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ON GRADED K-THEORY, ELLIPTIC OPERATORS AND THE FUNCTIONAL CALCULUS

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ABSTRACT. Let A be a graded C^*-algebra. We characterize Kasparov's K-theory group $\hat{K}_0(A)$ in terms of graded $*$-homomorphisms by proving a general converse to the functional calculus theorem for self-adjoint regular operators on graded Hilbert modules. An application to the index theory of elliptic differential operators on smooth closed manifolds and asymptotic morphisms is discussed.

1. Introduction

Let A be a graded σ-unital C^*-algebra with grading automorphism α. We characterize Kasparov's K-group in the category of graded C^*-algebras, $\hat{K}_0(A) = KK(\mathbb{C}, A)$, as the group of graded homotopy classes of graded $*$-homomorphisms from $C_0(\mathbb{R})$, the C^*-algebra of continuous functions on the real line with the even-odd function grading, to the graded tensor product $A\hat{\otimes}\mathcal{K}$, where $\mathcal{K} \cong M_2(\mathcal{K})$ is the C^*-algebra of compact operators graded into diagonal and off-diagonal matrices. Addition is given by direct sum.

The isomorphism is established in Section 3 by proving a general converse to the functional calculus theorem [11] for self-adjoint regular operators on graded Hilbert modules in Section 2. We will indicate in Section 4 how this characterization is useful in simplifying calculations with asymptotic morphisms of C^*-algebras and elliptic differential operators D with coefficients in a trivially graded C^*-algebra A over a smooth closed manifold M. The functional calculus will give an explicit formulation as (nontrivial) compatible graded $*$-homomorphisms of the generalized Fredholm index $\mathrm{Index}_A(D) \in K_0(A)$ and the symbol class $[\sigma(D)] \in K_A^0(T^*M)$ (the topological K-theory of vector A-bundles of the cotangent bundle T^*M) in a form which is suitable for composing directly with asymptotic morphisms, with no rescaling or suspensions as in the general theory. Since the product in E-theory is given by composition, this approach to index theory is simpler than using the Kasparov product in KK-theory [10], which can be very technical.

We should note that Kasparov's graded K-theory is unrelated to van Daele's version, except when A is trivially graded [19]. This paper represents work that partially began in the author's thesis [17], although the material in Section 2 is new. The author would like to thank his advisers Nigel Higson and Paul Baum for their invaluable help and encouragement and also Erik Guentner for helpful suggestions.

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2. Graded C^*-algebras and Hilbert modules

In this section we collect some definitions and results on graded C^*-algebras and Hilbert modules and fix notation. For a complete discussion, see the books [3], [9].

Let A be a C^*-algebra. Recall that A is *graded* if there is a $*$-automorphism $\alpha\colon A \to A$ such that $\alpha^2 = \mathrm{id}_A$. Equivalently, there is a decomposition of A as a direct sum $A = A_0 \oplus A_1$, where A_0 and A_1 are self-adjoint closed linear subspaces with the property that if $x \in A_m$ and $y \in A_n$ then $xy \in A_{m+n}$ (mod 2). In fact, $A_n = \{x \in A\colon \alpha(x) = (-1)^n x\}$. We write $\partial x = n$ if $x \in A_n$. If there is a self-adjoint unitary ϵ (called the *grading operator*) in the multiplier algebra $M(A)$ such that $\alpha(x) = \epsilon x \epsilon^*$, then A is said to be *evenly graded*. A $*$-homomorphism $\phi\colon A \to B$ of graded C^*-algebras is *graded* if $\phi(A_n) \subset B_n$ for $n = 0, 1$.

Example 2.1 The following are the main examples we will be concerned with.

(a) Every C^*-algebra A can be *trivially* graded by setting $A_0 = A$ and $A_1 = \{0\}$. This is an even grading with grading operator $\epsilon = 1$. The complex numbers $\mathbb{C}$ are always assumed to be trivially graded.
(b) The C^*-algebra $C_0(\mathbb{R})$ of continuous complex-valued functions on $\mathbb{R}$ vanishing at infinity is graded into the even and odd functions by defining $\alpha(f)(t) = f(-t)$ for all functions $f \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$.
(c) Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. By choosing an isomorphism $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ we obtain the standard even grading on the C^*-algebra of compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H}) \cong M_2(\mathcal{K})$, with grading operator

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is determined uniquely up to conjugation by a unitary homotopic to the identity.

Let A and B be graded C^*-algebras. Define a graded product and graded involution on the vector space tensor product $A \odot B$ by the formulas

$$\begin{aligned}(a\hat{\otimes}b)(a'\hat{\otimes}b') &= (-1)^{\partial b \partial a'}(aa'\hat{\otimes}bb') \\ (a\hat{\otimes}b)^* &= (-1)^{\partial a \partial b}(a^*\hat{\otimes}b^*).\end{aligned}$$

The resulting $*$-algebra is denoted by $A\hat{\odot}B$. A grading on $A\hat{\odot}B$ is defined by setting

$$\partial(a\hat{\otimes}b) = \partial a + \partial b \pmod 2.$$

Now faithfully represent A and B by ρ_1 and ρ_2 on graded Hilbert spaces H_1 and H_2 with grading operators ϵ_1 and ϵ_2, respectively. Then $A\hat{\odot}B$ is faithfully represented on $H_1 \otimes H_2$ (graded by $\epsilon_1 \otimes \epsilon_2$) via the formula

$$\rho(a\hat{\otimes}b) = \rho_1(a)\epsilon_1^{\partial a} \otimes \rho_2(b).$$

The C^*-algebra completion is denoted by $A\hat{\otimes}B$ and is called the (minimal) *graded* tensor product. It does not depend on the choice of representations. (There is also a maximal graded tensor product [3] but it will not be needed for our purposes since one of the factors will always be nuclear.)

