^* = B(o^*, R^*)$ does not contain critical points of the distance function $\mathrm{dist}(o, \ )$. Applying the Isotopy Lemma we then arrive at the following result.

Lemma 1. *Let $R^* = \mathrm{dist}(o, o^*)$, where o^* is the point of M^n having maximal distance to o. If $\pi/2 < R^*$, then for any $R \leq R^*$ the closed metric ball $B(o^*, R)$ does not contain critical points of the distance function* $\mathrm{dist}(o, \ )$ *and is diffeomorphic to the closed Euclidean ball B^n. The boundary $\Sigma^*(R)$ of $B(o^*, R)$ is diffeomorphic to the Euclidean sphere S^{n-1}.*

Consider again the comparison triangle $\triangle_{\bar{o}\bar{o}^*\bar{x}}$, where $x \in \Sigma^*(R^*)$. In this sphere triangle the angles $\bar{o}$ and $\bar{x}$ are equal, since $\mathrm{dist}(o^*, x) = \mathrm{dist}(o^*, o)$. Since these distances are equal to R^*, which is larger than $\pi/2$, these angles are strictly larger than $\pi/2$. A computation shows that

$$\measuredangle(\bar{o}\bar{x}\bar{o}^*) = \measuredangle(\bar{x}\bar{o}\bar{o}^*) > \pi/2 + c(R^*),$$

where $c(R^*) > 0$ is a function of R^* such that $c(R^*) \to 0$ as $R^* \to \pi/2$. If we fix $R^* > \pi/2$ and consider a sphere triangle $\triangle_{\bar{o}\bar{y}\bar{o}^*}$ such that $\mathrm{dist}(\bar{y}\bar{o}^*) = R < R^*$ and R is close to R^*, then the angles of this triangle are close to the angles of $\triangle_{\bar{o}\bar{o}^*\bar{x}}$. There exists R' such that for all $R' < R < R^*$ we have

$$\measuredangle(\bar{y}\bar{o}\bar{o}^*) > \measuredangle(\bar{x}\bar{o}\bar{o}^*) - c(R^*)/2,$$

and, since $\mathrm{dist}(y, o^*) < \mathrm{dist}(x, o^*)$, we have

$$\measuredangle(\bar{o}\bar{y}\bar{o}^*) > \measuredangle(\bar{x}\bar{o}\bar{o}^*).$$

Let $c = c(R^*)/2$ and fix R such that $R' < R < R^*$. By Lemma 1 the set $\Sigma^*(R)$ is diffeomorphic to the sphere S^{n-1}. Applying Proposition 1, in view of our choice of R, we obtain the following result.

LEMMA 2. *Let $\pi/2 < R < R^*$ be as above and let x be a point of $\Sigma^*(R)$, i.e.,* $\operatorname{dist}(o^*, x) = R < \operatorname{dist}(o^*, o) = R^*$. *Then, for any minimal geodesics γ_{ox} and γ_{oo^*} connecting o with x and o with o^*,*

$$\measuredangle(\gamma_{ox}, \gamma_{oo^*}) \geq \widetilde{\measuredangle}(\gamma_{ox}, \gamma_{oo^*}) > \pi/2 + c, \tag{1.6}$$

and

$$\measuredangle(\gamma_{xo}, \gamma_{xo^*}) \geq \widetilde{\measuredangle}(\gamma_{xo}, \gamma_{xo^*}) > \pi/2 + c,$$

for some $c > 0$.

2. Proof of Theorem A′

Let T_oM^n be a tangent space to M^n at the point o and S^{n-1} a sphere of unit vectors in T_oM^n. We first note that, using the same arguments as in the proof of the Isotopy Lemma, we have the following result (see [MM]).

LEMMA 3. *There exist continuous maps $V : \Sigma^*(R) \to S^{n-1}$ such that*

$$\measuredangle(V(x), \gamma_{oo^*}) \geq \pi/2 + c/2$$

for arbitrary $x \in \Sigma^*(R)$ and arbitrary minimal geodesics γ_{oo^*} connecting o and o^*.

