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TOTAL CURVATURES OF CONVEX HYPERSURFACES IN HYPERBOLIC SPACE

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ABSTRACT. We give sharp upper estimates for the difference circumradius minus inradius and for the angle between the radial vector (respect to the center of an inball) and the normal to the boundary of a compact h-convex domain in the hyperpolic space. We apply these estimates to get the limit at the infinity for the quotients Volume/Area and (Total k-mean curvature)/Area of a family of h-convex domains which expand over the whole space. The theorem for the first quotient gives an extension to arbitrary dimension of a result of Santaló and Yañez for the hyperbolic plane.

1. Introduction

In 1972, in the course of the study of some problems of geometric probability in $\mathbb{H}^2$, L. A. Santaló and I. Yañez [SY] proved the following result: *Let* $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ *be a family of compact h-convex domains in* $\mathbb{H}^2$ *which expands over the whole space. Then* $\lim_{t\to\infty} \frac{\text{Area}(\Omega(t))}{\text{Length}(\partial\Omega(t))} = 1$. We shall explain some of the concepts involved in this theorem. A domain in the hyperbolic space $\mathbb{H}^{n+1}$ of sectional curvature -1 (and dimension $n+1$) is a closed subset of $\mathbb{H}^{n+1}$ with interior not empty. An *h-convex domain* (or convex by horoballs in the terminology of [AC]) in the hyperbolic space $\mathbb{H}^{n+1}$ of sectional curvature -1 (and dimension $n+1$) is a domain $\Omega \subset \mathbb{H}^{n+1}$ with boundary $\partial\Omega$ such that, for every $p \in \partial\Omega$, there is a horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}$ through p such that Ω is contained in the horoball of $\mathbb{H}^{n+1}$ bounded by $\mathcal{H}$. This $\mathcal{H}$ is called a *supporting horosphere* of Ω (and of $\partial\Omega$). We say that *a family of domains* $\{C(t)\}_{t\in\mathbb{R}^+}$ *in* $\mathbb{H}^{n+1}$ *expands over the whole space* (e.o.w.s. in abreviated notation) if for any $x \in \mathbb{H}^{n+1}$ there is a $t_0 \in \mathbb{R}$ such that, for every $t > t_0$, $x \in C(t)$.

The above Santaló-Yañez Theorem is in hard contrast with the situation for convex domains in euclidean space, where $\lim_{t\to\infty} \frac{\text{Area}(\Omega(t))}{\text{Length}(\partial\Omega(t))} = \infty$. Santaló and Yañez conjectured that their result will be still true for convex domains in $\mathbb{H}^2$. By showing a counterexample, E. Gallego and A. Reventós [GR] have proved, in 1985, that this conjecture is not true.

No attempt to solve the problem in general dimension was made until very recently, when A.M. Naveira and A. Tarrío [NT] gave a version of the Santaló-Yañez Theo-

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rem for n odd and families $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ of h-convex regular (with smooth boundary) domains which expand *by parallels* over the whole space.

Here we shall prove a general version of the Santaló-Yañez Theorem for any family of h-convex domains e.o.w.s., and for any value of n. We shall even allow $\partial\Omega$ to be non-smooth. Our approach will be completely different from that of Santaló, Yañez, Naveira and Tarrío. In particular, we shall not use the isoperimetric nor the Gauss-Bonnet formulae.

The key facts of our proof will be the first two theorems that we state now and which have an independent interest. They give estimates of some metric invariants of h-convex domains. To get these estimates we define some model domains with two singular points (that we shall name the "worst h-convex domains" in Section 2). These domains realize the bounds that we shall state in Theorems 1 and 2, and will be compared with general h-convex domains to get the theorems.

We need to recall some definitions.

Given any domain $\Omega \subset \mathbb{H}^{n+1}$, an *inscribed ball* (*inball* for short) is a ball in $\mathbb{H}^{n+1}$ contained in Ω with maximum radius. Its radius is called the *inradius* of Ω. A *circumscribed ball* (or *circumball*) is a ball in $\mathbb{H}^{n+1}$ containing Ω and with minimum radius. Its radius is called *circumradius* of Ω. Our first theorem will be an estimate for the difference between the circumradius and the inradius of a compact h-convex domain.

Also recall that if Ω is a convex domain in $\mathbb{H}^{n+1}$, then $\partial\Omega$ is a topological embedded hypersurface, which is C^2 except for a set of zero measure.

THEOREM 1. (*a*) *Let Ω be a compact h-convex domain. Let o be the center of an inball of Ω. Let r be the inradius of Ω and $\tau = \tanh\frac{r}{2}$. Then the maximal distance* $\operatorname{maxd}(o, \partial\Omega)$ *between o and the points in $\partial\Omega$ satisfies the inequality*

$$\operatorname{maxd}(o, \partial\Omega) \leq r + \ln\frac{(1+\sqrt{\tau})^2}{1+\tau} < r + \ln 2.$$

In particular, if R is the circumradius of Ω, then

$$R - r \leq \ln\frac{(1+\sqrt{\tau})^2}{1+\tau} < \ln 2.$$

Moreover this bound is sharp.

(*b*) *If $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ is a family of h-convex domains expanding over the whole space, $o(t)$ is the center of an inball of $\Omega(t)$, and $r(t)$ and $R(t)$ are the inradius and the circumradius of $\Omega(t)$, respectively, then*

$$\lim_{t\to\infty}(\operatorname{maxd}(o(t), \partial\Omega(t)) - r(t)) = \lim_{t\to\infty}(R(t)) - r(t)) = \ln 2.$$

This theorem will be the basic fact for the next estimate, that we shall state after recalling another definition. Given an h-convex domain Ω and a point $p \in \partial\Omega$, a vector $N \in T_p\mathbb{H}^{n+1}$ is said to be normal to $\partial\Omega$ at p if it is normal to a supporting horosphere of Ω at p. When $\partial\Omega$ is smooth at point p, then N is normal to $\partial\Omega$ if and only if it is normal to $T_p\partial\Omega$.

THEOREM 2. *Let Ω be a compact h-convex domain. Let o be the center of an inball of Ω. Let r be the inradius of Ω and $\tau = \tanh\frac{r}{2}$. Let us denote by ℓ the distance to o and by ∂_ℓ its gradient (in $\mathbb{H}^{n+1}$). Then, at every $p \in \partial\Omega$ and for every unit vector N normal to $\partial\Omega$ at p, one has*

$$|\langle N, \partial_\ell\rangle(p)| \geq \frac{\tanh^2\frac{\ell}{2}(p) + \tau}{\tanh\frac{\ell}{2}(p)(1+\tau)} \geq \frac{2\sqrt{\tau}}{1+\tau}.$$

Moreover this bound is sharp.

