ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 3, Fall 2000

NON-SYMMETRIC CONVEX DOMAINS HAVE NO BASIS OF EXPONENTIALS

MIHAIL N. KOLOUNTZAKIS

ABSTRACT. A conjecture of Fuglede states that a bounded measurable set $\Omega \subset \mathbb{R}^d$, of measure 1, can tile $\mathbb{R}^d$ by translations if and only if the Hilbert space $L^2(\Omega)$ has an orthonormal basis consisting of exponentials $e_\lambda(x) = \exp 2\pi i \langle \lambda, x \rangle$. If Ω has the latter property it is called *spectral*. We generalize a result of Fuglede, that a triangle in the plane is not spectral, proving that every non-symmetric convex domain in $\mathbb{R}^d$ is not spectral.

Introduction

Let Ω be a measurable subset of $\mathbb{R}^d$ of measure 1 and Λ be a discrete subset of $\mathbb{R}^d$. We write

$$e_\lambda(x) = \exp 2\pi i \langle \lambda, x \rangle, \quad (x \in \mathbb{R}^d),$$
$$E_\Lambda = \{e_\lambda : \lambda \in \Lambda\} \subset L^2(\Omega).$$

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g \rangle_\Omega = \int_\Omega f\overline{g}, \quad \text{and} \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

Definition 1. The pair (Ω, Λ) is called a *spectral pair* if E_Λ is an orthonormal basis for $L^2(\Omega)$. A set Ω will be called *spectral* if there is $\Lambda \subset \mathbb{R}^d$ such that (Ω, Λ) is a spectral pair. The set Λ is then called a *spectrum* of Ω.

Example. If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $(Q_d, \mathbb{Z}^d)$ is a spectral pair.

We write $B_R(x) = \left\{y \in \mathbb{R}^d : |x - y| < R\right\}$.

Definition 2 (Density). (i) The set $\Lambda \subset \mathbb{R}^d$ has *uniformly bounded density* if for each $R > 0$ there exists a constant $C > 0$ such that Λ has at most C elements in each ball of radius R in $\mathbb{R}^d$.

Received January 19, 1999.
1991 Mathematics Subject Classification. Primary 41; Secondary 52.
Partially supported by the National Science Foundation.

(ii) The set $\Lambda \subset \mathbb{R}^d$ has *density* ρ, and we write $\rho = \text{dens}\,\Lambda$, if we have

$$\rho = \lim_{R\to\infty} \frac{|\Lambda \cap B_R(x)|}{|B_R(x)|}$$

uniformly for all $x \in \mathbb{R}^d$.

We define translational tiling for complex-valued functions below.

Definition 3. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be measurable and $\Lambda \subset \mathbb{R}^d$ be a discrete set. We say that f *tiles with* Λ *at level* $w \in \mathbb{C}$, and sometimes write $f + \Lambda = w\mathbb{R}^d$, if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = w \quad \text{for almost every (Lebesgue) } x \in \mathbb{R}^d, \tag{1}$$

with the sum above converging absolutely a.e. If $\Omega \subset \mathbb{R}^d$ is measurable we say that $\Omega + \Lambda$ *is a tiling* when $\mathbf{1}_\Omega + \Lambda = w\mathbb{R}^d$, for some w. If w is not mentioned it is understood to be equal to 1.

Remarks. 1. If $f \in L^1(\mathbb{R}^d)$ and Λ has uniformly bounded density one can easily show (see [KL96] for the proof in one dimension, which works in higher dimension as well) that the sum in (1) converges absolutely a.e. and defines a locally integrable function of x.

2. In the very common case when $f \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} f \neq 0$ the condition that Λ has uniformly bounded density follows easily from (1) and need not be postulated a priori.

3. It is easy to see that if $f \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f \neq 0$ and $f + \Lambda$ is a tiling then Λ has a density and the level of the tiling w is given by

$$w = \int_{\mathbb{R}^d} f \cdot \text{dens}\,\Lambda.$$

From now on we restrict ourselves to tiling with functions in L^1 and sets of finite measure.

Example. $Q_d + \mathbb{Z}^d$ is a tiling.

The following conjecture is still unresolved.