LEMMA 2.2 (Proposition 15.5.1 [3]). *If B is evenly graded, then $A\hat{\otimes}B \cong A\otimes B$. If A is also evenly graded, then under this isomorphism $A \otimes B$ is also evenly graded.*

COROLLARY 2.3. Let $\mathcal{K}$ have the standard even grading. Then $A\hat{\otimes}\mathcal{K} \cong M_2(A \otimes \mathcal{K})$. If A is evenly graded by ϵ, $A\hat{\otimes}\mathcal{K} \cong M_2(A \otimes \mathcal{K})$ with standard even grading given by $\eta = \mathrm{diag}(\epsilon \otimes 1, -\epsilon \otimes 1)$.

Let B be another graded C^*-algebra with grading β. Then $B[0, 1] = C([0, 1], B)$ canonically inherits a grading by the formula $\hat{\beta}(f)(t) = \beta(f(t))$. Two graded $*$-homomorphisms ϕ_0, ϕ_1: $A \to B$ are *graded homotopic* if there is a graded $*$-homomorphism Φ: $A \to B[0, 1]$ such that composition with the evaluation maps ev_t: $B[0, 1] \to B$ for $t = 0, 1$ are equal to ϕ_0 and ϕ_1, respectively. We shall denote by $[\![A, B]\!]$ the set of *graded* homotopy classes of *graded* $*$-homomorphisms from A to B. If ϕ: $A \to B$ is a graded $*$-homomorphism, then we denote by $[\![\phi]\!]$ its equivalence class in $[\![A, B]\!]$.

A Hilbert A-module $\mathcal{H}$ is *graded* if there is a Banach space decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that $\mathcal{H}_n \cdot A_m \subseteq \mathcal{H}_{n+m}$ and $\langle \mathcal{H}_n, \mathcal{H}_m \rangle \subseteq A_{n+m}$ (mod 2). We let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of all bounded A-linear maps T: $\mathcal{H} \to \mathcal{H}$ with an adjoint T^* and let $\mathcal{K}(\mathcal{H})$ denote the closed two-sided ideal of compact operators. The grading on $\mathcal{H}$ induces gradings on $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ via the identities $\partial T = m$ if $T(\mathcal{H}_n) \subset \mathcal{H}_{n+m}$. We let $\mathcal{H}^{\mathrm{op}}$ denote $\mathcal{H}$ with the opposite grading $\mathcal{H}_n^{\mathrm{op}} = \mathcal{H}_{1-n}$. Note that if A is trivially graded, $\mathcal{H}$ is the direct sum of two orthogonal A-modules. If ϕ: $B \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism, a closed submodule $\mathcal{E}$ of $\mathcal{H}$ is *ϕ-invariant* if $\phi(b)$: $\mathcal{E} \to \mathcal{E}$ for all $b \in B$.

3. The converse functional calculus

Let $\mathcal{H}$ be a (graded) Hilbert A-module. A *regular operator* on $\mathcal{H}$ is a densely defined closed A-linear map D: Domain$(D) \to \mathcal{H}$ such that the adjoint D^* is densely defined and $1 + D^*D$ has dense range.[1] D has *degree one* if $\partial(Dx) = \partial x + 1$ for all $x \in$ Domain(D).

PROPOSITION 3.1. *For any graded $*$-homomorphism ϕ: $C_0(\mathbb{R}) \to A$, there is a maximal ϕ-invariant closed graded Hilbert A-submodule A_ϕ of A and a self-adjoint regular operator D on A_ϕ of degree one such that for all $f \in C_0(\mathbb{R})$ we have $\phi(f)|_{A_\phi} = f(D)$.*

[1] If $\mathcal{H} = A$ then D is sometimes called an unbounded multiplier [4], [8], [11].

Proof. Given a graded $*$-homomorphism $\phi\colon C_0(\mathbb{R}) \to A$, define

$$A_\phi = C_0(\mathbb{R})\hat{\otimes}_\phi A = \overline{\phi(C_0(\mathbb{R}))A}$$

to be the closed right ideal generated by the image of ϕ. This is a closed graded Hilbert submodule of A (see Blackadar [3].) Let $C_c(\mathbb{R})$ denote the dense graded ideal of continuous functions on $\mathbb{R}$ with *compact support*. Define

$$\text{Domain}(D) = \phi(C_c(\mathbb{R}))A,$$

a dense graded submodule of A_ϕ. Let d denote the function $d(t) = t$ on $\mathbb{R}$. Define D: Domain$(D) \to A_\phi$ by the formula $D\phi(f)x = \phi(df)x$ where $f \in C_c(\mathbb{R})$ (so $df \in C_c(\mathbb{R})$) and extend linearly. Suppose that $\phi(f)x = \phi(g)y$ for some other $g \in C_c(\mathbb{R})$. Choose a function $d' \in C_c(\mathbb{R})$ such that $d = d'$ on the compact set $\text{supp}(f) \cup \text{supp}(g)$. Then we have

$$D\phi(f)x = \phi(d'f)x = \phi(d')\phi(f)x = \phi(d')\phi(g)y = \phi(d'g)y = D\phi(g)y.$$

It follows that D is well-defined and is clearly A-linear. Also, D is degree one since d is an odd function on $\mathbb{R}$. The computation

$$\begin{aligned}\langle D\phi(f)x, \phi(g)y\rangle &= x^*\phi(df)^*\phi(g)y = x^*\phi(d\bar{f}g)\\ &= x^*\phi(f)^*\phi(dg)y = \langle\phi(f)x, D\phi(g)y\rangle\end{aligned}$$

shows that D is symmetric on Domain(D). This implies that D is closeable, so we replace D by its closure $\bar{D}$. Consequently, $(D \pm i)$ are injective and have closed range by Lemma 9.7 [11]. Let $f \in C_c(\mathbb{R})$. For any $x \in A$ we have

$$(1 + D^2)\phi((1 + d^2)^{-1})\phi(f)x = \phi((1 + d^2)(1 + d^2)^{-1}f)x = \phi(f)x.$$

It follows that Range$(1 + D^2) \supseteq$ Domain(D) is dense and so D is regular. We will show D is self-adjoint by using a Cayley transform argument.