The estimate given by this theorem is the unique specific property of h-convex bodies that we shall use in the proof of the generalized Santaló-Yañez Theorem. *Our next theorems will be true for any family of compact convex domains e.o.w.s. and satisfying the estimate given by Theorem* 2. This is philosophically similar to the idea in [GR] of getting sufficient conditions on the support functions of convex domains e.o.w.s. to have a positive answer to the Santaló's conjecture in the hyperbolic plane. The generalization of the theorems of Santaló-Yañez and Naveira-Tarrío that we shall prove is the following:

THEOREM 3. *Let $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ be a family of h-convex domains expanding over the whole space. Then*

$$\lim_{t\to\infty} \frac{\text{volume}(\Omega(t))}{\text{volume}(\partial\Omega(t))} = \frac{1}{n}.$$

The next estimate is a natural continuation of the above research. It gives a bound on the quotient of the total k-mean curvature of an h-convex hypersurface by its volume. Again, this in contrast with the analog situation in the euclidean space: the limit for this quotient for a family e.o.w.s. is 0 for convex hypersurfaces in the euclidean space and 1 for h-convex hypersurfaces in $\mathbb{H}^{n+1}$. We shall state the theorem for not necessarily smooth h-convex hypersurfaces $\partial\Omega$, for which the concept of total k-mean curvature M_k still makes sense, generalizing the definitions for convex hypersurfaces in $\mathbb{R}^{n+1}$, as we shall detail in Section 6. Now we will only say that when $\partial\Omega$ is smooth,

$$M_k(\partial\Omega) = \int_{\partial\Omega} H_k\,\mu, \qquad \text{with} \qquad H_k = \frac{1}{\binom{n}{k}} S_k,$$

where S_k is the k-th elementary symmetric function on the principal curvatures of $\partial\Omega$ and μ is the volume element of $\partial\Omega$.

To prove the theorem we again use Theorem 2 and some special laplacians introduced by Reilly in [Re1]. The statement of the theorem is:

THEOREM 4. (*a*) *Let Ω be a compact h-convex domain. Let r be the inradius of Ω and $\tau = \tanh \frac{r}{2}$. Then*

$$\left(\frac{4\tau}{(1+\tau)^2}\right)^k \coth^k r \leq \frac{M_k(\partial\Omega)}{\operatorname{volume}(\partial\Omega)} \leq \left(\frac{1+\tau}{2\sqrt{\tau}}\right)^k \coth^k \left(r + \ln \frac{(1+\sqrt{\tau})^2}{1+\tau}\right).$$

Moreover this bound is sharp.

(*b*) *If $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ is a family of h-convex domains expanding over the whole space, then*

$$\lim_{t\to\infty} \frac{M_k(\partial\Omega(t))}{\operatorname{volume}(\partial\Omega(t))} = 1.$$

In the definition of h-convexity, we may substitute the supporting horosphere by a supporting equidistant hypersurface of constant normal curvature λ. Then we have a λ-convex domain in the hyperbolic space $\mathbb{H}^{n+1}$, where $0 \leq \lambda \leq 1$. For $\lambda = 0$ we get the usual convexity in $\mathbb{H}^{n+1}$, and $\lambda = 1$ gives the h-convexity. For λ-convexity ($\lambda \neq 0$), it is possible to generalize Theorem 2 and give some estimates (only for one side) for Theorems 3 and 4.

Although we have stated all our theorems for boundaries of convex domains, and then for embedded convex hypersurfaces, recent results on immersions which are h-convex made possible to state the theorems in a more general form. In fact, J. Currier (in [Cu]) has shown that h-convex immersions of smooth compact hypersurfaces are embedded spheres. This result has been generalized to immersions of non smooth manifolds by the first author and Vlasenko, which proved the following.

THEOREM [BV]. *Let F be a topological manifold of dimension $n \geq 2$, and let $f\colon F \longrightarrow \mathbb{H}^{n+1}$ be a topological immersion satisfying the following conditions*:

(*a*) *It is locally convex at any point.*
(*b*) *It has a locally supporting horosphere at any point.*
(*c*) *F is complete with the metric induced by f.*

Then f is an embedding and $f(F)$ is the boundary of an h-convex body Ω in $\mathbb{H}^{n+1}$. Moreover, either $\partial\Omega = f(F)$ is compact and homeomorphic to the sphere S^n or $\partial\Omega = f(F)$ is a standard horosphere.

Furthermore, suppose the immersion f satisfies the following condition:

(*d*) *At some point $\partial\Omega = f(F)$ has a strong locally supporting horosphere.*

Then $\partial\Omega = f(F)$ is also compact.

Then we have:

COROLLARY. *Let F be a topological manifold of dimension $n \geq 2$, and let $f\colon F \longrightarrow \mathbb{H}^{n+1}$ be a topological immersion satisfying the conditions (a) to (d) of the above theorem. Then $f(M)$ is compact and the boundary of an h-convex domain Ω, and the inequalities of Theorems* 1, 2 *and* 4 *are satisfied.*

The plan of the paper will be the following: In Section 2 we give some necessary background, we establish the model of $\mathbb{H}^{n+1}$ we shall work with, and will define and give some properties of the "worst h-convex domains". Sections 3 to 6 will be devoted to the proof of Theorems 1 to 4. In Section 7, bounds using the intrinsic distance in $\partial\Omega$ will be discussed.

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2. The "worst" h-convex domains

In this section we describe the h-convex domains generated by two points in $\mathbb{H}^{n+1}$. First, we recall some concepts and state some notation. A horosphere of $\mathbb{H}^2$ is called a *horocycle*. Let $p, q \in \mathbb{H}^{n+1}$ and let $\mathbb{H}^2$ be any 2-dimensional complete totally geodesic subspace of $\mathbb{H}^{n+1}$ containing p and q. There are only two horocycles in $\mathbb{H}^2$ passing through p and q. The set $\{p, q\}$ divides each one of these horocyles into three connected components. The bounded one is called a horocycle segment from p to q (or betwen p and q). It is known that a domain Ω of $\mathbb{H}^{n+1}$ is h-convex if and only if for every $p,q \in \Omega$ all the horocycle segments from p to q are contained in Ω.

