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KREIN'S ENTIRE FUNCTIONS AND THE BERNSTEIN APPROXIMATION PROBLEM

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ABSTRACT. We extend two theorems of Krein concerning entire functions of Cartwright class, and give applications for the Bernstein weighted approximation problem.

1. The Krein class and functions of bounded type

We start with two classical theorems of Krein concerning entire functions. An entire function f belongs to *the Cartwright class* if f has at most exponential type, that is, if

$$\log|f(z)| = O(|z|), \qquad |z| \to \infty,$$

and the logarithmic integral converges:

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2} dx < \infty.$$

THEOREM A (Krein [14]). *An entire function f belongs to the Cartwright class if and only if the function $\log^+|f|$ has (positive) harmonic majorants in both the upper and the lower half-planes.*

An entire function f belongs to *the Krein class* if its zeros λ_n are (simple and) real, and $1/f$ is represented as an absolutely convergent sum of simple fractions

$$\frac{1}{f(z)} = \sum_n \frac{1}{f'(\lambda_n)(z-\lambda_n)}, \qquad \sum_n \frac{1}{|f'(\lambda_n)|} < \infty. \tag{1.1}$$

THEOREM B (Krein [14]). *The Krein class is contained in the Cartwright class.*

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For the proofs see also [17]. These two results have numerous applications in operator theory and harmonic analysis (see, for example, [15], [16], [7], [8, Chapter IV], [13, Section VI F]), and were generalized in different directions (see [17, Section 26.4] and [9, Section VI.2]).

Let E be a non-empty closed subset of the real line. In what follows we assume that E is regular for the Dirichlet problem in $\mathbb{C}\setminus E$. A function f that is analytic in $\mathbb{C}\setminus E$ is said to be *of bounded type* if $\log^+|f|$ has a harmonic majorant in $\mathbb{C}\setminus E$. It is well known that if f and g are of bounded type and f/g is analytic in $\mathbb{C}\setminus E$, then f/g is also of bounded type there (see [21, Chapter VII] and [22, Theorem 19, p. 181]). It is worth mentioning that any function φ that is lower semicontinuous in the plane and has a positive harmonic majorant in $\mathbb{C}\setminus E$ satisfies

$$\int_E \varphi^+(x)\,\omega(i,dx,\mathbb{C}\setminus E)<\infty. \tag{1.2}$$

We are interested in conditions under which the asssertions of the above two theorems by Krein can be strengthened to guarantee that f is of bounded type in $\mathbb{C}\setminus E$. Note that every polynomial is of bounded type in $\mathbb{C}\setminus E$. Indeed, our conditions on E imply that E has positive capacity, and the identity function in $\mathbb{C}\setminus E$ therefore omits values from a set of positive capacity. Hence, by the Frostman theorem (see [21, Chapter X, Section 2.8] for the case of the unit disc, and use the uniformization argument in the general case), the identity function is of bounded type in $\mathbb{C}\setminus E$, and the statement for polynomials follows immediately.

For a regular set $E\subset\mathbb{R}$ we denote by $\mathcal{M}_E(z)$ the symmetric Martin function for $\mathbb{C}\setminus E$ with singularity at infinity, that is, a positive harmonic function in $\mathbb{C}\setminus E$ which vanishes on E and satisfies $\mathcal{M}_E(\bar z)=\mathcal{M}_E(z)$. A uniqueness theorem proved by Benedicks [3, Theorems 2 and 3] and Levin [19, Theorem 3.2] asserts that $\mathcal{M}_E$ exists and is unique up to a positive multiplicative constant. The function $\mathcal{M}_E$, extended to $\mathbb{C}$ by setting $\mathcal{M}_E \,\big|\, E=0$, is subharmonic in $\mathbb{C}$ and has order at most one and mean type $\mathcal{M}_E(z)=O(|z|)$ as $|z|\to\infty$.

Our first result describes the sets E for which every Cartwright class function is of bounded type in $\mathbb{C}\setminus E$. We say that a set $E\subset\mathbb{R}$ is an *Akhiezer–Levin set* if the function $\mathcal{M}_E$ is of mean type with respect to the order 1, that is, if

$$\sigma_{\mathcal{M}_E}\stackrel{\text{def}}{=}\limsup_{|z|\to\infty}\frac{\mathcal{M}_E(z)}{|z|}>0\,.$$

It is worth mentioning that in this case the limit

$$\sigma_{\mathcal{M}_E}=\lim_{|y|\to\infty}\frac{\mathcal{M}_E(iy)}{|y|}$$

exists, and $\mathcal{M}_E(z)\geq\sigma_{\mathcal{M}_E}|\operatorname{Im} z|$. The function $\mathcal{M}_E$, normalized by the condition $\sigma_{\mathcal{M}_E}=1$, is sometimes called the Phragmén–Lindelöf function.

The class of Akhiezer–Levin sets was introduced in [2]. Let us present two equivalent conditions. A set $E \subset \mathbb{R}$ is an Akhiezer–Levin set if and only if either of the following two properties holds:

(1) (Koosis [13, Section VIII A.2]) $\int_{\mathbb{R}} G(t, z)\, dt < \infty$, where G is the Green function for $\mathbb{C} \setminus E$ and $z \in \mathbb{C} \setminus E$.

(2) (Benedicks [3, Theorem 4]) $\int_{\mathbb{R}} \beta_E(t)/(1 + |t|)\, dt < \infty$, where $\beta_E(t)$ is the harmonic measure $\omega(t, \partial S_t, S_t \setminus E)$ of the boundary of the square $S_t = \{z = x + iy : |x - t| < t/2, |y| < t/2\}$ with respect to the domain $S_t \setminus E$ at the point t.

Next we present three metric tests:

(1) (Akhiezer–Levin [2, Section 3.VII], Kargaev [12, Theorem 6(a)]) If $\int_{\mathbb{R}\setminus E} dx/(1 + |x|) < \infty$, then E is an Akhiezer–Levin set.

(2) (Schaeffer [23, Lemma 1]) If E is relatively dense with respect to the Lebesgue measure dm (that is, if for some positive a and b and for every $x \in \mathbb{R}$, we have $m(E \cap [x, x + a]) \geq b$), then E is an Akhiezer–Levin set.

(3) (Kargaev [12, Theorem 4]) If E is an Akhiezer–Levin set, then $\int_{\mathbb{R}} \operatorname{dist}(x, E)/(1 + x^2)dx < \infty$.

Given a positive symmetric harmonic function h on $\mathbb{C} \setminus E$, set

$$C = \max\{c \geq 0 : h - c\mathcal{M}_E \text{ is non-negative on } \mathbb{C} \setminus E\},$$

and define the function $PI_{E,h}$ (the Poisson integral of a non-negative measure with support on E), which is non-negative, symmetric, and harmonic on $\mathbb{C}\setminus E$, by

$$h = PI_{E,h} + C\mathcal{M}_E. \tag{1.3}$$

Clearly,

$$\text{there is no } \varepsilon > 0 \text{ such that } PI_{E,h} \geq \varepsilon\mathcal{M}_E \text{ on } \mathbb{C} \setminus E. \tag{1.4}$$

The following lemma is possibly known. Since we were unable to find an appropriate reference, we will give a proof in Section 3.