Extend ϕ to $\phi^+\colon C_0(\mathbb{R})^+ \to A^+$ by adjoining a unit. Let $z \in C_0(\mathbb{R})^+$ denote the unitary

$$z(t) = \frac{t-i}{t+i} = 1 - 2ir_-(t) \text{ for } t \in \mathbb{R}$$

where $r_-(t) = (t-i)^{-1}$ denotes the resolvent. Let $U_D = \phi^+(z) = 1-2i\phi(r_-) \in A^+$. It is easy to check that for all $x \in$ Domain(D), the unitary U_D satisfies

$$U_D(D+i)x = (D+i)U_Dx = (D-i)x.$$

By Lemma 9.8 and the discussion following Proposition 10.6 in Lance [11], the closed symmetric regular operator D is self-adjoint and $U_D = (D+i)^{-1}(D-i)$.

To show $\phi(f)|_{A_\phi} = f(D)$, it suffices to show this for the resolvents $r_\pm(t) = (d \pm i)^{-1}(t)$. Let $\{f_n\}_{n=1}^{\infty}$ be an approximate unit for $C_0(\mathbb{R})$ consisting of compactly supported functions. Let $x \in A_\phi$ be given. Then $\phi(f_n)x \in \text{Domain}(D)$ for all n and $\phi(f_n)x \to x$ as $n \to \infty$. As $n \to \infty$,

$$(D \pm i)\phi((d \pm i)^{-1} f_n)x = \phi((d \pm i)(d \pm i)^{-1} f_n)x = \phi(f_n)x \to x.$$

Now since $\phi((d \pm i)^{-1} f_n)x = \phi((d \pm i)^{-1})\phi(f_n)x \to \phi((d \pm i)^{-1})x$ as $n \to \infty$ and $(D \pm i)$ is closed, we conclude that $\phi((d \pm i)^{-1})x = (D \pm i)^{-1}x$. Since $x \in A_\phi$ was arbitrary, we are done. □

Let B be a C^*-algebra. If $\mathcal{H}$ is a Hilbert B-module, a $*$-homomorphism $\phi\colon A \to \mathcal{L}(\mathcal{H})$ is called *nondegenerate* if $\phi(A)\mathcal{H}$ is dense in $\mathcal{H}$. It is called *strict* if $\{\phi(u_n)\}$ is Cauchy in the strict topology of $\mathcal{L}(\mathcal{H})$ for some approximate unit $\{u_n\}$ in A. Nondegeneracy implies strictness [11]. The following result may be considered the converse to the functional calculus for self-adjoint regular operators [2], [4], [11].

THEOREM 3.2 (Converse Functional Calculus). *Let $\phi\colon C_0(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be graded. There is a closed graded ϕ-invariant Hilbert submodule $\mathcal{H}_\phi$ of $\mathcal{H}$ and a self-adjoint regular operator D on $\mathcal{H}_\phi$ of degree one such that for all $f \in C_0(\mathbb{R})$ we have $\phi(f)x = f(D)x$ for all $x \in \mathcal{H}_\phi$. Moreover, if ϕ is strict then $\mathcal{H}_\phi$ is complemented and $\phi(f) = f(D) \in \mathcal{L}(\mathcal{H}_\phi) \subseteq \mathcal{L}(\mathcal{H})$. If ϕ is nondegenerate then $\mathcal{H} = \mathcal{H}_\phi$. And if $\phi(C_0(\mathbb{R})) \subset \mathcal{K}(\mathcal{H})$ then D has compact resolvents.*

Proof. Let $A = \mathcal{L}(\mathcal{H})$. Let $D'\colon \text{Domain}(D') \to A_\phi$ be the self-adjoint regular operator on $A_\phi = C_0(\mathbb{R})\hat{\otimes}_\phi A$ from the previous proposition such that $\phi(f) = f(D')$. Let $i\colon A \to \mathcal{L}(\mathcal{H})$ be the identity. Define $\mathcal{H}_\phi = \overline{\phi(C_0(\mathbb{R})\mathcal{H}} = A_\phi \hat{\otimes}_i \mathcal{H}$ which is a closed Hilbert submodule of $\mathcal{H}$. Define $D = D'\hat{\otimes}_i 1$ on

$$\text{Domain}(D) = \text{Domain}(D')\hat{\odot}_i \mathcal{H} \supseteq \phi(C_c(\mathbb{R}))\mathcal{H}.$$

By Proposition 10.7 [11], D extends to a self-adjoint regular operator on $\mathcal{H}_\phi$. ($D = i_*(D')$ in the notation of [11].) If $x \in \mathcal{H}_\phi$, we compute

$$f(D)x = f(D'\hat{\otimes}_i 1)x = (f(D')\hat{\otimes}_i 1)x = f(D')\hat{\otimes}_i x = \phi(f)x.$$

If ϕ is strict then $\mathcal{H}_\phi$ is a complemented submodule of $\mathcal{H}$ by Proposition 5.8 [11] and so $\mathcal{L}(\mathcal{H}_\phi)$ is included as a graded subalgebra of $\mathcal{L}(\mathcal{H})$. The result now easily follows. □

Note that if ϕ is the zero homomorphism then $\mathcal{H}_\phi = \{0\}$ and $D = 0$, so $f(D) = 0 = \phi(f)$.

4. Graded K-theory

Standing assumptions. Throughout this section, A will denote a complex σ-unital graded C^*-algebra and $C_0(\mathbb{R})$ and $\mathcal{K}$ will have the gradings as in Example 2.1.

Let H_A denote the Hilbert A-module of all sequences $\{a_n\}_1^\infty \subset A$ such that $\{\sum_1^n a_k^* a_k\}_1^\infty$ converges in A. It has a natural grading into sequences of even and odd elements. Let $\hat{H}_A = H_A \oplus H_A^{\text{op}}$, where H_A^{op} denotes H_A with the *opposite* grading. This is the standard graded Hilbert module for A. We have the following very important result of Kasparov in the theory of graded Hilbert modules.