In this paper, we use for $\mathbb{H}^{n+1}$ the model of a ball $\mathcal{B}$ in $\mathbb{R}^{n+1}$ with center at 0 and radius 2 endowed with the metric (cf. [dC, p. 179])

$$ds^2 = \frac{1}{(1 - \frac{1}{4}(x_1^2 + \cdots + x_{n+1}^2))^2}(dx_1^2 + \cdots + dx_{n+1}^2).$$

In this model, the horospheres are the spheres contained in $\mathcal{B}$ and tangent to its boundary, the straight lines through 0 are geodesics in $\mathbb{H}^{n+1}$, and the identity map on $\mathcal{B}$ preserves the angles corresponding to the hyperbolic and euclidean metrics on $\mathcal{B}$.

Given two points p, q in $\mathbb{H}^{n+1}$, we denote by $h(p, q)$ the h-convex domain generated by $\{p, q\}$, that is, the minimum h-convex domain containing $\{p, q\}$. Then $h(p, q)$ is the convex domain having as boundary all the horocycle segments from p to q. If we take the center 0 for $\mathcal{B}$ to be the midpoint o between p and q, $h(p, q)$ is the body enclosed by the revolution surface $\partial h(p, q)$, around the axis joining p and q, generated by the smallest arc of circle between p and q of a circle tangent to

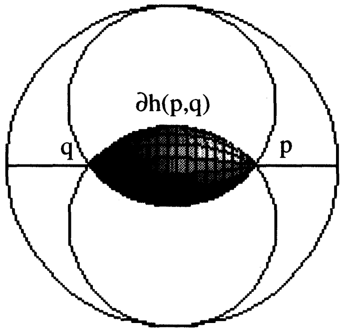

Figure 1

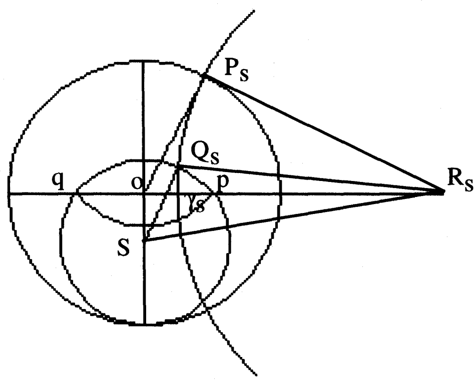

Figure 2

the boundary of $\mathcal{B}$ and passing through p and q (see Fig. 1). When we work with this model, by *euclidean radius, euclidean distance, ...* we mean the corresponding radius, distance, ... in $\mathcal{B}$ with the euclidean metric. The distance in the hyperbolic space will be denoted by "dist", and the euclidean distance will be denoted by "d_e".

In this section we shall give some properties of the domains $h(p, q)$. Given two points $p, q \in \mathbb{H}^{n+1}$, pq will denote one of the horocycle segments between p and q, and by $\overline{pq}$ we denote the geodesic segment from p to q.

PROPOSITION 2.1. *Let $p, q \in \mathbb{H}^{n+1}$, and let o be the midpoint of $\overline{pq}$. If s denotes the arclength of $\overline{pq}$ such that $\overline{pq}(0) = p$, we shall denote by γ_s the geodesic segment from $\overline{pq}$ to pq and orthogonal to $\overline{pq}$, and by $\varphi(s)$ the angle between γ_s and pq at the intersection point. Then we have*:

(*a*) $\varphi(\mathrm{dist}(p, o)) = \pi/2$ *and and* $\varphi(s) \neq \pi/2$ *for every* $s \neq \mathrm{dist}(p, o)$.

(*b*) *The function* l: $[0, \mathrm{dist}(p, q)] \to \mathbb{R}^+$ *defined by* $l(s) = \mathit{length}(\gamma_s)$ *has one and only one maximum at* $\mathrm{dist}(p, o)$ (*which corresponds to the geodesic* $\gamma_{\mathrm{dist}(p,o)}$ *starting from* o).

(*c*) *The inball of* $h(p, q)$ *is unique and has its center at* o.

Proof. Let $\mathbb{H}^2$ be the totally geodesic plane containing $\overline{pq}$ and pq. For any $s \in [0, \mathrm{dist}(p, o)[$, the geodesic segment γ_s is in a circle with center R_s at the axis $y = 0$ which intersects the boundary of the model ball at point P_s. Let S be the center of the circle containing the horocycle segment pq and let Q_s be the intersection point of γ_s and pq. Then $\varphi(s)$ is equal to the supplement of the interior angle θ at Q_s of the triangle SQ_sR_s. We observe that the triangles SQ_sR_s and oP_sR_s satisfy $d_e(P_s, R_s) = d_e(Q_s, R_s)$, $d_e(P_s, o) = 2 > d_e(Q_s, S)$, $d_e(o, R_s) < d_e(S, R_s)$. Then θ is greater than the interior angle at P_s of the triangle oP_sR_s, namely $\pi/2$, so $\varphi(s) < \pi/2$. A similar argument shows that $\varphi(s) > \pi/2$ for $s \in]\operatorname{dist}(p, o), \operatorname{dist}(p, q)]$. On the other hand, is obvious that $\varphi(\operatorname{dist}(p, o)) = \pi/2$, which finish the proof of part (a).

From the first variation formula it follows that

$$\frac{d}{ds} l(s) = \cos\varphi(s),$$

and part (b) follows from (a).