Proof. Indeed, given a point $x \in \Sigma^*(R)$, choose a minimal geodesic γ_{ox} and denote by $V(x)$ its direction at the base point o. Since $\operatorname{dist}(x, o^*) = R < R^* = \operatorname{dist}(o, o^*)$, the base point o does not belong to $\Sigma^*(R)$, i.e., $x \neq o$. Hence $V(x)$ is well defined and nonzero. Given $\lambda > 0$, define vectors $V_{x,\lambda}(y) = V(x)$ for any point y in a λ-neighborhood $B(x, \lambda, \Sigma^*(R))$ of x in $\Sigma^*(R)$. As $\lambda \to 0$, the points y from $B(x, \lambda, \Sigma^*(R))$ tend to the point x, and minimal geodesics connecting the base point o with points y tend to a minimal geodesics from o to x. Therefore, for any x there exists $\lambda(x) > 0$ such that for any point y in $B(x, \lambda(x), \Sigma^*(R))$ and any minimal geodesic γ_{oy} there exists a geodesic γ_{ox} having an angle less than $1/k$ with γ_{oy} at o for all $0 < \lambda < \lambda(x)$. Take any such minimal geodesic γ_{oo^*}. Then, by (1.6), we have

$$\measuredangle(V_{x,\lambda}(y), \Gamma) \geq \pi/2 + c - 1/k, \tag{2.1}$$

where Γ denotes the direction of γ_{oo^*} at the base point o. Since $\Sigma^*(R)$ is compact, the covering $\Sigma^*(R) \subset \cup B(x, \lambda(x), \Sigma^*(R))$ contains a finite subcovering $\Sigma^*(R) \subset \cup B(x_i, \lambda(x_i), \Sigma^*(R))$, and choosing a partition of unity $f_i : B(x_i, \lambda(x_i), \Sigma^*(R)) \to R$, $f_i > 0$, $\sum f_i(x) \equiv 1$, associated with this subcovering, we define a smooth map sending the point $x \in \Sigma^*(R)$ to the vector

$V_k(x) = \sum f_i(x) V_{x_i,\lambda(x_i)}(x)$. From the definition of this map and (2.1) we see that

$$\measuredangle(V_k(x), \Gamma) \geq \pi/2 + c - 1/k, \tag{2.2}$$

where x is any point from $\Sigma^*(R)$, γ_{oo^*} any minimal geodesic connecting o and o^*, and Γ the direction of this geodesic at o. Choosing k large enough so that $1/k < c/2$, we conclude from the last inequality that $V(x) = V_k(x)$ is nonzero. Therefore, we can divide the vector $V(x)$ by its length and then obtain a map from $\Sigma^*(R)$ into the unit sphere S^{n-1}, which we will denote in the same way by $V : \Sigma^*(R) \to S^{n-1}$. This completes the proof of Lemma 3. □

Next, we show that, for all $x \in \Sigma^*(R)$, the vector $W(x) = -V(x)$ satisfies the condition necessary to guarantee the existence of the integral trajectory of the field $X(\lambda)$ starting from the base point o in direction $W(x)$. Indeed, let $z(x, \epsilon) = \exp(\epsilon(W(x)))$ for some $\epsilon < r_{\text{inj}}(M^n)$. Then, for ϵ sufficiently small, the comparison angle $\widetilde{\psi}(x, \epsilon) = \widetilde{\measuredangle}(zoo^*)$ will be close to the actual angle $\psi(x) = \measuredangle(zoo^*)$ in the triangle $\triangle_{zoo^*}$ since M^n is a differentiable manifold and, by the first variation formula, $\widetilde{\psi}(x, \epsilon)$ tends to $\psi(x)$ as $\epsilon \to 0$. In fact, we can find $\epsilon_0 > 0$ such that for all $x \in \Sigma^*(R)$ and all $0 < \epsilon < \epsilon_0$ we have $\psi(x) - c/4 \leq \widetilde{\psi}(x, \epsilon) \leq \psi(x)$. By (2.2), for any minimal geodesic γ_{oo^*} from o to o^* the angle between $W(x)$ and the direction Γ of γ_{oo^*} is at most $\pi/2 - c/2$. Therefore, we have the following result.