Conjecture (Fuglede [F74]). If $\Omega \subset \mathbb{R}^d$ is bounded and has Lebesgue measure 1 then $L^2(\Omega)$ has an orthonormal basis of exponentials if and only if there exists $\Lambda \subset \mathbb{R}^d$ such that $\Omega + \Lambda = \mathbb{R}^d$ is a tiling.

Remark. It is not hard to show [F74] that $L^2(\Omega)$ has a basis Λ which is a *lattice* (i.e., $\Lambda = A\mathbb{Z}^d$, where A is a non-singular $d \times d$ matrix) if and only if $\Omega + \Lambda^*$ is a tiling. Here

$$\Lambda^* = \{\mu \in \mathbb{R}^d\colon \langle \mu, \lambda \rangle \in \mathbb{Z},\ \forall \lambda \in \Lambda\}$$

is the *dual lattice* of Λ (we have $\Lambda^* = A^{-\top}\mathbb{Z}^d$).

Fuglede [F74] showed that the disk and the triangle in $\mathbb{R}^2$ are not spectral domains. In this note we prove the following generalization of Fuglede's triangle result.

THEOREM 1. *Let Ω have measure* 1 *and be a convex, non-symmetric, bounded open set in $\mathbb{R}^d$. Then Ω is not spectral.*

The set Ω is called *symmetric* with respect to 0 if $y \in \Omega$ implies $-y \in \Omega$, and symmetric with respect to $x_0 \in \mathbb{R}^d$ if $y \in \Omega$ implies that $2x_0 - y \in \Omega$. It is called *non-symmetric* if it is not symmetric with respect to any $x_0 \in \mathbb{R}^d$. For example, in any dimension a simplex is non-symmetric.

It is known [V54], [M80] that every convex body that tiles $\mathbb{R}^d$ by translation is a centrally symmetric polytope and that each such body also admits a lattice tiling and, therefore (see the remark after Fuglede's conjecture above), its L^2 admits a lattice spectrum. Given Theorem 1, to prove Fuglede's conjecture restricted to convex domains, one still has to prove that any symmetric convex body that is not a tile admits no orthonormal basis of exponentials for its L^2.

In §1 we derive some necessary and some sufficient conditions for $f + \Lambda$ to be a tiling. These conditions roughly state that tiling is equivalent to a certain tempered distribution, associated with Λ being "supported" on the zero set of $\widehat{f}$ plus the origin. Similar conditions had been derived in [KL96] but here we have to work with less smoothness for $\widehat{f}$. To compensate for the lack of smoothness we work with compactly supported $\widehat{f}$ and nonnegative f and $\widehat{f}$, conditions which are fulfilled for our problem.

In §2 we restate the property that Ω is spectral as a tiling problem for $|\widehat{\mathbf{1}_\Omega}|^2$ and use the conditions derived in §1 to prove Theorem 1. What makes the proof work is that when Ω is a non-symmetric convex set the set $\Omega - \Omega$ has volume strictly larger than $2^d \operatorname{vol} \Omega$.

1. Fourier-analytic conditions for tiling

Our method relies on a Fourier-analytic characterization of translational tiling, which is a variation of the one used in [KL96]. We define the (generally unbounded) measure

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda,$$

where δ_λ represents a unit mass at $\lambda \in \mathbb{R}^d$. If Λ has uniformly bounded density then δ_Λ is a tempered distribution (for example, see [R73]) and therefore its Fourier Transform $\widehat{\delta_\Lambda}$ is defined and is itself a tempered distribution.

The action of a tempered distribution α (see [R73]) on a Schwartz function ϕ is denoted by $\alpha(\phi)$. The Fourier Transform of α is defined by the equation

$$\widehat{\alpha}(\phi) = \alpha(\widehat{\phi}).$$

The support $\operatorname{supp}\alpha$ is the smallest closed set F such that for any smooth ϕ of compact support contained in the open set F^c we have $\alpha(\phi) = 0$.