From the symmetry of $h(p, q)$ in the model, it follows that the points at maximal distance from $\partial h(p, q)$ are on the geodesic segment $\overline{pq}$. Among these, it also follows from (a) and the formula above, that the maximal distance between $\overline{pq}$ and $\partial h(p, q)$ is given by $l(\operatorname{dist}(o, p)) = \operatorname{dist}(o, Q_{\operatorname{dist}(o,p)})$, where we follow the notation in the proof of (a). Since the geodesics through o are the straight lines through o, the distance from o to any other point in $\partial h(p, q)$ is larger than $l(\operatorname{dist}(o, p))$. Then the ball with center o and radius $\operatorname{dist}(o, Q_{\operatorname{dist}(o,p)})$ is the unique inball of $h(p, q)$. □

PROPOSITION 2.2. *Let o be the midpoint of $\overline{pq}$. Let us take the model ball $\mathcal{B}$ (of $\mathbb{H}^{n+1}$) with center at o. If R and r are the circumradius and the inradius of $h(p, q)$, respectively, and 2τ is the euclidean radius of the inball, then $d_e(o, p) = 2\sqrt{\tau}$, $\tau = \tanh\frac{r}{2}$, and*

$$R - r = \ln\frac{(1+\sqrt{\tau})^2}{1+\tau}.$$

Proof. If $\mathbb{H}^2$ is any totally geodesic surface of $\mathbb{H}^{n+1}$ containing p and q, then $h(p, q) \cap \mathbb{H}^2$ is the h-convex domain generated by p and q in $\mathbb{H}^2$, that is, the set of points in the interior of the set bounded by the two horocycle segments in $\mathbb{H}_2$ between p and q. The inball B_2 has center at o and its boundary ∂B_2 is tangent to the above horocycle segments at $(0, 2\tau)$ and $(0, -2\tau)$. Then the horocycle $\mathcal{H}$ of $\mathbb{H}^2$ tangent to ∂B_2 at $(0, 2\tau)$ has center $(0, \tau - 1)$ and euclidean radius $1 + \tau$ (see Fig. 3). The intersection of $\mathcal{H}$ with the axis x consists of the points $(\pm 2\sqrt{\tau}, 0)$, the coordinates of the points p and q. Then

$$r = \operatorname{dist}(o, (0, 2\tau)) = \int_0^{2\tau} \frac{ds}{1 - \frac{1}{4}s^2} = \ln\frac{1+\tau}{1-\tau}$$

and

$$R = \operatorname{dist}(o, p) = \int_0^{2\sqrt{\tau}} \frac{ds}{1 - \frac{1}{4}s^2} = \ln\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}.$$

From these expressions we get the formula of the proposition. □

Given a ball B in $\mathbb{H}^{n+1}$ with center o at the center of the model ball $\mathcal{B}$, and with euclidean radius 2τ, a *worst h-convex domain generated by B* is any one of the h-convex domains $h(p, q)$ generated by two different points p and q on a geodesic through o and at euclidean distance $2\sqrt{\tau}$ from o. Then B is the inball of this $h(p, q)$.

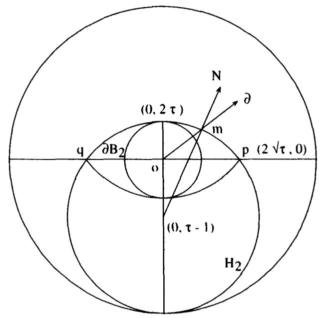

Figure 3

From now until Section 6, given any convex domain Ω, a unit vector N normal to $\partial\Omega$ will be chosen pointing outward. With this choice, for ∂_ℓ as in Theorem 2, we have $|\langle\partial_\ell, N\rangle| = \langle\partial_\ell, N\rangle$.

PROPOSITION 2.3. *Let o, p, q and τ be as in Proposition 2.2. Let $m \in \partial h(p,q)$, $N(m)$ be a unit vector normal to $\partial h(p,q)$ at m, and ∂_ℓ be the gradient of the function $\ell(m) = \mathrm{dist}(m, o)$. Then*

$$\langle N(m), \partial_\ell(m)\rangle = \frac{\tanh^2\frac{\ell}{2}(m) + \tau}{\tanh\frac{\ell}{2}(m)(1+\tau)} \geq \frac{2\sqrt{\tau}}{1+\tau}$$

and this bound is the limit of $\langle N(m), \partial_\ell(m)\rangle$ when m goes to p or q.

Proof. In the model, we have $|m| = d_e(o,m)$, and $\partial_\ell(m)$ is in the direction of m. Moreover, if $p \neq m \neq q$, then N is in the direction from the center of the horocycle $\mathcal{H}$ through p, m and q to m (see Fig. 3). In the plane containing p, q and m we will have the coordinates $p = (2\sqrt{\tau}, 0)$, $q = (-2\sqrt{\tau}, 0)$, and $(0, -1+\tau)$ for the center of $\mathcal{H}$. Since the radius of $\mathcal{H}$ is $1+\tau$, we have $|m - (0, -1+\tau)| = 1+\tau$ and $|m|^2 + (-1+\tau)^2 - 2\langle m, (0, -1+\tau)\rangle = (1+\tau)^2$. Then

$$\text{(2.3.1)} \qquad \langle N(m), \partial_\ell(m)\rangle = \left\langle \frac{m}{|m|}, \frac{m - (0,-1+\tau)}{|m - (0,-1+\tau)|}\right\rangle = \frac{|m|^2 + 4\tau}{2|m|(1+\tau)}.$$

This proves the equality, because $\ell(m) = \int_0^{|m|}(1/(1-\frac{1}{4}s^2))ds = 2\tanh^{-1}\frac{|m|}{2}$, so $|m| = 2\tanh\frac{\ell(m)}{2}$. The function $f(\mu) = \frac{\mu^2+4\tau}{2\mu(1+\tau)}$ is decreasing in μ if $\mu \leq 2\sqrt{\tau}$, because $f'(\mu) = \frac{\mu^2-4\tau}{2(1+\tau)^2} \leq 0$. Then, since $|m| \leq 2\sqrt{\tau}$, the minimum value of $\langle N(m), \partial_\ell(m)\rangle$ is that given in (2.3.1) for $|m| = 2\sqrt{\tau}$, and this proves the inequality. □