LEMMA 1.1. *For every positive symmetric harmonic function h on $\mathbb{C} \setminus E$ we have*

$$PI_{E,h}(iy) = o(\mathcal{M}_E(iy)), \qquad |y| \to \infty.$$

We next present an extension of Theorem A:

THEOREM 1.2. *If $E \subset \mathbb{R}$ is an Akhiezer–Levin set, then every function f in the Cartwright class is of bounded type in $\mathbb{C} \setminus E$. Conversely, let f be an entire function of non-zero exponential type belonging to the Cartwright class. If f is of bounded type in $\mathbb{C} \setminus E$, then E is an Akhiezer–Levin set.*

Proof. Let E be an Akhiezer–Levin set, and let f be in the Cartwright class and of exponential type $\sigma \geq 0$. We first suppose that $|f(x)| \leq 1$ for $x \in \mathbb{R}$. Applying the Phragmén–Lindelöf principle to the function $\log|f| - \sigma_1 \mathcal{M}_E$ with $\sigma_1 > \sigma\,\sigma_{\mathcal{M}_E}^{-1}$, in the upper and in the lower half-planes, we conclude that $\log|f| - \sigma_1\mathcal{M}_E$ is non-positive everywhere in $\mathbb{C}$, and therefore that $\sigma_1\mathcal{M}_E$ is a positive harmonic majorant for $\log|f|$ in $\mathbb{C}\setminus E$.

In the general case, we use the Beurling–Malliavin multiplier theorem [5]: there exists a function g in the Cartwright class with $(1+|f(x)|)|g(x)| \leq 1$ for $x \in \mathbb{R}$. Applying the previous argument, we obtain that g and fg, and hence f, are of bounded type in $\mathbb{C}\setminus E$.

Now, let f be an entire function of non-zero exponential type belonging to the Cartwright class. Suppose that f is of bounded type in $\mathbb{C}\setminus E$. Then the function $\log|f(z)|$ has a positive harmonic majorant $h(z)$; without loss of generality we may assume that h is symmetric, i.e., $h(z) = h(\overline{z})$. By (1.3), we have $h = PI_{E,h} + C\mathcal{M}_E$. Since the function f has non-zero exponential type, Lemma 1.1 implies that $\mathcal{M}_E(iy) \geq c|y|$ for large $|y|$. This implies that E is an Akhiezer–Levin set. □

Our next result extends Theorem B. We now assume that E is the union of disjoint closed intervals $I_m = [a_m, b_m]$ with $\operatorname{dist}(0, I_m) \to \infty$. Given an interval I we denote its length by $|I|$.

THEOREM 1.3. *Suppose f is a Krein class function with zeros λ_n on $E = \cup I_m$, and $|I_m| \geq c\,\operatorname{dist}(0, I_m)^{-M}$ for some constants $c > 0$ and $M < \infty$. Then f is of bounded type in $\mathbb{C}\setminus E$.*

Proof. We need to prove that the function $\sum_n 1/[f'(\lambda_n)(z-\lambda_n)]$ is of bounded type in $\mathbb{C}\setminus E$. Multiplying f by a polynomial with real zeros, if necessary, we obtain

$$\sum_n \frac{1+|\lambda_n|^M}{|f'(\lambda_n)|} < \infty\,. \tag{1.5}$$

Without loss of generality, we may assume that the numbers $f'(\lambda_n)$ are real. Furthermore,

$$\sum_n \frac{1}{f'(\lambda_n)(z-\lambda_n)} = \sum_{j=1}^{2} g_j(z) = \sum_{j=1}^{2}\sum_n \frac{c_{n,j}}{z-\lambda_n}, \tag{1.6}$$

where $c_{n,1} \geq 0$, $c_{n,2} \leq 0$, and

$$\sum_{j=1}^{2}\sum_n (1+|\lambda_n|^M)|c_{n,j}| < \infty.$$

It suffices to verify that each of the functions g_j in (1.6) is a function of bounded type in $\mathbb{C}\setminus E$. Consider the function g_1, say, and represent it as a sum of two functions,

$$g_1(z) = g_-(z) + g_+(z) = \sum_{\lambda_n \in E_-} \frac{c_{n,1}}{z-\lambda_n} + \sum_{\lambda_n \in E_+} \frac{c_{n,1}}{z-\lambda_n},$$

where $E_- = \bigcup [a_m, (a_m+b_m)/2)$ and $E_+ = \bigcup [(a_m+b_m)/2, b_m]$, so that $E = E_+ \cup E_-$. Let us verify that g_+ is of bounded type in $\mathbb{C} \setminus E$. Indeed, this function is analytic in $\mathbb{C} \setminus E$ and satisfies

$$\frac{\operatorname{Im}\, g_+(z)}{\operatorname{Im}\, z} < 0, \qquad z \in \mathbb{C} \setminus \mathbb{R},$$

and, for $x \in \mathbb{R} \setminus E$,

$$g_+(x) \geq -\sum_{\lambda_n \in E_+,\, \lambda_n > x} \frac{2c_{n,1}}{b_m - a_m} \geq -\frac{2}{c} \sum_n c_{n,1}(1+|\lambda_n|^M) = \tau > -\infty\,.$$

Thus, the image $g_+(\mathbb{C} \setminus E)$ omits the ray $(-\infty, \tau)$. Applying the Frostman theorem, we conclude that the function g_+ is of bounded type in $\mathbb{C} \setminus E$. (Alternatively, we could consider the functions $\theta(z) = (z-1)/(z+1)$, $\theta_1(z) = (1+z)/(1-z)$, and $\psi = \theta(\sqrt{g_+ - \tau})$. Since ψ has absolute values bounded by one in $\mathbb{C} \setminus E$, $g_+ = [\theta_1(\psi)]^2 + \tau$ is of bounded type there.) The same argument works for g_-, and we therefore conclude that $g_1 = g_- + g_+$ is of bounded type in $\mathbb{C} \setminus E$. Similarly, we see that g_2 is of bounded type in $\mathbb{C} \setminus E$. Hence the same holds for $1/f$. □

Remark 1.4. Using more information on the Krein class function f we can further weaken our conditions on $|I_m|$: the assertion of Theorem 1.3 holds if for some $c > 0$ and $M < \infty$ and for every zero λ_n of f we have

$$|I(\lambda_n)| \geq \frac{c}{(1+|\lambda_n|^M)|f'(\lambda_n)|},$$

where $I(\lambda_n)$ is the interval of E containing the point λ_n. It seems plausible that the assertion of Theorem 1.3 holds for any system of intervals of "non-quasianalytically decaying lengths". Namely, we may conjecture that if f is an entire function in the Krein class, of positive exponential type, with zeros λ_n, and if φ is a positive Lip 1 function on $\mathbb{R}$, then f is of bounded type in $\mathbb{C} \setminus E$ with

$$E = \bigcup_n \left[\lambda_n - \frac{1}{\varphi(\lambda_n)}, \lambda_n + \frac{1}{\varphi(\lambda_n)}\right]$$

if and only if $\int_{\mathbb{R}} \varphi(x)/(1+x^2)dx < \infty$. In some special cases, when the zero set of f is regularly distributed and φ satisfies additional regularity assumptions, this statement follows from results of Benedicks [3, Theorem 5].