Lemma 4. *There exists $\epsilon_0 > 0$ such that, given $x \in \Sigma^*(R)$ and $0 < \epsilon < \epsilon_0$, there exists a minimal geodesic γ_{oo^*} from o to o^*, such that for the point $z = z(x, \epsilon) = \exp(\epsilon W(x))$, the comparison angle $\widetilde{\measuredangle}(zoo^*)$ in the triangle $\triangle_{zoo^*}$ is at most $\pi/2 - c/4$.*

Let inj $= \min(\epsilon_0, r_{\text{inj}} M^n)$. According to [MM] the vector field $X(\lambda)$ can be defined in the same way as above inside the set $CA(\phi, d)$, which we define as follows.

Definition 2. We say that a point z belongs to $CA(\phi, d)$ if inj $< \text{dist}(o, z) \leq d$, $\phi < \pi/2$, and for some point p having distance d to o and a minimal geodesic γ_{op} the angle $\widetilde{\measuredangle} o$ of the comparison triangle $\triangle_{\bar{o}\bar{p}\bar{z}}$ for the triangle $\triangle_{opz}$ is at most ϕ.

Hence, by Lemma 4, all points $z(x, \text{inj})$ belong to $CA(\text{inj}, \pi/2 - c/4, R^*)$ with $p = o^*$. As we proved in [MM], for an arbitrary point z in this set there exists an integral trajectory $z(s)$ of the vector field $-X(\lambda)$ starting at this point and satisfying the following "comparison angle almost nonincreasing" property. [5]

[5] Note that we change the direction of our trajectories: instead of $X(\lambda)$ we consider the trajectories of $-X(\lambda)$ so that the parameter s increases as $z(s)$ approaches o^*; see (2.3).

Proposition 5 (Lemma 7 in [MM]). *Given arbitrarily small $\omega, \phi_0 > 0$, and $0 < \phi_0 < \phi_1 < \pi/2$, there exists $\lambda > 0$ such that for any trajectory $z(s)$ of the vector field $X(x, \lambda)$ defined above the left derivative of the comparison angle $\phi(s) = \widetilde{\measuredangle}(o^* o z(s))$ of the triangle $\triangle_{oo^* z(s)}$ exists and is almost nonpositive, i.e.,*

$$\phi'_-(\tau) \overset{def}{=} \lim_{s \nearrow \tau} \frac{\phi(\tau) - \phi(s)}{\tau - s} < \omega,$$

if only $\phi_0 \le \phi(\tau) \le \phi_1$ and inj $\le \operatorname{dist}(o, z(\tau)) \le R^*$.

We denote by $z(x, s)$ the trajectory starting at $z(x, \mathrm{inj})$ at the moment $s = \mathrm{inj}$. By Lemma 6 of [MM] the parameter s on the trajectory $z(s)$ is a monotone function of the distance to o^* such that

$$0 < C^{-1}(\phi(s)) \le \operatorname{dist}'_s(z(s), o^*) \le C(\phi(s)) \tag{2.3}$$

for some $C(\phi)$ of order $\sin^{-1}(\phi)$. (This also follows easily from (1.6) and (1.2)–(1.3), or from the similar estimates (1.4)–(1.5).) Hence we have the following result.

Lemma 5. *For every $x \in \Sigma^*(R)$ there exists $s(x) > inj$ which depends continuously on x, such that the point $z(x, s(x))$ belongs to $\Sigma^*(R)$.*

Since the function $s(x)$ is continuous, it is uniformly bounded by some constant s^* on $\Sigma^*(R)$. By our definition of inj, for all $z(x, \mathrm{inj})$ the comparison angle $\widetilde{\measuredangle}(z(x, \mathrm{inj}) o o^*)$ is at most $\pi/2 - c/4$ (see Lemma 4). By Proposition 5 above with $\omega = c/8s^*$, the comparison angles $\widetilde{\measuredangle}(z(x, s) o o^*)$ are almost nonincreasing functions along trajectories $z(x, s)$, $\mathrm{inj} < s \le s(x)$, with left derivatives at most ω. By our choice of ω this implies

$$\widetilde{\measuredangle}(z(x, s(x)) o o^*) \le \pi/2 - c/8. \tag{2.4}$$

Let $Z : \Sigma^*(R) \to \Sigma^*(R)$ be the map sending the point x to $z(x, s(x))$. Comparing the last inequality (2.4) with (1.6), we see that $z(x, s(x)) \ne x$ for all x; i.e., Z has no fixed points. If we identify $\Sigma^*(R)$ with a sphere S^{n-1} (via Lemma 1), we see that Z is homotopic to the antipodal map, has nonzero degree and therefore is "on". Hence we have:

Lemma 6. *The map $Z : \Sigma^*(R) \to \Sigma^*(R)$ sending the point x to $z(x, s(x))$ is "on".*

As in [MM], this implies:

Lemma 7. *The function* $\operatorname{dist}(o, \ )$ *has only two critical points, o and o^*.*

Proof. Choose any point $y \ne o$ outside $B(o^*, R^*)$ and consider the trajectory $z(x, s)$ starting in direction $W(x)$, the direction of a minimal geodesic γ_{oy} connecting o and y. The point y can only be critical if $\operatorname{dist}(o, y) \le \pi/2$.

Since, by Proposition 5, $z(x,s)$ is defined for all s such that $\operatorname{dist}(o, z(x,s)) \leq \operatorname{dist}(o, o^*) = R^* > \pi/2$, there exists $z = z(x, s')$ such that $\operatorname{dist}(o, y) = \operatorname{dist}(o, z)$. Since the comparison angle $\widetilde{\measuredangle}(yoz(x,s))$ is an almost nonincreasing function on s, we see as above that $\widetilde{\measuredangle}(yoz)) \leq s'\omega$, which tends to zero as $\lambda \to 0$ in the definition of the vector field $X(\lambda)$. Choose λ such that $s'\omega < c/16$. In the proof of Lemma 6 we have already seen that for an arbitrary trajectory $z(x,s)$ there exists a minimal geodesic γ_{oo^*} with direction Γ at the base point o and an angle less than $\pi/2 - c/8$. Hence, we deduce

$$\widetilde{\measuredangle}(\gamma_{oy}, \gamma_{oo^*}) < \pi/2 - c/16. \tag{2.5}$$

Now, comparing the sphere triangle $\triangle_{\bar{o}\bar{y}\bar{o}^*}$ with the triangle $\triangle_{oyo^*}$, the last inequality together with the inequalities $\operatorname{dist}(o, y) \leq \operatorname{dist}(o, o^*)$ and $\operatorname{dist}(o, o^*) > \pi/2$ imply that the angle $\bar{y}$ in the sphere comparison triangle is larger than $\pi/2$, or, by Proposition 1,

$$\widetilde{\measuredangle}(\gamma_{yo}, \gamma_{yo^*}) > \pi/2, \tag{2.6}$$

for any minimal geodesic γ_{oy}. This shows that y cannot be a critical point of the function $\operatorname{dist}(o, \ )$, since there are no minimal geodesics between y and o forming an angle at most $\pi/2$ with the direction of γ_{yo^*}. This completes the proof of Lemma 7. $\square$

By Lemma 7 the distance function $\operatorname{dist}(o, \ )$ has only two critical points on M^n. Theorem A$'$ now follows from the Isotopy Lemma in a standard way: both metric balls $B(o, R^*)$ and $B(o^*, R^*)$ are diffeomorphic to a Euclidean ball B^n, and since M^n is a union of these balls, it is homeomorphic to S^n. This completes the proof of Theorem A$'$.

3. Proof of Theorem A$''$

As above, the assertion of Theorem A$''$ follows from the fact that M^n can be represented as a union of two domains $B = B(o, \pi/2)$ and $B^* = B(o^*, R^*)$ that are diffeomorphic to the Euclidean ball B^n. To prove this, we proceed as before, i.e., we verify that both B and B^* do not contain critical points of the distance function $d(o, \ )$.

LEMMA 8. *If* $\operatorname{vol}(M^n) > \operatorname{vol}(S^n)/2$ *then for the base point* o *there exists a unique point* o^* *where the function* $\operatorname{dist}(o, \ )$ *attains its maximum, and* $\operatorname{dist}(o, o^*) > \pi/2$.