STABILIZATION THEOREM (Kasparov [10]). *If $\mathcal{H}$ is a countably generated graded Hilbert A-module then $\mathcal{H} \oplus \hat{H}_A \cong \hat{H}_A$.*

It is a standard result that $A\hat{\otimes}\mathcal{K}$ is graded $*$-isomorphic to $\mathcal{K}(\hat{H}_A)$, the C^*-algebra of compact operators on $\hat{H}_A$ (with the induced grading) (see 14.7.1 [3]). For the remainder of this section, we will identify $A\hat{\otimes}\mathcal{K}$ with $\mathcal{K}(\hat{H}_A)$. From stabilization, conjugation by the graded isomorphism $\hat{H}_A \cong \hat{H}_A \oplus \hat{H}_A$ determines a unitary in $\mathcal{L}(\hat{H}_A) = M(A\hat{\otimes}\mathcal{K})$ of degree zero.

LEMMA 4.1. *Let $u \in M(A\hat{\otimes}\mathcal{K})$ be a unitary of degree zero. There is a strictly continuous path of degree zero unitaries $\{U_t\}_{t\in[0,1]} \subset M(A\hat{\otimes}\mathcal{K})$ such that $U_1 = u$ and $U_0 = 1$.*

Proof. Write $\mathcal{K} = \mathcal{K}(H \oplus H)$ where $H = L^2[0,1]$. Then $M(A\hat{\otimes}\mathcal{K})$ contains a copy of $\mathcal{L}(H \oplus H)$. Let $\{v_t\}$ be a strictly continuous path of isometries in $\mathcal{L}(H)$ with $p_t = v_t v_t^* \to 0$ strongly as $t \to 0$ as in Proposition 12.2.2 [3]. Set $V_t = v_t \oplus v_t \in \mathcal{L}(H \oplus H)$ and note that each V_t has degree zero. Set $W_t = 1\hat{\otimes}V_t$ which also has degree zero and let

$$U_t = W_t u W_t^* + (1 - W_t W_t^*)$$

for $t > 0$ and $U_0 = 1$. It is easy to check that this works. □

Definition 4.2. Let A have grading automorphism α. Define

$$K'(A) = K'(A, \alpha) = [\![C_0(\mathbb{R}), A\hat{\otimes}\mathcal{K}]\!].$$

Define a binary operation on $K'(A)$ by direct sum $[\![\phi]\!] + [\![\psi]\!] = [\![\phi \oplus \psi]\!]$, where the direct sum is with respect to the graded isomorphism $\hat{H}_A \cong \hat{H}_A \oplus \hat{H}_A$

THEOREM 4.3. *$K'(A)$ is an abelian group under the direct sum operation and satisfies the relation*

$$-[\![\phi]\!] = [\![u\phi u^*]\!]$$

where $u = u^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\hat{H}_A = H_A \oplus H_A$.

Proof. It follows from Lemma 4.1 and the proof of Theorem 3.1 in Rosenberg [15] carried over to the graded case that $K'(A)$ is an abelian monoid with zero given by the zero (or any null-homotopic) $*$-homomorphism. We only need to show inverses.

Let ϕ: $C_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A)$ be graded. Let D be the regular operator on $\mathcal{H}_\phi \subset \hat{H}_A$ associated to ϕ from the converse functional calculus. Via stabilization $\mathcal{H}_\phi \oplus \hat{H}_A \cong \hat{H}_A$ and Lemma 4.1, we may assume (up to graded homotopy) that ϕ is strict by Proposition 5.8 [11]. Thus $\phi(f) = f(D)$ for all $f \in C_0(\mathbb{R})$. Then $D^{\mathrm{op}} = uDu^*$ on the Hilbert module $\mathcal{H}_\phi$ is the operator associated to $[\![u\phi u^*]\!]$ since by the functional calculus

$$f(D^{\mathrm{op}}) = f(uDu^*) = uf(D)u^* = u\phi(f)u^*.$$

Let ϵ be the grading on $\hat{H}_A$. For each $t \geq 0$, define

$$\mathbb{D}_t = \begin{pmatrix} D & t\epsilon \\ t\epsilon & D^{\mathrm{op}} \end{pmatrix}$$

on $\mathcal{H}_\phi \oplus \mathcal{H}_\phi^{\mathrm{op}} \subseteq \hat{H}_A$ and let $\mathbb{D}_t = 0$ on the complement. Define Φ_t: $C_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A)$ by

$$\Phi_t(f) = f(\mathbb{D}_t).$$

For $t = 0$ we have $\Phi_0(f) = f(\mathbb{D}_0) = \phi \oplus u\phi u^*$. Note that

$$\mathbb{D}_t^2 = \begin{pmatrix} D & t\epsilon \\ t\epsilon & D^{\mathrm{op}} \end{pmatrix}^2 = \begin{pmatrix} D^2 + t^2 & 0 \\ 0 & D^{\mathrm{op}2} + t^2 \end{pmatrix}$$

and so the spectrum of $\mathbb{D}_t$ is contained outside the interval $(-t, t)$. Therefore,

$$\|f(\mathbb{D}_t)\| \leq \sup\{|f(x)|: x \in \mathrm{spec}(\mathbb{D}_t)\} \to 0 \text{ as } t \to \infty$$

for all $f \in C_0(\mathbb{R})$ and the result follows. □

Definition 4.4. A *K-cycle* for a graded C^*-algebra A is an ordered pair $(\mathcal{H}, T)$, such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a countably generated graded Hilbert A-module and $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ is the graded C^*-algebra of all bounded A-linear operators on $\mathcal{H}$ with adjoint, which satisfies the following conditions:

(i) T is of degree one;
(ii) $T - T^* \in \mathcal{K}(\mathcal{H})$ is compact;
(iii) $T^2 - 1 \in \mathcal{K}(\mathcal{H})$ is compact.

The K-cycle is called *degenerate* if $T^2 = 1$.

By a standard argument we may assume that $T = T^*$ is self-adjoint. There is an obvious notion of *unitary equivalence* for two K-cycles [3], [10]. Two K-cycles $(\mathcal{H}_0, T_0)$ and $(\mathcal{H}_1, T_1)$ are *homotopic* if there is a K-cycle $(\mathcal{H}, T)$ for $A[0,1]$ such that $(\mathcal{H}\hat{\otimes}_{\mathrm{ev}_i} A, T\hat{\otimes}_{\mathrm{ev}_i} 1)$ are unitarily equivalent to $(\mathcal{H}_i, T_i)$ where ev_t: $A[0,1] \to A$ are the evaluation maps. A collection $\{(\mathcal{H}, T_t)\}_{t\in[0,1]}$ of K-cycles for A is called an *operator homotopy* if $t \mapsto T_t$ is norm continuous in t. An operator homotopy induces a homotopy $(\mathcal{H}', T)$ by defining $\mathcal{H}' = C([0,1], \mathcal{H})$ and $T(f)(t) = T_t(f(t))$ for $f\colon [0,1] \to \mathcal{H}$.

