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# ON CIRCLE ENDOMORPHISMS

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ABSTRACT. In [6], Shub and Sullivan posed the problem of finding complete measure theoretic invariants for analytic Lebesgue measure preserving expanding endomorphisms of $S^1$. In [2], the author gave necessary and sufficient conditions for two such endomorphisms to be isomorphic. These complete invariants were a mixture of a topological and measure-theoretic nature. Establishing them required finding a coboundary equation with no obvious method of construction. In this note we use a result by Arteaga to furnish a different set of complete isomorphism invariants, still of a mixed topological and measure-theoretic nature but far more easily checked than the ones established in [2].

## Introduction

For $i = 1, 2$ let $f_i$ be endomorphisms of the Lebesgue spaces $(X_1, B_i, \mu_i)$. We say that the two systems $(X_1, B_1, \mu_1, f_1)$ and $(X_2, B_2, \mu_2, f_2)$ are isomorphic if there are sets of measure zero $A_1 \subset X_1, A_2 \subset X_2$ and a one-to-one onto map $\phi$: $X_1 \setminus A_1 \to X_2 \setminus A_2$ such that $\phi f_1 = f_2\phi$ on $X_1 \setminus A_1$ and $\mu(\phi^{-1}E) = \mu_2(E)$ for all measurable $E \subset X_2 \setminus A_2$. The classification problem in ergodic theory is to determine when two given endomorphisms are isomorphic. As usual in measure theory, we do not distinguish between functions which coincide almost everywhere.

Let $1 \leq r \leq \omega$ and $f$: $S^1 \to S^1$ be a $C^r$ Lebesgue measure-preserving endomorphism. Then if $Df$ denotes the derivative of $f$ we say that $f$ is expanding if there exists $\lambda \in \mathbf{R}$ such that $|Df(z)| > \lambda > 1$ for all $z \in S^1$.

Countably many-to-one positively measurable non-singular maps have Jacobian derivatives (see [3], [4], [7] for details) which we denote by $|D|$. For $C^1$ Lebesgue measure-preserving endomorphisms, the Jacobian derivative is simply the absolute value of the derivative of the endomorphism. We say that the Jacobian derivatives $|Df|$ and $|Dg|$ are isomorphic if there is a Lebesgue measure-preserving automorphism $\phi$ of $S^1$ such that $|Df| = |Dg|\phi$. If $\phi$ is a Lebesgue measure-preserving automorphism of $S^1$ then $|D\phi| = 1$. Therefore, if $\phi$ is an isomorphism between $f$ and $g$, by the chain rule we have $|Df| = |Dg|\phi$ and so the Jacobians will be isomorphic. When our endomorphisms are real analytic and expanding, the following theorem of Shub and Sullivan shows that this invariant is nearly complete.

THEOREM 1 [6]. *Let $f$ and $g$ be real analytic expanding endomorphisms of $S^1$ which preserve Lebegue measure. Suppose that the Jacobian derivatives of $f$ and $g$ are isomorphic; then there are isometries $R_1$ and $R_2$ of $S^1$ such that $R_1^{-1}gR_1 = R_2f$.*

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Thus the problem posed by Shub and Sullivan in [2] was to find another measure-theoretic invarient to remove $R_2$, i.e., to find a complete set of measure-theoretic isomorphism invariants.

## The phase group

In [2] the author introduced the subgroup $G_f = \{\alpha \in S^1\colon \exists\, \beta \in S^1$ such that $f(\alpha z) = \beta f(z)$ for all $z \in S^1\}$. associated with a continuous surjection $f\colon S^1 \to S^1$, calling it the phase group. Being closed, it is either all of $S^1$ or the $m$-th roots of unity for some integer $m \geq 1$.

Examples were given of real analytic expanding Lebesgue measure-preserving endomorphisms of the circle with degree $d$ whose phase group has order $m$ for any integers $d \geq 2$ and $m \geq 1$. We will need:

LEMMA 1 [2]. *If $f\colon S^1 \to S^1$ is a continuous surjection with degree d then $f(z) = cz^d$ for some constant $c \in S^1$ if and only if $G_f = S^1$.*

LEMMA 2 [2]. *Suppose that $f\colon S^1 \to S^1$ is a continuous surjection with degree $d$; then if $\alpha \in G_f$, we have $f(\alpha z) = \alpha^d f(z)$* for all $z \in S^1$.