THEOREM 2. *Suppose that $f \geq 0$ is not identically 0, that $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \geq 0$ has compact support and $\Lambda \subset \mathbb{R}^d$. If $f + \Lambda$ is a tiling then*

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \left\{x \in \mathbb{R}^d : \widehat{f}(x) = 0\right\} \cup \{0\}. \tag{2}$$

Proof of Theorem 2. Assume that $f + \Lambda = w\mathbb{R}^d$ and let

$$K = \left\{\widehat{f} = 0\right\} \cup \{0\}.$$

We have to show that

$$\widehat{\delta_\Lambda}(\phi) = 0, \quad \forall \phi \in C_c^\infty(K^c).$$

Since $\widehat{\delta_\Lambda}(\phi) = \delta_\Lambda(\widehat{\phi})$ this is equivalent to $\sum_{\lambda\in\Lambda}\widehat{\phi}(\lambda) = 0$, for each such ϕ. Notice that $h = \phi/\widehat{f}$ is a continuous function, but not necessarily smooth. We shall need $\widehat{h} \in L^1$. This is a consequence of a well-known theorem of Wiener [R73, Ch. 11]. We denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus.

THEOREM (Wiener). *If $g \in C(\mathbb{T}^d)$ has an absolutely convergent Fourier series*

$$g(x) = \sum_{n\in\mathbb{Z}^d} \widehat{g}(n) e^{2\pi i\langle n,x\rangle}, \quad \widehat{g} \in \ell^1(\mathbb{Z}^d),$$

and if g does not vanish anywhere on $\mathbb{T}^d$ then $1/g$ also has an absolutely convergent Fourier series.

Assume that

$$\operatorname{supp}\phi,\ \operatorname{supp}\widehat{f} \subseteq \left(-\frac{L}{2}, \frac{L}{2}\right)^d.$$

Define the function F

(i) to be periodic in $\mathbb{R}^d$ with period lattice $(L\mathbb{Z})^d$,
(ii) to agree with $\widehat{f}$ on $\operatorname{supp}\phi$,
(iii) to be non-zero everywhere and,
(iv) to have $\widehat{F} \in \ell^1(\mathbb{Z}^d)$, i.e.,

$$\widehat{F} = \sum_{n\in\mathbb{Z}^d} \widehat{F}(n)\delta_{L^{-1}n},$$

is a finite measure in $\mathbb{R}^d$.

One way to define such an F is as follows. First, define the $(L\mathbb{Z})^d$-periodic function $g \geq 0$ to be $\widehat{f}$ periodically extended. The Fourier coefficients of g are $\widehat{g}(n) = L^{-d} f(-n/L) \geq 0$. Since $g, \widehat{g} \geq 0$ and g is continuous at 0 it is easy to prove that $\sum_{n\in\mathbb{Z}^d} \widehat{g}(n) = g(0)$, and therefore that g has an absolutely convergent Fourier series.

Let ϵ be small enough to guarantee that $\widehat{f}$ (and hence g) does not vanish on $(\operatorname{supp}\phi) + B_\epsilon(0)$. Let k be a smooth $(L\mathbb{Z})^d$-periodic function which is equal to 1 on $(\operatorname{supp}\phi) + (L\mathbb{Z}^d)$ and equal to 0 off $(\operatorname{supp}\phi + B_\epsilon(0)) + (L\mathbb{Z}^d)$, and satisfies $0 \leq k \leq 1$ everywhere. Finally, define

$$F = kg + (1-k).$$

Since both k and g have absolutely summable Fourier series and this property is preserved under both sums and products, it follows that F also has an absolutely summable Fourier series. And by the nonnegativity of g it follows that F is never 0, since $k = 0$ on $Z(\widehat{f}) + (L\mathbb{Z}^d)$.

By Wiener's theorem, $\widehat{F^{-1}} \in \ell^1(\mathbb{Z}^d)$, i.e., $\widehat{F^{-1}}$ is a finite measure on $\mathbb{R}^d$. We now have

$$\left(\frac{\phi}{\widehat{f}}\right)^\wedge = \widehat{\phi F^{-1}} = \widehat{\phi} * \widehat{F^{-1}} \in L^1(\mathbb{R}^d).$$

This justifies the interchange of the summation and integration below:

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda) &= \sum_{\lambda\in\Lambda} \left(\frac{\phi}{\widehat{f}}\widehat{f}\right)^\wedge(\lambda) \\
&= \sum_{\lambda\in\Lambda} \left(\frac{\phi}{\widehat{f}}\right)^\wedge * \widehat{\widehat{f}}(\lambda) \\
&= \sum_{\lambda\in\Lambda} \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge(y) f(y-\lambda)\, dy \\
&= \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge(y) \sum_{\lambda\in\Lambda} f(y-\lambda)\, dy \\
&= w \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge(y)\, dy \\
&= w \frac{\phi}{\widehat{f}}(0) \\
&= 0,
\end{aligned}$$

as we had to show. □

For a set $A \subseteq \mathbb{R}^d$ and $\delta > 0$ we write

$$A_\delta = \left\{x \in \mathbb{R}^d : \operatorname{dist}(x, A) < \delta\right\}.$$

We shall need the following partial converse to Theorem 2.

THEOREM 3. *Suppose that $f \in L^1(\mathbb{R}^d)$, and that $\Lambda \subset \mathbb{R}^d$ has uniformly bounded density. Suppose also that $O \subset \mathbb{R}^d$ is open and*

$$\operatorname{supp} \widehat{\delta_\Lambda} \setminus \{0\} \subseteq O \text{ and } O_\delta \subseteq \{\widehat{f} = 0\} \tag{3}$$

for some $\delta > 0$. Then $f + \Lambda$ is a tiling at level $\widehat{f}(0) \cdot \widehat{\delta_\Lambda}(\{0\})$.

Proof. Let $\psi: \mathbb{R}^d \to \mathbb{R}$ be smooth, have support in $B_1(0)$ and $\widehat{\psi}(0) = 1$ and for $\epsilon > 0$ define the approximate identity $\psi_\epsilon(x) = \epsilon^{-d}\psi(x/\epsilon)$. Let

$$f_\epsilon = \widehat{\psi_\epsilon} f,$$

which has rapid decay.

First we show that $(\int f_\epsilon)^{-1} f_\epsilon + \Lambda$ is a tiling. That is, we show that the convolution $f_\epsilon * \delta_\Lambda$ is a constant. Let ϕ be any Schwartz function. Then

$$f_\epsilon * \delta_\Lambda(\phi) = \widehat{f_\epsilon}\widehat{\delta_\Lambda}(\widehat{\phi}(-x)) = \widehat{\delta_\Lambda}(\widehat{\phi}(-x)\widehat{f_\epsilon}).$$

The function $\widehat{\phi}(-x)\widehat{f_\epsilon}$ is a Schwartz function whose support intersects $\operatorname{supp}\widehat{\delta_\Lambda}$ only at 0, since, for small enough $\epsilon > 0$,

$$\operatorname{supp}\widehat{\phi}\widehat{f_\epsilon} \subseteq \operatorname{supp}\widehat{f_\epsilon} \subseteq (\operatorname{supp}\widehat{f})_\epsilon \subseteq O^c.$$

Hence, for each Schwartz function ϕ,

$$f_\epsilon * \delta_\Lambda(\phi) = \widehat{\phi}(0)\widehat{f_\epsilon}(0)\widehat{\delta_\Lambda}(\{0\}),$$

which implies

$$f_\epsilon * \delta_\Lambda(x) = \widehat{f_\epsilon}(0)\widehat{\delta_\Lambda}(\{0\}), \quad \text{a.e.}(x).$$

We also have $\sum_{\lambda\in\Lambda} |f(x-\lambda)|$ finite a.e. (see Remark 1 following the definition of tiling), hence, for almost every $x \in \mathbb{R}^d$,

$$\sum_{\lambda\in\Lambda} |f(x-\lambda) - f_\epsilon(x-\lambda)| = \sum_{\lambda\in\Lambda} |f(x-\lambda)| \cdot |1 - \widehat{\psi_\epsilon}(x-\lambda)|,$$

which tends to 0 as $\epsilon \to 0$. This proves

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = \widehat{f}(0) \cdot \widehat{\delta_\Lambda}(\{0\}), \quad \text{a.e.}(x). \qquad \square$$

2. Proof of the main result

We now make some remarks that relate the property "E_Λ is a basis for $L^2(\Omega)$" to a certain function tiling $\mathbb{R}^d$ with Λ.