PROPOSITION 4.5 (Theorem 4.1 [10]). *The set $KK(\mathbb{C}, A)$ of all equivalence classes of K-cycles for A under the equivalence relation (generated by) homotopy is an abelian group under the relations*

$$\begin{aligned}(\mathcal{H}_1, T_1) + (\mathcal{H}_2, T_2) &= (\mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus D_2),\\ -(\mathcal{H}, T) &= (\mathcal{H}^{\mathrm{op}}, -T).\end{aligned}$$

The class of any degenerate K-cycle is zero in $KK(\mathbb{C}, A)$.

Let $u = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ be the degree one unitary with respect to the grading on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

LEMMA 4.6. $-(\mathcal{H}, T) = (\mathcal{H}, T^{\mathrm{op}}) \in KK(\mathbb{C}, A)$, *where* $T^{\mathrm{op}} = uTu^*$.

Proof. In the complex world, $(\mathcal{H}, T) = (\mathcal{H}, -T)$ since they are operator homotopic (but not through self-adjoint K-cycles in general.) It follows that

$$-(\mathcal{H}, T) = (\mathcal{H}^{\mathrm{op}}, -T) = (\mathcal{H}^{\mathrm{op}}, T) = (\mathcal{H}, uTu^*) = (\mathcal{H}, T^{\mathrm{op}})$$

since $u\colon \mathcal{H}^{\mathrm{op}} \to \mathcal{H}$ implements a unitary equivalence. □

THEOREM 4.7. *$K'(A)$ is isomorphic to $KK(\mathbb{C}, A)$.*

Proof. Let $G(t) = t(t^2+1)^{-1/2}$ which defines a degree one, self-adjoint element in $C_b(\mathbb{R}) = M(C_0(\mathbb{R}))$, the continuous *bounded* functions on $\mathbb{R}$. Define a map $K'(A) \to KK(\mathbb{C}, A)$ via

$$[\![\phi]\!] \mapsto (\mathcal{H}_\phi, G(D))$$

where D is the regular operator associated to $\phi\colon C_0(\mathbb{R}) \to \mathcal{K}(\mathcal{H}_\phi) \subset \mathcal{K}(\hat{H}_A)$ via the converse functional calculus. (As in Theorem 4.3, we may assume that ϕ is strict.) The operator $G(D)$ is a degree one, self-adjoint element of $M(\mathcal{K}(\hat{H}_A)) = \mathcal{L}(\hat{H}_A)$ and $G(D)^2 - 1$ is compact since

$$G(D)^2 - 1 = (D^2+1)^{-1} = \phi(G) \in \mathcal{K}(\mathcal{H}_\phi).$$

This map is easily seen to be well-defined since applying the construction to a graded homotopy $\Phi\colon C_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A)[0,1]$ yields a homotopy of K-cycles by using the graded isomorphism

$$\mathcal{K}(\hat{H}_A)[0,1] \cong (A \hat{\otimes} \mathcal{K})[0,1] \cong A[0,1] \hat{\otimes} \mathcal{K} \cong \mathcal{K}(\hat{H}_{A[0,1]}).$$

It is also distributes over direct sums and maps

$$-[\![\phi]\!] = [\![u\phi u^*]\!] \mapsto (\mathcal{H}_\phi, G(D)^{\mathrm{op}}) = -(\mathcal{H}_\phi, G(D))$$

via properties of the functional calculus and Lemma 4.6.

The reverse map is defined using the techniques of Baaj and Julg [2]. Let $(\mathcal{H}, F)$ be a K-cycle for A. We may assume that $F = F^*$ and $\mathcal{H} = \hat{H}_A$. Let $T > 0$ be a strictly positive element of $\mathcal{K}(\hat{H}_A)$ of degree zero which commutes with F. Any two such operators are operator homotopic via the straight line homotopy. Let $D = FT^{-1}$. Note that Domain(D) = Range(T) is a dense submodule of $\hat{H}_A$. One has that $D = D^*$ and $(D^2+1)^{-1} = T^2(F^2+T^2)^{-1}$ is compact. We have the identity $G(D) = F(F^2 + T^2)^{-1/2}$ and so it also follows that $(\hat{H}_A, F)$ and $(\hat{H}_A, G(D))$ are operator homotopic. It follows from the identity

$$(D \pm i)^{-1} = D(D^2 + 1)^{-1} \mp i(D^2 + 1)^{-1}$$

that the resolvents are also compact. Define

$$KK(\mathbb{C}, A) \to K'(A)$$

by sending $(\hat{H}_A, F)$ to the graded homotopy class of the graded $*$-homomorphism

$$\phi\colon\ f \mapsto f(D) \in \mathcal{K}(\hat{H}_A).$$

As above, $\mathcal{K}(\hat{H}_{A[0,1]}) \cong \mathcal{K}(\hat{H}_A)[0, 1]$, so a homotopy $(\hat{H}_{A[0,1]}, F)$ is mapped to a homotopy $\Phi\colon C_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A)[0, 1]$. Thus the reverse map is well-defined. One checks easily that these two maps are inverses of each other. □

If A is trivially graded and unital then $A\hat{\otimes}\mathcal{K} \cong M_2(A\otimes\mathcal{K})$ with even grading given by $\epsilon = \mathrm{diag}(1, -1)$. That is, $M_2(A \otimes \mathcal{K})$ is graded into diagonal and off-diagonal matrices. It follows from the above that

$$K'(A) = [\![C_0(\mathbb{R}), A \otimes \mathcal{K}]\!] \cong K_0(A).$$

We will describe the isomorphism directly via the more familiar language of projections. It is a standard result that $K_0(A)$ is the group of formal differences of homotopy classes of projections $p = p^* = p^2 \in A \otimes \mathcal{K}$ with addition given by direct sum $[p] + [q] = [p' + q']$ where $p \sim_h p', q \sim_h q'$ and $p' \perp q'$. Let $u \in M_2(M(A \otimes \mathcal{K}))$ be the degree one unitary

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recall that for any self-adjoint involution w (i.e., $w^* = w, w^2 = 1$) there is an associated projection $p(w) = \frac{1}{2}(w + 1)$.