Combining Theorems 1.2 and 1.3, we obtain a sufficient condition for E to be an Akhiezer–Levin set.

Corollary 1.5. *If f is a Krein class function of positive exponential type with zeros λ_n, and if M is a constant, then the set*

$$E=\bigcup_n \left[\lambda_n-\frac{1}{(1+|\lambda_n|^M)|f'(\lambda_n)|}\,,\lambda_n+\frac{1}{(1+|\lambda_n|^M)|f'(\lambda_n)|}\right]$$

is an Akhiezer–Levin set.

2. The Bernstein problem on subsets of the real line

Fix a *weight* W, that is, a lower semicontinuous function $W:\mathbb{R}\to[1,+\infty]$ such that $\lim_{|x|\to\infty}|x|^n/W(x)=0$ for $n\geq 0$. Consider the space $C(W)$ of functions f that are continuous on $\mathbb{R}$ and satisfy $\lim_{|x|\to\infty}|f(x)|/W(x)=0$, and set

$$\|f\|_{C(W)}=\sup_{x\in\mathbb{R}}\frac{|f(x)|}{W(x)}.$$

The Bernstein problem consists of determining whether the set $\mathcal{P}$ of all polynomials is dense in $C(W)$. From now on, we suppose that *the weight W is finite on a subset of $\mathbb{R}$ having a finite limit point.* In this case the polynomials are simultaneously dense, or not dense, in every space $C(W_r)$ with $W_r(x)=W(x)(1+|x|)^r$, $r\in\mathbb{R}$ (see [20, Subsection 24]). Denote by X_W the set of polynomials P such that $||P||_{C(W)}\leq 1$. We define the *Hall majorant* M_W as

$$M_W(z)=\sup\{|P(z)|:P\in X_W\}.$$

Since the function $\varphi\equiv 1$ belongs to X_W, we have $M_W(z)\geq 1$ for $z\in\mathbb{C}$. Furthermore, we have $M_W(x)\leq W(x)$ for $x\in\mathbb{R}$, and $\log M_W$ is lower semicontinuous in the plane.

General criteria for the density of polynomials in weighted spaces were obtained by Akhiezer–Bernstein and by Pollard and Mergelyan in the early 1950s (see [1, 20, 13]). We shall use the following result.

Theorem C. *The polynomials are dense in $C(W)$ if and only if one of the following three equivalent conditions holds:*

(1) (Akhiezer–Bernstein)

$$\sup_{P\in X_W}\int_{\mathbb{R}}\frac{\log^+|P(x)|}{1+x^2}\,dx=+\infty\,;$$

(2) (Mergelyan)

$$\int_{\mathbb{R}}\frac{\log M_W(x)}{1+x^2}\,dx=+\infty\,;$$

(3) (Mergelyan) $M_W(z)=+\infty$ *for some (all)* $z\in\mathbb{C}\setminus\mathbb{R}$.

Another criterion was proposed in [6].

Theorem D (de Branges). *The polynomials are not dense in $C(W)$ if and only if there exists an entire function F of zero exponential type, $F \notin \mathcal{P}$, with (simple) real zeros λ_n, such that*

$$\sum_n \frac{W(\lambda_n)}{|F'(\lambda_n)|} < +\infty.$$

Such a function F belongs to the Krein class and satisfies

$$\sum_n \frac{P(\lambda_n)}{F'(\lambda_n)} = 0$$

for every polynomial P.

In [6] the weight W is assumed to be continuous, but the result holds also for lower semicontinuous weights W, see [24].

From now on, we suppose that $W(x) = \infty$ for $x \in \mathbb{R} \setminus E$, where E is a subset of $\mathbb{R}$ of the kind considered in the previous section, i.e., $E = \cup I_m$, where the I_m are disjoint closed intervals in $\mathbb{R}$ and $\operatorname{dist}(0, I_m) \to \infty$ as $m \to \infty$. Following Benedicks [4] and Koosis [13, Section VIIIA] we try to solve the Bernstein problem for $C(W)$ in terms of $M_W\big|E$, replacing the form $(1+x^2)^{-1}dx$ by the harmonic measure $\omega_E(dx) = \omega(i, dx, \mathbb{C} \setminus E)$.

Theorem 2.1. *Suppose that $|I_m| \ge c(\operatorname{dist}(0, I_m))^{-M}$ for some $c > 0$ and $M < \infty$. The polynomials are dense in $C(W)$ if and only if*

$$\int_E \log M_W(x)\, \omega_E(dx) = +\infty. \tag{2.1}$$

Furthermore, if the polynomials are not dense in $C(W)$, then the function $\log M_W$ has a (positive) harmonic majorant in $\mathbb{C} \setminus E$.

Proof. To obtain the result in one direction, we prove that

$$\log|P(z)| \le \int_E \log^+ |P(x)|\, \omega(z, dx, \mathbb{C} \setminus E), \qquad P \in \mathcal{P}. \tag{2.2}$$

First, observe that our assumptions on E imply that

$$\log|y| = o(\mathcal{M}_E(iy)), \qquad y \to \infty. \tag{2.3}$$

Indeed, take a sequence of points $x_k \in E$ tending to ∞ sufficiently rapidly (for example, $|x_{k+1}| > 2|x_k|$ suffices), and consider the entire function $B(z) = \prod_k (1 - z/x_k)$. Then B is in the Krein class and satisfies $\log|B(iy)|/\log|y| \to \infty$ as $y \to \infty$. By Theorem 1.3, $\log|B|$ has a positive harmonic majorant h in $\mathbb{C} \setminus E$. Using the representation (1.3) for h together with Lemma 1.1 we obtain (2.3).

Applying a standard Phragmén–Lindelöf argument to the subharmonic functions

$$\log|P(z)| - \int_E \log^+|P(x)|\,\omega(z,dx,\mathbb{C}\setminus E) - c\mathcal{M}_E(z)\,, \qquad c>0$$

in the domain $\mathbb{C}\setminus E$, we obtain (2.2).

By (2.2), we have

$$\log M_W(i) \le \int_E \log M_W(x)\,\omega_E(dx).$$

If the last integral is finite, then $M_W(i)<\infty$, and by Theorem C the polynomials are not dense in $C(W)$.