Proof. Indeed, if for all $x \in M^n$ we have $\operatorname{dist}(o, x) \leq \pi/2$, then $M^n \subset B(o, \pi/2)$. Since, by Proposition 3, $\operatorname{vol}(B(o, \pi/2)) \leq \operatorname{vol}(\bar{B}(\bar{o}, \pi/2)) = \operatorname{vol}(S^n)/2$, this would imply $\operatorname{vol}(M^n) \leq \operatorname{vol}(S^n)/2$, contradicting the assumption of the lemma. Hence, for some x, we have $\operatorname{dist}(o, x) > \pi/2$. Take a point o^* at which $\operatorname{dist}(o, \ )$ attains its maximum. Then $\operatorname{dist}(o, o^*) > \pi/2$, and using Lemma 1 and Proposition 1 we easily conclude that o^* is unique. This completes the proof of Lemma 8. $\square$

LEMMA 9. *We have $M^n \backslash B^* \subset B$, where $B = B(o, \pi/2)$ and $B^* = B(o^*, R^*)$ with $R^* = \operatorname{dist}(o, o^*)$.*

Proof. If the assertion of the lemma is not true, then there exists a point x such that $\operatorname{dist}(o, x) > \pi/2$ and $\operatorname{dist}(o^*, x) > R^*$. Since o^* is a critical point for the distance function to o, for any minimal geodesic γ_{o^*x} there exists a minimal geodesic γ_{o^*o} from o^* to o having angle at most $\pi/2$ with γ_{o^*x}. As in the proof of Lemma 1, applying Proposition 1 to the triangle $\triangle_{oxo^*}$, we deduce from $\operatorname{dist}(o, o^*) > \pi/2$ that $\operatorname{dist}(o, x) \leq \pi/2$, which is a contradiction, proving the lemma. $\square$

By Lemma 1, B^* is diffeomorphic to the Euclidean ball B^n. We now show that B is also diffeomorphic to the Euclidean ball B^n.

Let $U \subset T_o M^n$ be a star-shaped open disk domain whose boundary is the tangential cut locus to o. Setting $\widetilde{U} = \exp_o(U)$, we see that $M^n \backslash \widetilde{U} = \operatorname{cutlocus}(o)$ has no interior points and

$$\operatorname{vol}(\widetilde{U}) = \operatorname{vol}(M^n). \tag{3.1}$$

Define a map $W : S^n \backslash \{\bar{o}\} \to M^n$ by $W(\bar{x}) = \exp_o \circ I \circ \exp_{\bar{o}}^{-1}(\bar{x})$, where $I : T_{\bar{o}} S^n \to T_o M^n$ is an isometry. Setting $\bar{U} = \exp_{\bar{o}} \circ I^{-1}(U)$, we see that W is a diffeomorphism between $\bar{U}$ and $\widetilde{U}$. It is easy to see that Proposition 1 is equivalent to the assertion that W does not increase distances on $\bar{U}$. Therefore,

$$\operatorname{vol}(\widetilde{U}) = \operatorname{vol}(W(\bar{U})) \leq \operatorname{vol}(\bar{U}). \tag{3.2}$$

Given an arbitrary point $\bar{x}$ of S^n such that $\operatorname{dist}(\bar{o}, \bar{x}) \leq \pi/2$, let $B(\bar{x})$ be the closed $\pi/2$-ball $\bar{B}(\hat{x}, \pi/2)$ with center $\hat{x}$, such that $\operatorname{dist}(\bar{x}, \hat{x}) = \pi/2$ and $\bar{o}$ belongs to the minimal geodesic $\gamma_{\bar{x}\hat{x}}$. Another definition for $B(\bar{x})$ is

$$B(\bar{x}) = \{\bar{y} \in S^n \mid \measuredangle(\bar{o}\bar{x}\bar{y}) \leq \pi/2\}.$$

Given a point $x \in B$, let $\Gamma(x)$ be the set of unit directions of all minimal geodesics from o to x. If x is a critical point of the distance function $\operatorname{dist}(o, \ )$ then $x \in \operatorname{cutlocus}(o)$, $\Gamma(x)$ consists of more than one vector, and the set $\Lambda(x)$ of unit vectors of $\Gamma(x)$ multiplied by $\operatorname{dist}(o, x)$ belongs to the boundary of U. Let $\bar{\Lambda}(x)$ be the image of $\Lambda(x)$ under the map $\exp_{\bar{o}} \circ I$.