Let $x = [p] - [q] \in K_0(A)$ where p and q are projections in $A \otimes \mathcal{K}$. Define a map

$$\phi_x\colon C_0(\mathbb{R}) \to M_2(A \otimes \mathcal{K})$$

by the formula

$$\phi_x(f) = \begin{pmatrix} f(0)p & 0 \\ 0 & f(0)q \end{pmatrix}, \quad f \in C_0(\mathbb{R}).$$

This defines a $*$-homomorphism since $p = p^2 = p^*$ (similarly for q) and is graded since $f(0) = 0$ for any odd function. Note that the homotopy class of ϕ_x depends only on the homotopy classes of p and q. Now we define a map $\mu\colon K_0(A) \to K'(A)$ by mapping

$$x \longmapsto [\![\phi_x]\!].$$

It also follows that

$$\begin{aligned} \phi_{[p]}(f) \oplus \phi_{[q]}(f) &= \begin{pmatrix} f(0)\,\mathrm{diag}(p,q) & \mathrm{diag}(0,0) \\ \mathrm{diag}(0,0) & \mathrm{diag}(0,0) \end{pmatrix} \sim_h \begin{pmatrix} f(0)(p'+q') & 0 \\ 0 & 0 \end{pmatrix} \\ &= \phi_{[p'+q']}(f) \end{aligned}$$

and so it is additive. For $x = [p] - [q]$, $-x = [q] - [p]$ maps to

$$\phi_{-x}(f) = \begin{pmatrix} f(0)q & 0 \\ 0 & f(0)p \end{pmatrix} = u\phi_x(f)u^*.$$

Thus, $\mu(-x) = [\![u\phi_x u^*]\!] = -[\![\phi_x]\!] = -\mu(x)$. One should note that with the grading present ϕ_x and ϕ_{-x} are *not* homotopic through *graded* $*$-homomorphisms since u has degree one and the identity has degree zero.

Conversely, given $[\![\phi]\!] \in K'(A)$, extend ϕ to a graded $*$-homomorphism

$$\phi^+\colon C_0(\mathbb{R})^+ \to (A \otimes \mathcal{K})^+$$

by adjoining a unit. Let z denote the unitary given by the "Cayley transform"

$$z(t) = \frac{t+i}{t-i} = 1 + 2ir_-(t)$$

where $r_-(t) = (t-i)^{-1}$ is the resolvent function. Let u_ϕ denote the unitary

$$u_\phi = \phi^+(z) = 1 + 2i\phi(r_-) \in (A \otimes \mathcal{K})^+$$

A simple computation shows that $(\epsilon u_\phi)^2 = 1$ and $(\epsilon u_\phi)^* = \epsilon u_\phi$. We also have $\epsilon^* = \epsilon$ and $\epsilon^2 = 1$. Consider the associated projections

$$p(\epsilon),\ p(\epsilon u_\phi) \in (A \otimes \mathcal{K})^+.$$

By the definition of u_ϕ above, we see that $p(\epsilon) - p(\epsilon u_\phi) = 2i\phi(r_+) \in A \otimes \mathcal{K}$. Also, a homotopy of ϕ induces a homotopy of the unitary u_ϕ and thus of $p(\epsilon u_\phi)$. We define ν: $K'(A) \to K_0(A)$ by

$$\nu(\llbracket\phi\rrbracket) = [p(\epsilon)] - [p(\epsilon u_\phi)] \in K_0(A).$$

A simple computation shows that $\nu \circ \mu = 1$. We only need to show μ is onto. It then follows that $\nu = \mu^{-1}$ is a homomorphism.

Since A is trivially graded, $\hat{H}_A = H_A \oplus H_A$ with each factor determining the grading. Again identify $A \hat{\otimes} \mathcal{K}$ with $\mathcal{K}(\hat{H}_A)$. Let $\llbracket\phi\rrbracket \in K'(A)$. Up to graded homotopy we may assume that ϕ: $C_0(\mathbb{R}) \to \mathcal{K}(\hat{H}_A)$ is strict (via stabilization). Let

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix}$$

on $\hat{H}_A$ be the self-adjoint regular operator of degree one with compact resolvents from the converse functional calculus such that $\phi(f) = f(D)$. Let $G(D) = D(D^2+1)^{-\frac{1}{2}}$ which is a self-adjoint bounded operator of degree one on $\hat{H}_A$ with $G(D)^2 - 1$ compact. By a graded homotopy, we may assume that $\phi(f) = (f \circ G)(D) = f(G(D))$. (Note that the diffeomorphism G: $\mathbb{R} \to (-1, 1)$ is the homotopy inverse to the inclusion $(-1, 1) \subset \mathbb{R}$.) Thus, we can write

$$G(D) = \begin{pmatrix} 0 & G_+^* \\ G_+ & 0 \end{pmatrix}$$

on $H_A \oplus H_A$ where G_+: $H_A \to H_A$ is a generalized Fredholm operator [18]. Up to a compact perturbation of G_+ (which would induce a graded homotopy), we may assume that $\mathrm{Ker}(G(D)) = \mathrm{Ker}(G_+) \oplus \mathrm{Ker}(G_+^*)$ is a finite projective A-module in $\hat{H}_A$, and is thus complemented. Note that for $x \in \mathrm{Ker}(G(D))$ we have $f(G(D))x = f(0)x$. Since A is trivially graded, $\mathrm{Ker}(G_+)$ and $\mathrm{Ker}(G_+^*)$ are finite projective A-modules. Let $P_+^{(*)} \in \mathcal{K}(H_A)$ be the compact projections onto $\mathrm{Ker}(G_+^{(*)})$. Let $x = [P_+] - [P_+^*] = \mathrm{Index}_A(G_+) \in K_0(A)$ [18]. A graded homotopy connecting ϕ to the graded $*$-homomorphism

$$\phi_x(f) = \begin{pmatrix} f(0)P+ & 0 \\ 0 & f(0)P_+^* \end{pmatrix} \in \mathcal{K}(H_A \oplus H_A) = \mathcal{K}(\hat{H}_A)$$

is given by

$$\Phi_t(f) = \begin{cases} f(t^{-1}G(D)), & t > 0, \\ \phi_x(f), & t = 0. \end{cases}$$

Thus, $\mu(x) = \llbracket\phi\rrbracket$ and so μ is onto as was desired.