Now suppose that the polynomials are not dense in $C(W)$ and therefore not dense in $C(W_r)$, where r will be chosen later. We apply Theorem D to obtain an entire function F in the Krein class, $F\notin\mathcal{P}$, of zero exponential type with zeros $\lambda_n\in E$ such that

$$\sum_n \frac{W_r(\lambda_n)}{|F'(\lambda_n)|}\le 1 \tag{2.4}$$

and for every polynomial P and every $z\in\mathbb{C}$,

$$\sum_n \frac{P(z)-P(\lambda_n)}{(z-\lambda_n)F'(\lambda_n)}=0.$$

Using relation (1.1) we get

$$\frac{P(z)}{F(z)}=\sum_n \frac{P(\lambda_n)}{(z-\lambda_n)F'(\lambda_n)}\,.$$

Theorem 1.3 implies that the function F is of bounded type in $\mathbb{C}\setminus E$ and hence, by (1.2),

$$0\le h(z)\stackrel{\text{def}}{=}\int_E \log^+|F(t)|\,\omega(z,dt,\mathbb{C}\setminus E)<\infty.$$

From this and (2.2) we obtain

$$\begin{aligned}\log|P(z)| &\le h(z)+\int_E \log^+\Big|\sum_n \frac{P(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|\,\omega(z,dt,\mathbb{C}\setminus E),\\ \log M_W(z) &\le h(z)+\int_E \log^+\sup_{P\in X_W}\Big|\sum_n \frac{P(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|\,\omega(z,dt,\mathbb{C}\setminus E),\end{aligned}$$

for $z\in\mathbb{C}\setminus E$, where X_W is, as above, the set of polynomials P with $\|P\|_{C(W)}\le 1$. Therefore, to complete the proof of the theorem, we need only verify that

$$\int_E \log^+\sup_{P\in X_W}\Big|\sum_n \frac{P(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|\,\omega_E(dt)<\infty. \tag{2.5}$$

Then, using Harnack's inequality, we conclude that $\log M_W$ has a harmonic majorant in $\mathbb{C}\setminus E$, and by (1.2) we see that condition (2.1) does not hold.

Let us prove (2.5). By Jensen's inequality we have, for every $d>0$,

$$\frac{1}{d}\int_E \log^+ \sup_{P\in X_W}\Big|\sum_n \frac{P(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|^d \omega_E(dt) \\ \le \frac{1}{d}\log^+\int_E \sup_{P\in X_W}\Big|\sum_n \frac{P(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|^d \omega_E(dt).$$

Furthermore, from (2.4) we get, for $0<d<1$,

$$\sup_{P\in X_W}\Big|\sum_n \frac{P(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|^d \le \sup_{P\in X_W}\sum_n\Big|\frac{P(\lambda_n)}{W_r(\lambda_n)}\Big|^d\Big|\frac{W_r(\lambda_n)}{(t-\lambda_n)F'(\lambda_n)}\Big|^d \\ \le \sum_n \frac{1}{(1+|\lambda_n|)^{rd}|t-\lambda_n|^d}.$$

Therefore, to prove (2.5) it remains to check that, for some d with $0<d<1$, we have

$$\sum_n\int_E \frac{1}{(1+|\lambda_n|)^{rd}|t-\lambda_n|^d}\,\omega_E(dt)<\infty. \tag{2.6}$$

Since the numbers λ_n are the zeros of an entire function of zero exponential type, we have $1+|\lambda_n|\ge cn$ for some $c>0$. Thus, (2.6) follows from an estimate of the form

$$\int_E \frac{1}{|t-\lambda|^d}\,\omega_E(dt)\le C(1+|\lambda|)^{rd-2} \tag{2.7}$$

with a constant C that is independent of $\lambda\in E$ and some d with $0<d<1$.

Our conditions on E imply that, for some $c>0$ and some $M<\infty$, and for every $\lambda\in E$, there exists δ with $c(1+|\lambda|)^{-M}\le\delta\le 10c(1+|\lambda|)^{-M}$ such that for $I=[\lambda-\delta,\lambda+\delta]$ we have

$$E\cap I = J_1\cup J_2,$$

where the intervals $J_k=[a_k,b_k]$ satisfy $|J_k|\ge c(1+|\lambda|)^{-M}$, $k=1,2$. The elementary inequality for the harmonic measure

$$\omega_E(dt)=\omega(i,dt,\mathbb{C}\setminus E)\le\omega(i,dt,\mathbb{C}\setminus I_k)\le\frac{C\,dt}{\sqrt{(b_k-t)(t-a_k)}},\quad t\in J_k,$$

shows that for $d=1/4$,

$$\int_{J_k}\frac{1}{|t-\lambda|^d}\,\omega_E(dt) \\ \le\int_{a_k}^{b_k}\frac{C\,dt}{(b_k-t)^{1/2}(t-a_k)^{1/2}|t-\lambda|^{1/4}}\le C(1+|\lambda|)^{M/4},\quad k=1,2.$$

Furthermore,

$$\int_{E\setminus I} \frac{1}{|t-\lambda|^{1/4}}\,\omega_E(dt) \le \sup_{t\in E\setminus I} \frac{1}{|t-\lambda|^{1/4}} \le C(1+|\lambda|)^{M/4}.$$

Thus, the inequality (2.7) holds with $d = 1/4$ and any $r \ge M + 8$. □

Remark 2.2. The same proof shows that the polynomials are not dense in $C(W)$ as soon as (2.1) fails and there exists a sequence $m_k \to \infty$ such that $|I_{m_k}| \ge c(\operatorname{dist}(0, I_{m_k}))^{-M}$ for some $c > 0$ and $M < \infty$.

Remark 2.3. In the general case when the polynomials are not dense in $C(W)$, every function f in the closure $\operatorname{Clos}_{C(W)}\mathcal{P}$ of the polynomials has an analytic continuation to the entire complex plane and satisfies $|f(z)| \le M_W(z)||f||_{C(W)}$ for $z \in \mathbb{C}$. Therefore, under the assumptions of Theorem 2.1, every function from $\operatorname{Clos}_{C(W)}\mathcal{P}$ is of bounded type in $\mathbb{C} \setminus E$.

Remark 2.4. As in Theorem C, condition (2.1) is equivalent to

$$\sup_{P\in X_W} \int_E \log^+ |P(x)|\,\omega_E(dx) = +\infty\,.$$

The following examples show that the assertions of Theorem 2.1 are not valid if the condition $|I_m| \ge c(\operatorname{dist}(0, I_m))^{-M}$ is not fulfilled.