LEMMA 10. *If $x \in B$ is a critical point of the distance function $\operatorname{dist}(o, \ )$ different from o, then $U \subset W(\cap_{\bar{x} \in \bar{\Lambda}(x)} B(\bar{x}))$.*

Proof. For any point $y \in \widetilde{U}$ the minimal geodesic γ_{oy} is unique. Take a minimal geodesic γ_{ox} between o and x and let v be its direction at o. Let $\bar{x}$ and $\bar{y}$ denote $\exp_{\bar{o}} \circ I(\operatorname{dist}(o, x)v)$ and $\exp_{\bar{o}} \circ I(\operatorname{dist}(o, y)w)$, respectively, where

w is the direction of γ_{oy} at o. It is easy to see that the assertion of the lemma is equivalent to the statement that, for all possible choices of the minimal geodesic γ_{ox} and all corresponding v, we have $\bar{y} \in B(\bar{x})$.

Take an arbitrary γ_{ox} (and v) and consider in S^n a two-dimensional totally geodesic (big) sphere S^2 containing points $\bar{x}$, $\hat{x}$ and $\bar{y}$. The point $\bar{o}$ belonging to a minimal geodesic between $\bar{x}$ and $\hat{x}$ also belongs to S^2 as well as a minimal geodesics $\gamma_{\bar{o}\bar{y}}$. Find in S^2 a point $\hat{y}$ such that $\operatorname{dist}(\bar{o}, \hat{y}) = \operatorname{dist}(\bar{o}, \bar{y})$ and $\operatorname{dist}(\bar{x}, \hat{y}) = \operatorname{dist}(x, y)$. Then the triangle $\triangle_{\bar{o}\bar{x}\hat{y}}$ is a comparison triangle for the triangle $\triangle_{oxy}$. Since, for any triangle $\triangle_{oxy}$ its comparison triangle in S^2 is defined by distances between vertices of $\triangle_{oxy}$, the comparison triangle is independent of the choice of γ_{ox}. Since x is a critical point for every γ_{xy}, there exists a minimal geodesic γ_{ox} between o and x having angle at most $\pi/2$ with γ_{xy}. By Proposition 1 we conclude

$$\measuredangle(\bar{o}\bar{x}\hat{y}) \leq \pi/2, \tag{3.3}$$

i.e., the point $\hat{y}$ belongs to $B(\bar{x})$. Since, by definition, $\measuredangle(xoy) = \measuredangle(\bar{x}\bar{o}\bar{y})$, another application of Proposition 1 shows that

$$\measuredangle(\bar{o}\bar{x}\hat{y}) \leq \measuredangle(\bar{o}\bar{x}\bar{y}),$$

or

$$\measuredangle(\hat{x}\bar{o}\bar{y}) \leq \measuredangle(\hat{x}\bar{o}\hat{y}) \tag{3.4}$$

as $\bar{o}$ belongs to $\gamma_{\bar{x}\hat{x}}$. Since $\operatorname{dist}(\bar{o}, \bar{y}) = \operatorname{dist}(\bar{o}, \hat{y})(= \operatorname{dist}(o, y))$ by the last inequality, an easy argument in sphere trigonometry shows that $\operatorname{dist}(\hat{x}, \bar{y}) \leq \operatorname{dist}(\hat{x}, \hat{y})$, and hence that $\bar{y}$ (like $\hat{y}$) belongs to $B(\bar{x})$. Since the choice of γ_{ox} was arbitrary, it follows that $\bar{y}$ belongs to $B(\bar{x})$ for all possible choices, and we obtain the assertion of the lemma. $\square$

Finally, the volume of an arbitrary ball $B(\bar{x})$ equals $\operatorname{vol}(S^n)/2$. Since the map W does not increase distances, we see that an arbitrary set

$$W(\cap_{\bar{x} \in \bar{\Lambda}(x)} B(\bar{x}))$$

has volume no larger than $\operatorname{vol}(S^n)/2$. If we assume the existence of a critical point $x \in B$, then by Lemma 10 and (3.2) we would have $\operatorname{vol}(\widetilde{U}) \leq \operatorname{vol}(S^n)/2$, which, by (3.1), contradicts the condition $\operatorname{vol}(M^n) > \operatorname{vol}(S^n)/2$ of Theorem A$''$. This proves that the distance function $\operatorname{dist}(o, \ )$ has only two critical points, o and o^*, that both B and B^* are diffeomorphic to the Euclidean ball B^n, and that M^n is homeomorphic to the sphere S^n. The proof of Theorem A$''$ is therefore complete.

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