Let $f^n$ denote the $n$-fold composition of $f$. In [2], complete measure theoretic invariants for $f$ and $\alpha g$ to be isomorphic were given, where $\alpha$ is an element of the finite group $G_g$:

THEOREM 2 [2]. *Let $f$ and $g$ be real analytic Lebesgue measure-preserving expanding endomorphisms of $S^1$ with the same degree. Then $g$ is isomorphic to $\alpha f$, where $\alpha \in G_f$, if and only if there exists a Lebesgue measure-preserving automorphism $\phi$ of $S^1$ such that $|Df|(z) = |Dg|(\phi(z))$ and $|Df^2|(z) = |Dg^2|(\phi(z))$.*

## Main result

We will need the following result by Arteaga which tells us when a topological conjugacy is differentiable:

THEOREM 3 [1]. *For $r \geq 2$ let $f, g$ be $C^r$ orientation preserving endomorphisms of $S^1$ and let $\phi\colon S^1 \to S^1$ be a topological conjugacy between $f$ and $g$. Then $\phi$ is a $C^r$ diffeomorphism if and only if*

$$(f^n)'(z) = (g^n)'(\phi(z)) \text{ for all } n \in \mathbf{N} \textit{ and } z \in S^1 \textit{ satisfying } f^n(z) = z.$$

A result by Shub and Sullivan shows that under certain conditions this diffeomorphism is an isometry:

THEOREM 4 [6]. *Let $f$ and $g$ be expanding endomorphisms of $S^1$ which preserve Lebesgue measure. Suppose that either $|Df|(z)$ or $|Dg|(z)$ is not constant and that $fh = hg$ for an absolutely continuous conjugacy $h$. Then there exists an isometry $R$ of $S^1$ such that $h = R$ a.e.*

In [4] Shub showed that any two $C^1$ expanding maps of the circle with the same degree are topologically conjugate. Therefore the $f$ and $\alpha f$ from Theorem 2 are topologically conjugate and we denote this topological conjugacy by $\Psi_{f,g}$ (as the $\alpha$ is dependent on $g$).

We now state the main result of this paper.

THEOREM 5. *Let $f$ and $g$ be real analytic Lebesgue measure preserving endomorpisms of $S^1$ with the same degree. Then $f$ is isomorpic to $g$ if and only if there exists a Lebesgue measure preserving automorphism $\phi$ of $S^1$ such that:*

(i) $|Df|(z) = |Dg|(\phi(z))$ for all $z \in S^1$.
(ii) $|Df^2|(z) = |Dg^2|(\phi(z))$ for all $z \in S^1$.
(iii) $(f^n)'(z) = (f^n)'(\Psi_{f,g}(z))$ for all $n \in N$ *and* $z \in S^1$ *satisfying* $f^n(z) = z$.

*Proof.* Let the degree of the functions be $d \in \mathbf{Z}$.

$\Rightarrow$ If $f$ is isomorphic to $g$ then $f^2$ will be isomorphic to $g^2$. The chain rule will then give conditions (i) and (ii).

Also, if $f$ is isomorphic to $g$ then $\Psi_{f,g} = \mathrm{Id}$. In that case, condition (iii) is trivially satisfied.

$\Leftarrow$ If one of the Jacobians is constant a.e., then the other one is also constant a.e. since they are isomorphic. In that case, $f(z) = \gamma z^d$ and $g(z) = \beta z^d$ for some $\gamma, \beta \in S^1$. Now let $\eta$ satisfy $\eta^{d-1} = \frac{\beta}{\gamma}$ and let $R(z) = \eta z$. Then $fR = Rg$ and so the two functions will be isomorphic. Hence we can assume the Jacobians are not constant a.e.

From conditions (i) and (ii) and Theorem 2 it follows that $g$ is isomorphic to $\alpha f$ for some $\alpha \in G_f$. Now, using Lemma 2 we get

$$((\alpha f)^n)'(z) = (\alpha^{1+d+\ldots d^n} f^n)'(z) = (f^n)'(z) \text{ for all } n \in \mathbf{N} \text{ and } z \in S^1.$$

Combining this with condition (iii) gives

$$(f^n)'(z) = ((\alpha f)^n)'(\Psi_{f,g}(z)) \text{ for all } n \in \mathbf{N} \text{ and } z \in S^1 \text{ satisfying } f^n(z) = z.$$

From Theorem 3 we can conclude that $\Psi_{f,g}$ is a diffeomorphism. From Theorem 4 this diffeomorphism must be an isometry; hence $f$ is isomorphic to $\alpha f$. Finally, if $f$ is isomorphic to $\alpha f$ and $g$ is isomorphic to $\alpha f$ then $f$ is isomorphic to $g$ and the result is proved.

*Remarks.* Supppose that $|G_f| = m$ and $\deg(f) = d$.

(1) It was shown in [2] that if g.c.d$(d-1, m) = 1$ then, just conditions (i) and (ii) of Theorem 5 are sufficient for isomorphism (cf. Corollaries 2.1 and 2.2 of [2]).

(2) If g.c.d$(d-1, m) \neq 1$ then, since $\Psi_{f,g}$ is an isometry, to check (iii) it suffices to find a $\beta \in S^1$ which satisfies one of the following:

$$(f^n)'(z) = (f^n)'(\beta z) \text{ for all } n \in \mathbf{N} \text{ and } z \in S^1 \text{ satisfying } f^n(z) = z$$

or

$$(f^n)'(z) = (f^n)'(\tfrac{\beta}{z}) \text{ for all } n \in \mathbf{N} \text{ and } z \in S^1 \text{ satisfying } f^n(z) = z.$$

Note that the first condition deals with the case where $\Psi_{f,g}$ is orientation preserving and the second with $\Psi_{f,g}$ orientation reversing. Note that $\Psi_{f,g}$ does not have to be explicitely found in either case.

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