Assume that Ω is a bounded open set of measure 1. First, notice that

$$\langle e_\lambda, e_x \rangle_\Omega = \widehat{\mathbf{1}_\Omega}(x - \lambda).$$

The set E_Λ is an orthonormal basis for $L^2(\Omega)$ if and only if for each $f \in L^2(\Omega)$,

$$\|f\|_\Omega^2 = \sum_{\lambda \in \Lambda} |\langle e_\lambda, f \rangle_\Omega|^2,$$

and, by the completeness of the exponentials in L^2 of a large cube containing Ω, it is necessary and sufficient that

$$\sum_{\lambda \in \Lambda} \left|\widehat{\mathbf{1}_\Omega}(x - \lambda)\right|^2 = 1 \tag{4}$$

for each $x \in \mathbb{R}^d$. In other words a necessary and sufficient condition for (Ω, Λ) to be a spectral pair is that $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling at level 1. Notice also that $\left|\widehat{\mathbf{1}_\Omega}\right|^2$ is the Fourier Transform of $\mathbf{1}_\Omega * \widetilde{\mathbf{1}_\Omega}$ which has support equal to the set $\overline{\Omega - \Omega}$. We use the notation $\widetilde{f}(x) = \overline{f(-x)}$.

Proof of Theorem 1. Write $K = \Omega - \Omega$, which is a symmetric, open convex set. Assume that (Ω, Λ) is a spectral pair. We can clearly assume that $0 \in \Lambda$. It follows that $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling and hence that Λ has uniformly bounded density, has density equal to 1 and $\widehat{\delta_\Lambda}(\{0\}) = 1$.

By Theorem 2 (with $f = \left|\widehat{\mathbf{1}_\Omega}\right|^2$, $\widehat{f} = \mathbf{1}_\Omega * \widetilde{\mathbf{1}_\Omega}(-x)$) it follows that

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup K^c.$$

Let $H = K/2$ and write

$$f(x) = \mathbf{1}_H * \widetilde{\mathbf{1}_H}(x) = \int_{\mathbb{R}^d} \mathbf{1}_H(y) \mathbf{1}_H(y - x)\, dy.$$

The function f is supported in $\overline{K}$ and has nonnegative Fourier Transform

$$\widehat{f} = \left|\widehat{\mathbf{1}_H}\right|^2.$$

We have

$$\int_{\mathbb{R}^d} \widehat{f} = f(0) = \operatorname{vol} H$$

and

$$\widehat{f}(0) = \int_{\mathbb{R}^d} f = (\mathrm{vol}\, H)^2.$$

By the Brunn-Minkowski inequality (for example, see [G94, Ch. 3]), for any convex body Ω,

$$\mathrm{vol}\, \frac{1}{2}(\Omega - \Omega) \geq \mathrm{vol}\, \Omega,$$

with equality only in the case of symmetric Ω. Since Ω has been assumed to be non-symmetric it follows that

$$\mathrm{vol}\, H > 1.$$

For

$$1 > \rho > \left(\frac{1}{\mathrm{vol}\, H}\right)^{1/d}$$

consider

$$g(x) = f(x/\rho)$$

which is supported properly inside K, and has

$$g(0) = f(0) = \mathrm{vol}\, H, \quad \int_{\mathbb{R}^d} g = \rho^d \int_{\mathbb{R}^d} f = \rho^d (\mathrm{vol}\, H)^2.$$

Since $\mathrm{supp}\, g$ is properly contained in K, Theorem 3 implies that $\widehat{g} + \Lambda$ is a tiling at level $\int \widehat{g} \cdot \mathrm{dens}\, \Lambda = \int \widehat{g} = g(0) = \mathrm{vol}\, H$. However, the value of $\widehat{g}$ at 0 is $\int g = \rho^d (\mathrm{vol}\, H)^2 > \mathrm{vol}\, H$, and, since $\widehat{g} \geq 0$ and $\widehat{g}$ is continuous, this is a contradiction. □

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Department of Mathematics, 1409 W. Green St, University of Illinois, Urbana IL 61801

kolount@math.uiuc.edu

Current address: Department of Mathematics, University of Crete, 714 09 Iraklio, Greece
kolount@math.uch.gr