Proposition 2.5. (a) *There exist a weight W and a subset E of $\mathbb{R}$ such that $W(x) = \infty$ for $x \in \mathbb{R} \setminus E$, the polynomials are dense in $C(W)$ and*

$$\int_E \log M_W(x)\,\omega_E(dx) < \infty.$$

(b) *There exist a weight W and a subset E of $\mathbb{R}$ such that $W(x) = \infty$ for $x \in \mathbb{R} \setminus E$, the polynomials are not dense in $C(W)$ and*

$$\int_E \log M_W(x)\,\omega_E(dx) = +\infty.$$

Proof. (a) Consider a set of disjoint intervals I_n such that $|I_n| \le 1$, $\exp(n) \in I_n$ for $n \ge 1$, and $\omega(i, I_n, \mathbb{C} \setminus (I_1 \cup I_n)) < n^{-2}\exp(-n)$ for $n > 1$. We define a weight W by setting $W\big|I_n \equiv \exp\exp(n)$, $n \ge 1$, and defining $W(x) = +\infty$ for $x \notin \cup I_n$. By Theorem D, the polynomials are not dense in $C(W)$. Indeed, no entire function F of zero exponential type, with real zeros $\lambda_n \to \infty$, satisfies the condition

$$|F'(\lambda_n)| \ge c\exp(\lambda_n).$$

Finally, since $M_W(x) \le W(x)$ for $x \in \mathbb{R}$, we obtain

$$\int_E \log M_W(x)\,\omega_E(dx) \le \int_{I_1} \log W(x)\,\omega_E(dx) + \sum_{n>1} \int_{I_n} \log W(x)\,\omega_E(dx)$$
$$\le C + \sum_{n>1} \omega(i, I_n, \mathbb{C} \setminus (I_1 \cup I_n)) \sup_{I_n} \log W \le C + \sum_{n>1} n^{-2} e^{-n} e^n < \infty.$$

(b) We use the following bound for the harmonic measure.

LEMMA 2.6. *Let I_n be intervals of length* 1 *such that* $\operatorname{dist}(0, I_n) = (1 + o(1)) \exp(n)$ *as $n \to +\infty$, and let $E = \cup I_n$. Then there exist constants $\varepsilon > 0$ and $C > 0$ that are independent of E such that*

$$\omega(i, I_n, \mathbb{C} \setminus E) \ge C \cdot e^{(\varepsilon - 1)n}.$$

We postpone the proof of this lemma to the next section.

Fix ρ with $1 - \varepsilon < 2\rho < 1$, consider the canonical product

$$B_\rho(z) = \prod_{n=1}^{\infty} \Big(1 - \frac{z}{n^{1/\rho}}\Big),$$

and define an entire function F_ρ of zero exponential type by $F_\rho(z) = B_\rho(z^2)$. Denote by Λ the zero set of F_ρ, i.e., $\Lambda = \big\{\lambda_{\pm n} = \pm n^{1/(2\rho)},\, n \ge 1\big\}$. It follows from a result G. H. Hardy [10] that the function

$$\frac{B_\rho(z)}{z^{-1/2} \sin(\pi z^\rho) \exp((\pi \cot \pi\rho) z^\rho)}$$

tends to a finite non-zero limit as $|z| \to \infty$ with $|\arg z| \le \pi/2$. Put $f(z) = \sin(\pi z^\rho)$ and $g = B_\rho / f$. Since $B_\rho'(\lambda) = g(\lambda) f'(\lambda)$ for $\lambda = n^{1/\rho}$, $n \ge 1$, we obtain, for some $c > 0$,

$$|B_\rho'(\lambda))| = (c + o(1)) \lambda^{\rho - 3/2} \exp((\pi \cot \pi\rho) \lambda^\rho), \quad \lambda = n^{1/\rho},\, n \to \infty,$$
$$|F_\rho'(\lambda)| = (2c + o(1)) |\lambda|^{2\rho - 2} \exp((\pi \cot \pi\rho) |\lambda|^{2\rho}), \quad \lambda \in \Lambda,\, |\lambda| \to \infty,$$

whence

$$\log |F_\rho'(\pm \exp(t))| = \log(2c) + o(1) + (2\rho - 2)t + (\pi \cot \pi\rho) \exp(2\rho t),$$
$$\exp(t) \in \Lambda,\, t \to \infty.$$

We define an even weight W by

$$\log W(\pm \exp(t)) = (\pi \cot \pi\rho) \exp(2\rho t). \tag{2.8}$$

Since

$$\sum_{\lambda \in \Lambda} \frac{W(\lambda)}{|\lambda|^2 |F_\rho'(\lambda)|} < \infty,$$

Theorem D implies that the polynomials are not dense in $C(W_r)$. The even log-convex function W is increasing on the positive half-line. Let us verify that

$$\log W(x) = (1+o(1))\log M_W(x), \qquad x\to+\infty. \tag{2.9}$$

First, without loss of generality, we assume that W is C^2-smooth. By the convexity of $\varphi=\log W\circ\exp$, we have $\varphi(s)+\varphi'(s)(r-s)\le\varphi(r)$ for every r and s. Now fix $t=\log x$. If $\varphi'(r)=n\le\varphi'(t)<n+1$, then $\varphi(t)\ge n(t-r)$. Furthermore, for some $\xi\in(r,t)$ we have $\varphi(t)-\varphi(r)=\varphi'(\xi)(t-r)$. Since $\varphi'(\xi)\le\varphi'(t)<n+1=\varphi'(r)+1$, we get $\varphi(t)-\varphi(r)-\varphi'(r)(t-r)\le t-r$, and as a result, $\varphi(r)+\varphi'(r)(t-r)\ge(1-1/n)\varphi(t)$. Therefore, $x^n\exp(\varphi(r)-rn)\ge W(x)^{1-1/n}$; also, for every $y>0$ and $r\in\mathbb{R}$, we have $W(y)\ge y^n\exp(\varphi(r)-rn)$. This proves (2.9).

Relations (2.8) and (2.9) imply that

$$\log M_W(x) = (1+o(1))(\pi\cot\pi\rho)|x|^{2\rho}, \qquad |x|\to\infty. \tag{2.10}$$

Choose a sequence of intervals I_n with $|I_n|=1$ and $\operatorname{dist}(0,I_n)=(1+o(1))\exp(n)$, $n\to+\infty$, and set $E=\cup I_n$. Apply Lemma 2.6 and add to E a union E' of small intervals such that $\Lambda\subset E'$ and

$$\omega\big(i,I_n,\mathbb{C}\setminus(E\cup E')\big)\ge C_1\cdot e^{(\varepsilon-1)n}.$$

Then by (2.10),

$$\begin{aligned}\int_{E\cup E'}&\log M_W(x)\,\omega\big(i,dx,\mathbb{C}\setminus(E\cup E')\big)\\ &\ge\sum_n\omega\big(i,I_n,\mathbb{C}\setminus(E\cup E')\big)\inf_{I_n}\log M_W\\ &\ge\sum_n C_2\cdot e^{(\varepsilon-1)n}e^{2\rho n}=+\infty.\end{aligned}$$

□

As corollaries to Theorem 2.1 we obtain some results of Levin and Benedicks. First, suppose that $W\big|E=\widetilde{W}\big|E$ for an even log-convex function $\widetilde{W}$ that is increasing on the positive half-line, $W\big|\mathbb{R}\setminus E\equiv+\infty$. Applying Theorem 2.1 and using relation (2.9) for $\widetilde{W}$, we arrive at the following result which is essentially equivalent to a theorem of Levin [18, Theorem 3.23]:

Theorem E. *Suppose W is a weight as above and E a set satisfying the conditions of Theorem* 2.1. *Then the polynomials are dense in $C(W)$ if and only if*

$$\int_E\log W(x)\,\omega_E(dx)=+\infty.$$

Benedicks ([4]; see also the discussion in [13, Section VIII A.4]) investigated the weighted polynomial approximation on the sets

$$E = \cup_{n\in\mathbb{Z}}\big[|n|^p \operatorname{sgn} n - \delta, |n|^p \operatorname{sgn} n + \delta\big] \tag{2.11}$$

for $p > 1$ and $\delta < 1/2$, and announced the following result:

THEOREM F. *Suppose that E is a set of the form* (2.11) *and W is a weight such that $W(x) = +\infty$ for $x \in \mathbb{R}\setminus E$. The polynomials are dense in $C(W)$ if and only if*

$$\sup_{P\in X_W}\int_E \frac{\log|P(x)|}{1+|x|^{1+1/p}}\,dx = +\infty\,.$$

In [4] Benedicks gave a proof of the "only if" part of this theorem based on the following estimate of the harmonic measure $\omega_E(dx) = \omega(i, dx, \mathbb{C}\setminus E)$ for sets E of the form (2.11):

LEMMA 2.7 (Benedicks [4]). *If E has the form* (2.11), *then*

$$\begin{aligned} &\frac{c}{1+|x|^{1+1/p}}\frac{1}{\sqrt{\delta^2-(x-|n|^p \operatorname{sgn} n)^2}} \\ &\qquad \le \frac{\omega_E(dx)}{dx} \le \frac{C}{1+|x|^{1+1/p}}\frac{1}{\sqrt{\delta^2-(x-|n|^p \operatorname{sgn} n)^2}} \end{aligned} \tag{2.12}$$

for $x \in \big[|n|^p \operatorname{sgn} n - \delta, |n|^p \operatorname{sgn} n + \delta\big]$, where c and C are positive constants that do not depend on x and n.

A more accessible reference for the upper bound in (2.12) is [13, Section VIII A.4], where a sketch of the proof is given. We will give a proof for the lower bound in the next section.

Theorem 2.1 together with the lower bound in (2.12) immediately yields the "if" part of Theorem F:

COROLLARY 2.8. *Suppose that W is a weight and E is a set of the form* (2.11) *such that $W(x) = \infty$ for $x \in \mathbb{R}\setminus E$. If the polynomials are not dense in $C(W)$, then*

$$\int_E \frac{\log M_W(x)}{1+|x|^{1+1/p}}\,dx < \infty.$$

3. Harmonic estimation in slit domains

In this section we prove Lemmas 1.1 and 2.6 and the lower estimate in Lemma 2.7.

Proof of Lemma 1.1. Suppose that for a sequence $y_k \to +\infty$ we have

$$\infty > PI_{E,h}(iy_k) \ge \mathcal{M}_E(iy_k).$$

By Harnack's inequality there exist positive constants c_1 and c_2, independent of k, such that

$$\left.\begin{aligned} PI_{E,h}(z) &\geq c_1 PI_{E,h}(iy_k) \\ \mathcal{M}_E(z) &\leq c_2 \mathcal{M}_E(iy_k) \end{aligned}\right\}, \qquad |z - iy_k| < y_k/2. \tag{3.1}$$

Let c_3 be a positive constant and consider the function $u = c_3\mathcal{M}_E - PI_{E,h}$, which is harmonic on $\mathbb{C} \setminus E$ and satisfies

$$u(z) \leq -(c_1 - c_2c_3)\mathcal{M}_E(iy_k), \qquad |z - iy_k| < y_k/2. \tag{3.2}$$

We use the following fact (see Lemma 1 of [23] and Lemma 6 of [3]):

$$\text{the function} \quad y \to \mathcal{M}_E(x+iy) \quad \text{is increasing for} \quad y \geq 0. \tag{3.3}$$

For the sake of completeness, we give a proof of this result, following [19]. Since the function $\mathcal{M}_E$ is positive, harmonic and symmetric in $\mathbb{C}\setminus E$, is subharmonic in the plane and has order at most one and mean type there, the subharmonic version of the Hadamard representation (see [11, Section 4.2]) implies that, for a finite positive measure μ on $\mathbb{R}$,

$$\begin{aligned} \mathcal{M}_E(z) = &\int_{|t|\geq 1} \Bigl(\log\Bigl|1 - \frac{z}{t}\Bigr| + \frac{\text{Re } z}{t}\Bigr)\, d\mu(t) \\ &+ \int_{|t|\leq 1} \log|t - z|\, d\mu(t) + c_1 + c_2 \text{ Re } z, \quad z \in \mathbb{C} \setminus E. \end{aligned}$$

Property (3.3) now follows immediately.

Using (3.3) and the second inequality in (3.1), we get

$$u(z) \leq c_3\mathcal{M}_E(z) \leq c_2c_3\mathcal{M}_E(iy_k), \quad |\text{Re } z| = y_k/2,\ |\text{Im } z| < y_k. \tag{3.4}$$

Denote by H the union of two horizontal sides of the domain $S = \{z \in \mathbb{C} : |\text{Re } z| < y_k/2, |\text{Im } z| < y_k\}$, and by V the union of its two vertical sides. An estimate for the harmonic measure in $S \setminus E$ shows that there exists a positive constant C that is independent of k and E such that

$$\frac{\omega(z, H, S \setminus E)}{\omega(z, V, S \setminus E)} \geq \frac{1}{C}, \qquad |z| < y_k/5. \tag{3.5}$$

To obtain this estimate we use the following claim, which is an easy generalization of a lemma of Benedicks [3, Lemma 7] (see also [13, p.436]): *For every square $S_{x,t} = \{z \in \mathbb{C} : |\text{Re } z - x| < t, |\text{Im } z| < t\}$ with horizontal sides $H_{x,t}$ and vertical sides $V_{x,t}$, we have*

$$\omega(x+iy, H_{x,t}, S_{x,t} \setminus E) \geq \omega(x+iy, V_{x,t}, S_{x,t} \setminus E),\ x+iy \in S_{x,t}. \tag{3.6}$$