3. Proof of Theorem 1

Let B be the inball with center o and radius r. Let $p' \in \partial\Omega$ such that $\max d(o, \partial\Omega) = \mathrm{dist}(o, p')$. Let us suppose that $\mathrm{dist}(o, p') > r + \ln \frac{(1+\sqrt{\tau})^2}{1+\tau}$. Let σ be the geodesic line through o and p', and let p be the point in σ between o and p' and at distance $r + \ln \frac{(1+\sqrt{\tau})^2}{1+\tau}$ from o. Let q be the point ($\neq p$) in σ such that $\mathrm{dist}(o, p) = \mathrm{dist}(o, q)$ (see Fig. 4). From Proposition 2.2, B is also an inball of $h(p, q)$. Let $\mathbb{H}^n$ be the totally geodesic hypersurface through o orthogonal to σ. Let $D = B \cap \mathbb{H}^n$. It follows from Proposition 2.3 that the horocycle segments from p to ∂D are orthogonal to the geodesics starting from o. The horocycle segments β from p' to ∂D belong to Ω (since it is h-convex), and they lie outside $h(p, q)$ (because, if q' is the intersection—different from p'—of the horocycle containing β with the geodesic σ, then $h(p', q')$ is an h-convex domain containing p and ∂D, and so contains all the horocycle segments between p and ∂D). Then the angles at points $Q \in \partial D$ between the horocycle segments β and the geodesics starting from o are greater than $\pi/2$. Therefore, it follows from Proposition 2.1 that the inradius r' of $h(p', q')$ is greater than r. From the h-convexity of Ω, it follows that the part $h_+(p', q')$ of $h(p', q')$, which lies in the side bounded by $\mathbb{H}^n$ which contains p and p', is contained in Ω. From Proposition 2.1 it follows that the center o' of the inball B' of $h(p', q')$ lies in σ between o and p' (and is different from o), so $o' \in h_+(p', q') \subset \Omega$.

From the description of the inball B' of $h(p', q')$ given in Section 2, we see that all the points in $h(p', q')$ at distance r' from o' are in the geodesics starting from o' and orthogonal to σ. Then distance $r_0 := \mathrm{dist}(o', \partial D) > r'$, because the geodesic from o' to any $Q \in \partial D$ is not orthogonal to σ.

Let Q be a point in the boundary S_- of $B_- = B - h_+(p', q')$. The geodesic triangle with vertices $oo'Q$ has an interior angle at o greater than $\pi/2$. Then, from the cosinus law,

$$\cosh(\mathrm{dist}(o', Q)) > \cosh(\mathrm{dist}(o, o')) \cosh r = \cosh r_0, \tag{3.1}$$

where the last equality follows again from the cosinus law applied to the triangle $o'oP$, with $P \in \partial D = \partial S_-$, which has interior angle at o equal to $\pi/2$. From the inequality (3.1) we have $\mathrm{dist}(o', Q) > r_0 > r'$. Then $B' \subset B_- \cup h_+(p', q') \subset \Omega$, but the radius of B' is larger than the inradius of Ω, which is a contradiction. This finishes the proof of Theorem 1.1(a).

To prove part (b) we shall need the following result.

LEMMA 3.2. *Let $\Omega(t)$ be a family of h-convex domains e.o.w.s., and let $r(t)$ be the inradius of $\Omega(t)$. Then, for every $r_0 > 0$, there is a $t_0 > 0$ such that $r(t) \geq r_0$ for every $t \geq t_0$.*

Proof. Given $r_0 > 0$, let B_0 be a ball with center o and radius r_0 in $\mathbb{H}^{n+1}$. Let $h(p, q)$ be a worst h-convex domain generated by B_0. Since $\{\Omega(t)\}$ e.o.w.s., there is

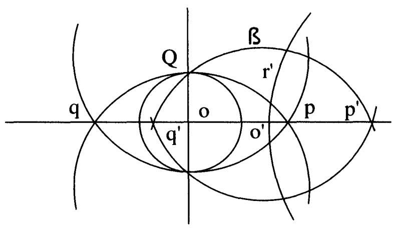

Figure 4

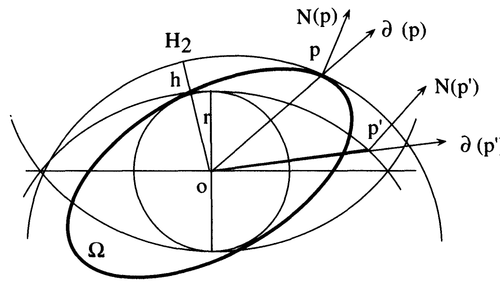

Figure 5

a $t_0 \in \mathbb{R}^+$ such that $p, q \in \Omega(t)$ for every $t \geq t_0$. But, since the $\Omega(t)$ are h-convex, we have $B_0 \subset h(p, q) \subset \Omega(t)$. Then $r(t) \geq r_0$ for every $t \geq t_0$. □

From this lemma we have

$$\lim_{t\to\infty}(R(t) - r(t)) = \lim_{r(t)\to\infty}(R(t) - r(t)) \leq \lim_{\tau\to 1} \ln \frac{(1+\sqrt{\tau})^2}{1+\tau} = \ln 2,$$

and this is the best bound, which is attained, among others, at the family $\{h(p(t), q(t))\}_{t\in\mathbb{R}^+}$, where $p(t) = (2\sqrt{\tau}, 0)$, $q(t) = (-2\sqrt{\tau}, 0)$ and $\tau = \tanh(t/2)$.

4. Proof of Theorem 2

Given $p \in \partial\Omega$, let $\mathcal{H}$ be a horosphere which is orthogonal to $N(p)$ at p and bounds a horoball which contains Ω. Let $\mathcal{H}_2$ be the horocycle intersection of $\mathcal{H}$ and the totally geodesic hyperbolic plane $\mathbb{H}^2$ through p tangent to $\partial_\ell(p)$ and $N(p)$ (see Fig. 5). (If $N(p)$ and $\partial_\ell(p)$ have the same direction, we take as $\mathbb{H}^2$ any of the totally geodesic hyperbolic planes through p tangent to $\partial_\ell(p)$.) Obviously, $\mathbb{H}^2$ contains the geodesic through p tangent to $\partial_\ell(p)$. Then it contains o. If $h = \operatorname{dist}(o, \mathcal{H}_2)$, then

$$r \leq h \leq \ell(p) \leq \operatorname{maxd}(o, \partial\Omega) \leq r + \ln \frac{(1+\sqrt{\tau})^2}{1+\tau}.$$

Let $K(r)$ be a worst h-convex domain in $\mathbb{H}^2$ generated by the disk B_r of center o and radius r. From Proposition 2.2, the function ℓ on $\partial K(r)$ takes values in $[r, r + \ln \frac{(1+\sqrt{\tau})^2}{1+\tau}]$, so there is a p' in $\partial K(r)$ such that $\ell(p') = \ell(p)$. Also, let us denote by $N(p')$ the unit vector normal to $\partial K(r)$ at p' and pointing outward. Then we have:

PROPOSITION 4.1. $\langle N(p'), \partial_\ell(p')\rangle \leq \langle N(p), \partial_\ell(p)\rangle$.