COROLLARY 4.8. *If A is unital and trivially graded then the maps μ and ν are inverses.*

5. Elliptic operators over C^*-algebras

In this section, we will show how the previous results and the functional calculus give explicit realizations as graded $*$-homomorphisms of the K-theory symbol class and Fredholm index of an elliptic differential operator with coefficients in a trivially graded C^*-algebra.

Let A be a trivially graded *unital* C^*-algebra and M a smooth closed Riemannian manifold. Let $E \to M$ and $F \to M$ be smooth vector A-bundles, that is, smooth locally trivial fiber bundles on M whose fibers E_p and F_p are finite projective A-modules for each $p \in M$. Let $C^\infty(E)$ denote the vector space of smooth sections of E, which is a module over A, and similarly for $C^\infty(F)$. Let $D\colon C^\infty(E) \to C^\infty(F)$ be an elliptic differential A-operator of order n on M [13], [17]. (If $A = \mathbb{C}$ then D is an ordinary differential operator.) Let $\sigma = \sigma(D)\colon \pi^*(E) \to \pi^*(F)$ denote the principal symbol of D which is a homomorphism of vector A-bundles, where $\pi\colon T^*M \to M$ is the cotangent bundle. The condition of ellipticity is the requirement that for each non-zero cotangent vector $\xi \neq 0 \in T_p^*M$ the principal symbol $\sigma_\xi(D)\colon E_p \to F_p$ is an isomorphism of A-modules.

Equipping the fibers E_p (and F_p) with smoothly varying Hilbert A-module structures

$$\langle\cdot,\cdot\rangle_p\colon E_p \times E_p \to A$$

defines a pre-Hilbert A-module structure on $C^\infty(E)$ via the formula

$$\langle s, s'\rangle = \int_M \langle s(p), s'(p)\rangle_p \,\mathrm{dvol}_M \in A$$

for $s, s' \in C^\infty(E)$, where dvol_M is the Riemannian volume measure on M. (And any two such structures are homotopic via the straight line homotopy.) It follows that an adjoint differential operator $D^t\colon C^\infty(F) \to C^\infty(E)$ exists and is of the same order as D. The principal symbol of the adjoint is the adjoint of the principal symbol $\sigma_\xi(D^t) = \sigma_\xi^*(D) \in \mathcal{L}(F_p, E_p)$ for $\xi \in T_p^*M$. Consider the formally self-adjoint differential A-operator of degree one

$$\mathbb{D} = \begin{pmatrix} 0 & D^t \\ D & 0 \end{pmatrix}\colon C^\infty(E) \oplus C^\infty(F) \to C^\infty(E) \oplus C^\infty(F)$$

on the graded pre-Hilbert A-module $C^\infty(E) \oplus C^\infty(F)$. The principal symbol of $\mathbb{D}$ is the self-adjoint bundle morphism of degree one

$$\sigma = \sigma(\mathbb{D}) = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}\colon \pi^*(E) \oplus \pi^*(F) \to \pi^*(E) \oplus \pi^*(F)$$

on the graded pull-back vector A-bundle $\pi^*(E) \oplus \pi^*(F)$.

LEMMA 5.1. *The resolvents $(\sigma \pm i)^{-1}\colon \pi^*(E) \oplus \pi^*(F) \to \pi^*(E) \oplus \pi^*(F)$ are vector A-bundle morphisms which vanish at infinity on T^*M in the operator norm induced by the Hilbert A-module structures on the fibers $E_p \oplus F_p$.*

Proof. This follows from homogeneity $\sigma(p, t\xi) = t^n\sigma(p, \xi)$ and ellipticity. □

Form the Cayley transform [14]

$$\mathbf{u} = (\sigma + i)(\sigma - i)^{-1} = 1 + 2i(\sigma - i)^{-1}.$$

By complementing the vector A-bundles E and F , e.g. $E \oplus G \cong M \times A^n$, we may embed $\pi^*(E \oplus F)$ in a trivial A-bundle

$$\mathbb{A} = T^*M \times (A^n \oplus A^n).$$

Now extend the automorphism $\mathbf{u}$ to the A-bundle $\mathbb{A}$ by defining it to be equal to the identity on the complement of $\pi^*E \oplus \pi^*F$ in $\mathbb{A}$. From the lemma above, it follows that $\mathbf{u}$ extends continuously to the trivial A-bundle on the one-point compactification $(T^*M)^+$ by setting $\mathbf{u}(\infty) = I$.

Let $\epsilon = \mathrm{diag}(1, -1)$ be the grading of the trivial A-bundle $(T^*M)^+ \times (A^n \oplus A^n)$. Since $\epsilon\sigma = -\sigma\epsilon$ it follows, as in the previous section, that $(\mathbf{u}\epsilon)^2 = 1$ and $(\mathbf{u}\epsilon)^* = \mathbf{u}\epsilon$. (Obviously we also have $\epsilon^* = \epsilon$ and $\epsilon^2 = 1$.)