To verify (3.6) we note first that, by symmetry, we have $\omega(\cdot, H_{x,t}, S_{x,t}) = \omega(\cdot, V_{x,t}, S_{x,t})$ on the diagonals. Therefore, applying the maximum principle to the difference of these functions, we get

$$\omega(r, H_{x,t}, S_{x,t}) \le \omega(r, V_{x,t}, S_{x,t}), \qquad r \in (x-t, x+t),$$
$$\omega(x+iy, H_{x,t}, S_{x,t}) \ge \omega(x+iy, V_{x,t}, S_{x,t}), \qquad x+iy \in S_{x,t},$$

and hence

$$\begin{aligned}
&\omega(x+iy, H_{x,t}, S_{x,t} \setminus E) \\
&\quad = \omega(x+iy, H_{x,t}, S_{x,t}) - \int_E \omega(r, H_{x,t}, S_{x,t})\omega(x+iy, dr, S_{x,t} \setminus E) \\
&\quad \ge \omega(x+iy, V_{x,t}, S_{x,t}) - \int_E \omega(r, V_{x,t}, S_{x,t})\omega(x+iy, dr, S_{x,t} \setminus E) \\
&\quad = \omega(x+iy, V_{x,t}, S_{x,t} \setminus E), \qquad x+iy \in S_{x,t}.
\end{aligned}$$

To deduce (3.5), note that the function $\omega(z, H, S \setminus E)$ is positive and continuous in $S \setminus E$. Therefore, for $A = \{x \pm iy_k/5 : |x| \le 2y_k/5\}$ we have $\min_{z \in A} \omega(z, H, S \setminus E) > 0$. Hence, if C is sufficiently large,

$$\varphi(z) \stackrel{\text{def}}{=} C\omega(z, H, S \setminus E) - \omega(z, V, S \setminus E) \ge 1, \qquad z \in A.$$

For every $z = x+iy$ with $|z| < y_k/5$ consider the square $S_{x,t}$ with $t = y_k/5$, and note that $H_{x,t} \subset A$ and $S_{x,t} \subset S$. Therefore $\varphi\big|H_{x,t} \ge 1$, $\varphi\big|E \equiv 0$, and $\varphi\big|V_{x,t} \ge -1$, and the estimate (3.6) implies that $\varphi(x+iy) \ge 0$. Thus, property (3.5) is proved.

Now, if c_3 is sufficiently small, then applying the theorem on two constants to the symmetric harmonic function u in the domain $S \setminus E$ and using (3.2), (3.4), (3.5), and the relation

$$\limsup_{z \to w} u(z) = -\liminf_{z \to w} PI_{E,h}(z) \le 0, \qquad w \in E,$$

we obtain

$$u(z) \le 0, \qquad |z| < y_k/5.$$

Thus, $c_3\mathcal{M}_E(z) - PI_{E,h}(z) = u(z) \le 0$, $z \in \mathbb{C} \setminus E$, which contradicts (1.4). □

Proof of the lower bound in Lemma 2.7.

Step A. For $t \ge 0$ denote by K_t the square $\{z \in \mathbb{C} : |\operatorname{Re} z - t| \le t/2, |\operatorname{Im} z| \le t/2\}$. In what follows we use the function $u(z) = \log|z + \sqrt{z^2-1}|$, which is positive and harmonic in $\mathbb{C} \setminus [-1,1]$, vanishes on $[-1,1]$, and satisfies $u(z) = \log|z| + O(1)$ as $|z| \to \infty$.

The function $v(z) = u(Ct^{1-1/p} \sin \pi z^{1/p})$ vanishes on a closed set F with $K_t \cap E \subset F$, for a suitable choice of C (independent of t). Furthermore, v is non-negative on K_t and harmonic on $K_t \setminus F$. We estimate the function $z \to |t^{1-1/p} \sin \pi z^{1/p}|$ using the following inequalities:

$$|t^{1-1/p} \sin \pi z^{1/p}| \le \exp(Ct^{1/p}), \qquad z \in K_t,$$
$$|t^{1-1/p} \sin \pi t^{1/p}| \ge Cn^{p-1}, \qquad t = (n+1/2)^p,\ n \ge 0.$$

The asymptotic relation $u(z) = \log|z| + O(1)$, $|z| \to \infty$, implies now that, for some positive constant c, we have $v(z) \le ct^{1/p}$ for $z \in K_t$, and $v((n+1/2)^p) \ge c\log n$, $n \ge 0$. Next, we choose $t = (n+1/2)^p$, and apply the theorem on two constants to the function $v(z)$ in $K_t \setminus F$:

$$c\log n \le v((n+1/2)^p) \le \omega(t, \partial K_t, K_t \setminus F) \sup_{\partial K_t} v \le c\, n\, \omega(t, \partial K_t, K_t \setminus F)\,.$$

Hence, for $t = (n+1/2)^p$, $n > 1$, we have

$$\omega(t, \partial K_t, K_t \setminus E) \ge \omega(t, \partial K_t, K_t \setminus F) \ge c\,\frac{\log n}{n}\,.$$

Let H_t be the union of the two horizontal sides of K_t. By the lemma of Benedicks mentioned in the proof of Lemma 1.1 we have

$$\omega(t, H_t, K_t \setminus E) \ge \frac{1}{2}\omega(t, \partial K_t, K_t \setminus E).$$

Therefore, for $n > 1$,

$$\omega((n+1/2)^p, H_{(n+1/2)^p}, K_{(n+1/2)^p} \setminus E) \ge c\frac{\log n}{n}. \tag{3.7}$$

Step B. The Green function $G(z,i)$ for $\mathbb{C} \setminus E$ is positive, bounded and harmonic on $\{z : |\operatorname{Im} z| > 2\}$. Therefore, applying the Poisson formula in the half-planes $\{z : \pm \operatorname{Im} z > 2\}$ we get

$$\begin{aligned} G(z,i) &\ge \frac{1}{\pi}\int_{-\infty}^{\infty} G(x \pm 2i, i)\frac{|\operatorname{Im} z| - 2}{(|\operatorname{Im} z| - 2)^2 + x^2}\, dx \\ &\ge \frac{c}{|\operatorname{Im} z|}, \quad 3|\operatorname{Im} z| \ge |\operatorname{Re} z| \ge 10. \end{aligned} \tag{3.8}$$

The inequalities (3.7) and (3.8) imply that for $n > 1$,

$$G((n+1/2)^p, i) \ge \omega((n+1/2)^p, H_{(n+\frac12)^p}, K_{(n+\frac12)^p} \setminus E) \inf_{H_{(n+\frac12)^p}} G \ge c\,\frac{\log n}{n^{p+1}}.$$

Set $I_n = [n^p - \delta, n^p + \delta]$. Since $G(z,i)$ is positive and harmonic on $\{z : |z - n^p| \le n^{p-1}\} \setminus I_n$, Harnack's inequality gives

$$G(z,i) \ge c\frac{\log n}{n^{p+1}}, \qquad |z - n^p| = n^{p-1}.$$

Step C. Finally, consider the auxiliary function $w(z) = u((z - n^p)/\delta)$, which is harmonic on $\mathbb{C} \setminus I_n$, vanishes on I_n, and satisfies $w(z) \le c\log n$ for $|z - n^p| = n^{p-1}$. Therefore, for $n > 1$, we have

$$G(z,i) \ge \frac{c}{n^{p+1}} w(z), \qquad |z - n^p| \le n^{p-1},$$

and

$$\frac{\omega(i,dx,\mathbb{C}\setminus E)}{dx} = \frac{1}{\pi}\frac{\partial G(x+iy,i)}{\partial y}\Big|_{y=0}$$
$$\geq \frac{c}{1+|x|^{1+1/p}}\frac{1}{\sqrt{\delta^2-(x-|n|^p \operatorname{sgn} n)^2}}\,dx, \qquad x\in I_n.$$