Proof. Let us denote by $|p|$ and $|p'|$ the euclidean distances from p and p' to o. Then $\ell(p) = \ell(p')$ implies $|p| = |p'|$. Let $2\overline{\tau}$ be the euclidean distance between o and $\mathcal{H}_2$. Since p is in the horosphere of center $(0, -1 + \overline{\tau})$ and radius $1 + \overline{\tau}$, then $|p - (0, -1 + \overline{\tau})| = 1 + \overline{\tau}$, and, from this, we get $-\langle p, (0, -1 + \overline{\tau})\rangle = (4\overline{\tau} - |p|^2)/2$. Then

$$\langle N(p), \partial_\ell(p)\rangle = \left\langle \frac{p}{|p|}, \frac{p - (0, -1 + \overline{\tau})}{|p - (0, -1 + \overline{\tau})|} \right\rangle = \frac{|p|^2 + 4\overline{\tau}}{2|p|(1 + \overline{\tau})}.$$

And, since $|p| = |p'|$, similar computations give the same expression for $\langle N(p'), \partial_\ell(p')\rangle$ but with $\overline{\tau}$ replaced by τ. But, if $f(t) = \frac{|p|^2+4t}{2|p|(1+t)}$, its derivative is given by $f'(t) = \frac{4-|p|^2}{2|p|(1+t)^2} \geq 0$ (since $|p| \leq 2$), and, since $\overline{\tau} \geq \tau$ (because $h \geq r$), we have $f(\overline{\tau}) \geq f(\tau)$. □

Now Theorem 2 follows from Proposition 4.1 by applying Proposition 2.3 to $\partial K(r)$.

5. Proof of Theorem 3

Let Ω be a compact h-convex domain , and let o be the center of an inball of Ω. Let us consider spherical geodesic coordinates of $\mathbb{H}^{n+1}$ with origin at o. In these coordinates, the volume form ω of Ω has the expression $\omega = \sinh^n(\ell)d\ell \wedge \nu$, where ℓ is the distance to o and ν is the volume form of the unit sphere S^n in $\mathbb{R}^{n+1}$ (see, for instance, [Gr2]). From the definition of ν we have $\nu(\partial_\ell, \ldots) = 0$. Then

$$\nu = \iota_{\partial_\ell}(d\ell \wedge \nu) \text{ where } \iota_{\partial_\ell} \text{ denotes the interior contraction with } \partial_\ell. \tag{5.1}$$

For any u in the unit sphere $S^n \subset T_o\mathbb{H}^{n+1}$, let $l(u)$ be the length of the geodesic segment from o to $\partial\Omega$ tangent to u at o. Since Ω is h-convex, the map $u \mapsto \exp_o l(u)u$, from S^n to $\partial\Omega$, is a $C^{0,1}$ homeomorphism, and it defines a system of spherical coordinates for $\partial\Omega$. In order to compute the expression in these coordinates of the volume form μ of $\partial\Omega$, we take an orthonormal basis $e_1, \ldots, e_n$ of $T_p\partial\Omega$ at any point $p \in \partial\Omega$ where $\partial\Omega$ is C^2. By (5.1), we get

$$\begin{aligned}
\nu(e_1, \ldots, e_n) &= (d\ell \wedge \nu)(\partial_\ell, e_1, \ldots, e_n) = (d\ell \wedge \nu)(\langle \partial_\ell, N(p)\rangle N(p), e_1, \ldots, e_n) \\
&= \frac{\langle \partial_\ell, N(p)\rangle}{\sinh^n(\ell(p))} \sinh^n(\ell(p))\,(d\ell \wedge \nu)(N(p), e_1, \ldots, e_n) \\
&= \frac{\langle \partial_\ell, N(p)\rangle}{\sinh^n(\ell(p))}
\end{aligned}$$

because $N(p), e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{H}^{n+1}$ and $\sinh^n(\ell)d\ell \wedge \nu$ is the volume form of $\mathbb{H}^{n+1}$. From the above formula and the fact that $\mu(e_1, \ldots, e_n) = 1$, we get

$$\mu = \frac{\sinh^n \ell(p)}{\langle \partial_\ell, N(p)\rangle}\nu. \tag{5.2}$$

Then, from (5.2), the expression of the volume in these coordinates, and the estimate of Theorem 2, we have

$$\frac{\text{volume}(\Omega)}{\text{volume}(\partial\Omega)} = \frac{\int_{S^n}\int_0^{l(u)} \sinh^n \ell \, d\ell \wedge \nu}{\int_{S^n} \frac{\sinh^n l(u)}{\langle \partial_\ell, N\rangle}\nu} \geq \frac{2\sqrt{\tau}}{1+\tau} f(l(u)) \tag{5.3}$$

where

$$f(l(u)) = \frac{\int_{S^n} g(l(u)) \sinh^n l(u) \nu}{\int_{S^n} \sinh^n l(u) \nu} \quad \text{and} \quad g(l(u)) = \int_0^{l(u)} \frac{\sinh^n \cdot \ell}{\sinh^n l(u)} \, d\ell.$$

Computing as above, but using $\langle \partial_\ell, N\rangle \leq 1$ instead of Theorem 2, we have

$$\frac{\text{volume}(\Omega)}{\text{volume}(\partial\Omega)} \leq f(l(u)). \tag{5.4}$$

But a straightforward computation gives

$$\lim_{l(u)\to\infty} g(l(u)) = \frac{1}{n},$$

and, from this, it is easy to check that

$$\lim_{l(u)\to\infty} f(l(u)) = \frac{1}{n}. \tag{5.5}$$

Now, it follows from Lemma 3.2 and formulas (5.3), (5.4) and (5.5) that if $\Omega(t)$ is a family of h-convex domains e.o.w.s., then

$$\frac{1}{n} \geq \lim_{t\to\infty} \frac{\text{volume}(\Omega(t))}{\text{volume}(\partial\Omega(t))} \geq \frac{1}{n}.$$

This finishes the proof of Theorem 3.

6. Proof of Theorem 4

First we shall prove the theorem when $\partial\Omega$ is C^∞. In this case we recall that the k-th mean curvature H_k of $\partial\Omega$ at $x \in \partial\Omega$ is defined by $S_k = \binom{n}{k} H_k$, where S_k is the k-th symmetric function of the principal curvatures of $\partial\Omega$ at x for a given orientation.