Therefore, we obtain two projection-valued sections

$$p(\epsilon),\ p(\mathbf{u}\epsilon)\colon (T^*M)^+ \to \mathrm{End}(\mathbb{A})$$

on $(T^*M)^+$ which are equal at infinity. We can view them as projection-valued functions $(T^*M)^+ \to M_2(M_n(A)) \cong M_{2n}(A)$. Both define elements in $K_0(C(T^*M^+) \otimes A)$ and so their difference defines an element

$$\Sigma(D) = [p(\epsilon)] - [p(\epsilon\mathbf{u})] \in K_0(C_0(T^*M) \otimes A).$$

This is the *symbol class* of the elliptic A-operator D as constructed in [7], [14], [17]. By Corollary 4.8 and stability, it follows that

$$K_0(C_0(T^*M) \otimes A) \cong [\![C_0(\mathbb{R}), C_0(T^*M) \otimes M_{2n}(A))]\!]$$

and $\Sigma(D)$ is identified with the graded homotopy class of the graded $*$-homomorphism

$$\Phi_\sigma\colon C_0(\mathbb{R}) \to C_0(T^*M, M_{2n}(A)) \cong M_{2n}(C_0(T^*M) \otimes A)$$

given fiber-wise by the ordinary matrix functional calculus

$$\Phi_\sigma(f)(\xi) = f(\sigma_\xi(\mathbb{D})) \in M_{2n}(A)), \text{ for } \xi \in T^*M.$$

The principal symbol $\sigma(D)\colon \pi^*(E) \to \pi^*(F)$ determines a class $[\sigma(D)] \in K_A^0(T^*M)$ (the topological K-theory of T^*M defined via vector A-bundles) since it is a bundle morphism that is an isomorphism off the compact zero-section $M \subset T^*M$. By the Mingo-Serre-Swan Theorem [12], [16], we have $K_A^0(T^*M) \cong K_0(C_0(T^*M) \otimes A)$, which is induced via the action of taking sections as for the case $A = \mathbb{C}$. It thus follows from this and the constructions in the previous section that all three of these symbol classes can be identified.

PROPOSITION 5.2. $[\sigma(D)] = \Sigma(D) = [\![\Phi_\sigma]\!] \in K_A^0(T^*M) \cong K_0(C_0(T^*M) \otimes A)$.

Let $L^2(E)$ denote the completion of the pre-Hilbert A-module $C^\infty(E)$. The differential A-operator $\mathbb{D}$ defines an (essentially) self-adjoint regular operator of degree one on the graded Hilbert A-module $\mathcal{H}_D = L^2(E) \oplus L^2(F)$. (We replace $\mathbb{D}$ by its closure $\bar{\mathbb{D}}$ which is self-adjoint.) Since $\mathbb{D}$ is elliptic, the resolvents $(\mathbb{D} \pm i)^{-1}$ are compact. (This follows from the parallel Sobolev theory for differential A-operators [13].) The complementation of the bundles E and F above (with the previous constructions) allows the coherent inclusion

$$\mathcal{H}_D \subset L^2(\mathbb{A}) \cong L^2(M) \otimes A^{2n}$$

which induces a graded inclusion of C^*-algebras $\mathcal{K}(\mathcal{H}_D) \hookrightarrow M_{2n}(\mathcal{K} \otimes A)$. By the functional calculus for self-adjoint regular operators [11] we obtain a graded $*$-homomorphism

$$\Phi_D\colon C_0(\mathbb{R}) \to M_{2n}(\mathcal{K} \otimes A)\colon f \mapsto f(\mathbb{D})$$

Recall that the usual definition of the generalized Fredholm (analytic) index, $\mathrm{Index}_A(D)$ in terms of kernel and cokernel modules requires compact perturbations for a general C^*-algebra A [13], [18]. This is incorporated in the computations in the proof of Corollary 4.8, so we see that the functional calculus for $\mathbb{D}$ gives this index.

PROPOSITION 5.3. $\mathrm{Index}_A(D) = [\![\Phi_D]\!] \in K_0(A)$.

Naturally associated to M and A is an asymptotic morphism of C^*-algebras

$$\{\Psi_t\}_{t\in[1,\infty)}\colon C_0(T^*M) \otimes A \to \mathcal{K}(L^2M) \otimes A,$$

which is defined via Fourier transforms and a partition of unity up to asymptotic equivalence. (For complete details on the construction see [5], [7], [17].) The induced map

$$\Psi_*\colon K_A^0(T^*M) \cong K_0(C_0(T^*M) \otimes A) \to K_0(A)$$

on K-theory is useful for doing index-theoretic and K-theoretic calculations with elliptic operators. If $M = \mathbb{R}^n$, the induced map is Bott periodicity $K_0(C_0(\mathbb{R}^{2n}) \otimes A) \cong K_0(A)$ [17]. The following result implies the exact form of the Mishchenko-Fomenko index theorem [13], hence the Atiyah-Singer index theorem [1] when $A = \mathbb{C}$ as proved originally by Higson [7].

THEOREM 5.4 (Lemma 4.6 [17]). *If D is an elliptic differential A-operator of order one on M then*

$$\Psi_*([\sigma(D)]) = \mathrm{Index}_A(D) \in K_0(A).$$

The proof reduces to composing the graded symbol homomorphism

$$\Phi_\sigma\colon C_0(\mathbb{R}) \to M_{2n}(C_0(T^*M) \otimes A)\colon f \mapsto f(\sigma)$$

with the matrix inflation of this "fundamental" asymptotic morphism for M and A,

$$\{\Psi_t\}\colon M_{2n}(C_0(T^*M) \otimes A) \to M_{2n}(\mathcal{K} \otimes A),$$

and comparing this to the continuous family of graded operator $*$-homomorphisms

$$\{\Phi_D^t\}_{t\in[1,\infty)}\colon C_0(\mathbb{R}) \to M_{2n}(A \otimes \mathcal{K})\colon f \mapsto f(t^{-1}\mathbb{D}).$$

One then proves [17] via Fourier analysis and a compactness argument that for any $f \in C_0(\mathbb{R})$,

$$\lim_{t\to\infty} \|\Psi_t(f(\sigma)) - f(t^{-1}\mathbb{D})\| = 0$$

and so the composition $\{\Psi_t \circ \Phi_\sigma\}$ is asymptotically equivalent to $\{\Phi_D^t\}$. Therefore, by stability and homotopy invariance of the induced map [5], [6],

$$\Psi_*[\![\Phi_\sigma]\!] = [\![\Phi_D^t]\!] = [\![\Phi_D]\!] \in K_0(A).$$

The result now follows by Propositions 5.2 and 5.3.

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