The proof for the case $n \leq 1$ is analogous. □

Proof of Lemma 2.6. As in the preceding proof, we consider the Green function $G(z,i)$ for $\mathbb{C}\setminus E$, which is positive, harmonic and bounded in $\mathbb{C}\setminus(E\cup\{z:|z-i|<1\})$. Set

$$h_n = \int_0^{\exp(n)} G(x,i)\,dx, \qquad n\geq 0.$$

Applying the Poisson formula in the lower half-plane we get

$$G(-ie^n,i) \geq \frac{1}{\pi}\int_0^{\exp(n)} \frac{e^n}{x^2+e^{2n}} G(x,i)\,dx \geq c\cdot e^{-n}h_n, \qquad n\geq 0.$$

By Harnack's inequality, on at least one half of the length of the interval $[e^n, e^{n+1}]$, we have the bound $G(x,i)\geq c\cdot G(-ie^n,i)\geq c\cdot e^{-n}h_n$. Therefore, for some $c>0$, we have $h_{n+1}>(1+c)h_n$ for $n\geq 0$. Consequently, $h_{n+1} > c(1+c)^n$ for $n\geq 0$. Applying again the Poisson formula, we obtain, for some $\varepsilon>0$,

$$G(-ix,i) > c\cdot x^{\varepsilon-1}, \qquad x\geq 1.$$

Denote by c_n the center of the interval I_n. Arguing as in step C of the preceding proof, we compare $G(z,i)$ with $w(z)=u(2(z-c_n))$ in $\{z:|z-c_n|\leq e^{n-1}\}\setminus I_n$ and deduce

$$\omega(i,I_n,\mathbb{C}\setminus E) \geq c\cdot e^{(\varepsilon-1)n}/n, \qquad n\geq 1.$$

□

Remark 3.1. To estimate the harmonic measure ω_E from above, we may use Theorem 1.3 or Theorems D and E. In particular, under the conditions of Lemma 2.6, we have

$$\omega(i,I_n,\mathbb{C}\setminus E) \leq \exp(-c\sqrt{n}), \qquad n\geq 1.$$

for some positive constant c.

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References

[1] N. I. Akhiezer, *On the weighted approximation of continuous functions by polynomials on the entire real axis*, Uspekhi Mat. Nauk **11** (1956), 3–43; English translation: Amer. Math. Soc. Transl. **22** (1962), 95–137.

[2] N. I. Akhiezer, B. Ya. Levin, *Generalization of S. N. Bernšteĭn's inequality for derivatives of entire functions*, Issledovanija po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960, pp. 111–165; French translation: Fonctions d'une variable complexe. Problèmes contemporains (A. Marcouchevitch, ed.), Gauthier–Villars, Paris, 1962, pp. 109–161.

[3] M. Benedicks, *Positive harmonic functions vanishing on the boundary of certain domains in* $\mathbf{R}^n$, Ark. Mat. **18** (1980), no. 1, 53–72.

[4] ———, *Weighted polynomial approximation on subsets of the real line*, Preprint 1981:11, Uppsala University, Mathematical Department (1981), 12 pp.

[5] A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support*, Acta Math. **107** (1962), 291–309.

[6] L. de Branges, *The Bernstein problem*, Proc. Amer. Math. Soc. **10** (1959), 825–832.

[7] ———, *Hilbert spaces of entire functions*, Prentice-Hall, 1968.

[8] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear non-selfadjoint operators*, Transl. Math. Monogr., vol. 18, American Mathematical Society, 1969.

[9] A. A. Goldberg and I. V. Ostrovskiĭ, *The distribution of values of meromorphic functions* (in Russian), Nauka, Moscow, 1970.

[10] G. H. Hardy, *On the function* $P_\rho(x)$, Quart. J. Math. **37** (1906), 146–172.

[11] W. K. Hayman and P. B. Kennedy, *Subharmonic functions. Vol. I*, London Mathematical Society Monographs, No. 9., Academic Press, London, 1976.

[12] P. P. Kargaev, *Existence of the Phragmén-Lindelöf function and some conditions for quasianalyticity*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **126** (1983), 97–108; English translation: J. Soviet Math. **27** (1984), 2486–2495.

[13] Paul Koosis, *The logarithmic integral. I*, Cambridge University Press, Cambridge, 1988.

[14] M. G. Krein, *A contribution to the theory of entire functions of exponential type* (in Russian), Izv. Akad. Nauk SSSR **11** (1947), 309–326.

[15] ———, *The fundamental propositions of the theory of representations of Hermitian operators with deficiency index* (m, m), Ukrain. Mat. Žurnal **1** (1949), no. 2, 3–66; English translation: Amer. Math. Soc. Transl., vol. 97, American Mathematical Society, Providence, 1970, pp. 75–143.

[16] ———, *On the indeterminate case of the Sturm-Liouville boundary problem in the interval* $(0, \infty)$ (in Russian), Izvestiya Akad. Nauk SSSR. Ser. Mat. **16** (1952), 293–324.

[17] B. Ya. Levin, *Lectures on entire functions*, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, 1996.

[18] ———, *Density of systems of functions, quasianalyticity and subharmonic majorants*, Zap. Nauchn. Seminarov LOMI **170** (1989), 102–156; English translation: J. Soviet Math. **63** (1993), 171–201.

[19] ———, *Majorants in classes of subharmonic functions,* II*, The relation between majorants and conformal mapping,* III*, The classification of the closed sets on* $\mathbb{R}$ *and the representation of the majorants*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **52** (1989), 3–33; English translation: J. Soviet Math. **52** (1990), 3351–3372.

[20] S. N. Mergelyan, *Weighted approximation by polynomials*, Uspekhi Mat. Nauk **11** (1956), 107–152; English translation: Amer. Math. Soc. Transl., vol. 10, American Mathematical Society, Providence, 1958, pp. 59–106.

[21] Rolf Nevanlinna, *Analytic functions*, Springer-Verlag, New York, 1970.

[22] M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann*, Ann. Inst. Fourier Grenoble **3** (1951), 103–197 (1952).

[23] A. C. Schaeffer, *Entire functions and trigonometric polynomials*, Duke Math. J. **20** (1953), 77–88.

[24] M. Sodin and P. Yuditskiĭ, *Another approach to de Branges' theorem on weighted polynomial approximation*, in: Proceedings of the Ashkelon Workshop on Complex Function Theory (May 1996), L. Zalcman, ed., Israel Mathematical Conferences Proceedings, vol. 11, American Mathematical Society, Providence, 1997, pp. 221-227.

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