We shall use the ψ-laplacian (the laplacian associated to a $(1,1)$-tensor ψ over $\partial\Omega$ which is self-adjoint and has divergence zero). This laplacian (cf. [Re1,2], [CY], [Ro] and [Mi]) is defined by

$$(\Box_\psi f)(p) = -\sum_{i=1}^{n} \langle \psi(\nabla_{e_i} \operatorname{grad} f), e_i\rangle, \tag{6.1}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of $T_p\partial\Omega$, $p \in \partial\Omega$ and grad is the gradient in $\partial\Omega$. It satisfies

$$\Box_\psi f\,\mu = -d\iota_{\psi(\operatorname{grad} f)}\mu \quad \text{and then} \quad \int_{\partial\Omega} \Box_\psi f\,\mu = 0, \tag{6.2}$$

where we recall that μ is the volume form of $\partial\Omega$.

Let us denote by I the identity map on each tangent space of $\partial\Omega$, by L the Weingarten map of $\partial\Omega$ associated to N, and let T_k be the $(1,1)$-tensor defined inductively by

$$T_k = S_k\, I - L \circ T_{k-1} \text{ and } T_0 = I, \tag{6.3}$$

or, more explicitely, $T_k = S_k\, I - S_{k-1}\, L + \cdots + (-1)^k L^k$. Then T_k is self-adjoint and has divergence zero (see [Re1,2] and [Ro]), and the following identities hold:

$$(k+1)S_{k+1} = \operatorname{tr}(L \circ T_k) \text{ and } \operatorname{tr} T_k = (n-k)S_k, \tag{6.4}$$

We denote by $\partial_\ell^\top$ the gradient of ℓ in $\partial\Omega$, which is the component of ∂_ℓ tangent to $\partial\Omega$. In general, for any vector X tangent to $\mathbb{H}^{n+1}$ at a point in $\partial\Omega$, $X^\top$ will denote the projection of X onto the tangent vector space to $\partial\Omega$ at this point.

It is known that the Weingarten map $S(\ell)$ of a geodesic sphere of radius ℓ in $\mathbb{H}^{n+1}$ is $S(\ell) = -\coth(\ell)I$ (cf. [Gr2]). Then, if $\overline{\nabla}$ and ∇ are the covariant derivatives in $\mathbb{H}^{n+1}$ and $\partial\Omega$ respectively, we have

$$\begin{aligned}
\nabla_{e_i}\partial_\ell^\top &= (\overline{\nabla}_{e_i}\partial_\ell)^\top - (\overline{\nabla}_{e_i}\langle\partial_\ell, N\rangle N)^\top = -(S(\ell)(e_i - \langle e_i, \partial_\ell\rangle\partial_\ell))^\top + \langle\partial_\ell, N\rangle L e_i \\
&= \coth\ell(e_i - \langle e_i, \partial_\ell\rangle\partial_\ell^\top) + \langle\partial_\ell, N\rangle L e_i,
\end{aligned}$$

and, from this formula, (6.1) and (6.4), we have

$$\begin{aligned}
\Box_{T_k}\ell &= -\sum_{i=1}^n \langle(\nabla_{e_i}\partial_\ell^\top, T_k e_i\rangle \\
&= -\coth\ell\sum_{i=1}^n\{\langle e_i, T_k e_i\rangle - \langle e_i, \partial_\ell\rangle\langle\partial_\ell^\top, T_k e_i\rangle\} - \sum_{i=1}^n\langle\partial_\ell, N\rangle\langle L e_i, T_k e_i\rangle \\
&= -\coth\ell\{(n-k)S_k - \langle\partial_\ell^\top, T_k\partial_\ell^\top\rangle\} - (k+1)\langle\partial_\ell, N\rangle S_{k+1}.
\end{aligned} \tag{6.5}$$

Since Ω is convex, the eigenvalues of L have constant sign. Now we choose the direction N in such a way that all of them are positive (this is just the opposite direction of that used in the preceeding sections). With this choice, all the S_k are positive, and it follows from the recurrence formulae (6.3) that all the eigenvalues λ_i of T_k are positive. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p\partial\Omega$ diagonalizing T_k (or L, cf. [Re1,2]), we have

$$\begin{aligned}
\langle T_k\partial_\ell^\top, \partial_\ell^\top\rangle &= \sum_{i=1}^n\langle\langle\partial_\ell^\top, e_i\rangle T_k e_i, \partial_\ell^\top\rangle = \sum_{i=1}^n \lambda_i\langle\partial_\ell^\top, e_i\rangle^2 \\
&\le \sum_{i=1}^n \lambda_i|\partial_\ell^\top|^2 = \operatorname{tr} T_k\,|\partial_\ell^\top|^2 = (n-k)S_k|\partial_\ell^\top|^2.
\end{aligned} \tag{6.6}$$

On the other hand, by integration of (6.5) along $\partial\Omega$, and using (6.2), we have

$$\int_{\partial\Omega}(-(n-k)S_k+\langle T_k\partial_\ell^{\top},\partial_\ell^{\top}\rangle)\coth\ell\,\mu=\int_{\partial\Omega}\langle\partial_\ell,N\rangle(k+1)S_{k+1}\,\mu. \tag{6.7}$$

But it follows from inequality (6.6) that

$$-(n-k)S_k+\langle T_k\partial_\ell^{\top},\partial_\ell^{\top}\rangle\le(n-k)S_k(-1+|\partial_\ell^{\top}|^2)\le 0. \tag{6.8}$$

From this inequality and (6.7) we get

$$\begin{aligned}\int_{\partial\Omega}&(n-k)S_k\langle N,\partial_\ell\rangle^2\coth\ell\,\mu=-\int_{\partial\Omega}(n-k)S_k(-1+|\partial_\ell^{\top}|^2)\coth\ell\,\mu\\ &\le-\int_{\partial\Omega}\langle\partial_\ell,N\rangle(k+1)S_{k+1}\,\mu=-\int_{\partial\Omega}(-(n-k)S_k+\langle T_k\partial_\ell^{\top},\partial_\ell^{\top}\rangle)\coth r\,\mu\\ &=\int_{\partial\Omega}((n-k)S_k-\langle T_k\partial_\ell^{\top},\partial_\ell^{\top}\rangle)\coth\ell\,\mu\le\int_{\partial\Omega}(n-k)S_k\coth\ell\,\mu,\end{aligned} \tag{6.9}$$

because the election of N makes $\langle T_k\partial_\ell^{\top},\partial_\ell^{\top}\rangle\ge 0$. If we write the quotient to be bounded as

$$\frac{\int_{\partial\Omega}S_{k+1}\,\mu}{\operatorname{volume}(\partial\Omega)}=\frac{\int_{\partial\Omega}S_{k+1}\,\mu}{-\int_{\partial\Omega}\langle\partial_\ell,N\rangle(k+1)S_{k+1}\,\mu}\,\frac{-\int_{\partial\Omega}\langle\partial_\ell,N\rangle(k+1)S_{k+1}\,\mu}{\operatorname{volume}(\partial\Omega)},$$

from (6.9) and the expression (5.2) we get

$$\begin{aligned}&\frac{\int_{S^n}S_{k+1}\frac{\sinh^n l(u)}{\langle\partial_\ell,N\rangle}\,\nu}{\int_{S^n}(k+1)S_{k+1}\sinh^n l(u)\,\nu}\,\frac{\int_{\partial\Omega}(n-k)S_k\langle N,\partial_\ell\rangle^2\coth\ell\,\mu}{\operatorname{volume}(\partial\Omega)}\\ &\le\frac{\int_{\partial\Omega}S_{k+1}\,\mu}{\operatorname{volume}(\partial\Omega)}\\ &\le\frac{\int_{S^n}S_{k+1}\frac{\sinh^n l(u)}{\langle\partial_\ell,N\rangle}\,\nu}{\int_{S^n}(k+1)S_{k+1}\sinh^n l(u)\,\nu}\,\frac{\int_{\partial\Omega}(n-k)S_k\coth\ell\,\mu}{\operatorname{volume}(\partial\Omega)}.\end{aligned}$$

Taking into account that $r\le l(u)$, $\ell\le\operatorname{maxd}(o,\partial\Omega)$, and using Theorem 1 and the fact that, by Theorem 2, $1\ge-\langle N,\partial_\ell\rangle\ge\frac{2\sqrt{\tau}}{1+\tau}$, we have

$$\begin{aligned}&\frac{4\tau}{(1+\tau)^2}\coth r\,\frac{n-k}{k+1}\,\frac{\int_{\partial\Omega}S_k\,\mu}{\operatorname{volume}(\partial\Omega)}\\ &\le\frac{\int_{\partial\Omega}S_{k+1}\,\mu}{\operatorname{volume}(\partial\Omega)}\\ &\le\frac{1+\tau}{2\sqrt{\tau}}\,\frac{n-k}{k+1}\coth\left(r+\ln\left(1+\frac{2\sqrt{\tau}}{1+\tau}\right)\right)\frac{\int_{\partial\Omega}S_k\,\mu}{\operatorname{volume}(\partial\Omega)}.\end{aligned}$$

When $k = 0$ we get

$$\frac{4\tau}{(1+\tau)^2} n \coth r \le \frac{\int_{\partial\Omega} S_1 \,\mu}{\text{volume}(\partial\Omega)} \le \frac{1+\tau}{2\sqrt{\tau}} n \coth\left(r + \ln\left(1 + \frac{2\sqrt{\tau}}{1+\tau}\right)\right).$$

Then, by induction, we have

$$\begin{aligned}\left(\frac{4\tau}{(1+\tau)^2}\right)^k \binom{n}{k} (\coth r)^k &\le \frac{\int_{\partial\Omega} \binom{n}{k} H_k \,\mu}{\text{volume}(\partial\Omega)} \\ &\le \left(\frac{1+\tau}{2\sqrt{\tau}}\right)^k \binom{n}{k} \left(\coth\left(r + \ln\left(1 + \frac{2\sqrt{\tau}}{1+\tau}\right)\right)\right)^k.\end{aligned}$$

This proves the inequalities of Theorem 4. If we have a family $\Omega(t)$ of h-convex domains e.o.w.s., taking $\tau \to 1$ and $r \to \infty$ in the last inequalities we have

$$\lim_{t\to\infty} \frac{\int_{\partial\Omega} H_k \,\mu}{\text{volume}(\partial\Omega)} = 1,$$

which finishes the proof of Theorem 4 when $\partial\Omega$ is C^∞.

When $\partial\Omega$ is not C^∞, we first have to define $M_k(\partial\Omega)$. The motivation for our definition is similar to that in the euclidean space $\mathbb{R}^{n+1}$ (cf. [Sch, page 202], where the total k-mean curvature is called the k-th generalized measure curvature). In $\mathbb{R}^{n+1}$, the volume of a hypersurface F_t parallel to a convex hypersurface F at distance t is given by a polynomial in t of the form

$$\text{volume}(F_t) = \sum_{i=0}^{n} M_i \, t^i,$$

and the coefficients M_i are the total i-th mean curvatures of F.

To give the analog definition for convex hypersurfaces F in $\mathbb{H}^{n+1}$, let us consider the projective model (also called the Cayley-Klein or Beltrami model) for $\mathbb{H}^{n+1}$ (cf. [Ba]). In this model, the geodesic segments in $\mathbb{H}^{n+1}$ are straight line segments in a ball of $\mathbb{R}^{n+1}$. Then convex domains in $\mathbb{H}^{n+1}$ are convex domains in $\mathbb{R}^{n+1}$, and the following results for compact convex domains in $\mathbb{R}^{n+1}$ (cf. [BF] or [Sch]) are also true for compact convex domains in $\mathbb{H}^{n+1}$:

(i) Given a convex hypersurface $\partial\Omega$ in $\mathbb{H}^{n+1}$, there is a sequence $\{\partial\Omega^m\}$ of C^∞ convex hypersurfaces which converges (in the Hausdorff distance on the space of compact subsets in $\mathbb{H}^{n+1}$) to $\partial\Omega$.

(ii) If r_m and r are, respectively, the inradius of Ω^m and Ω, then $\lim_{m\to\infty} r_m = r$.

(iii) $\mathrm{Lim}_{m\to\infty} \text{volume}(\partial\Omega_t^m) = \text{volume}(\partial\Omega_t)$.

It is known that for any C^∞ hypersurface $\{\partial\Omega^m\}$ in $\mathbb{H}^{n+1}$ (see [Sa, page 321] and [Gr1, Th. 4.4]) we have

$$\text{volume}(\partial\Omega_t^m) = \sum_{k=0}^{n} M_k(\partial\Omega^m) \cosh^{n-k} t \sinh^k t, \tag{6.10}$$

where

$$M_k(\partial\Omega^m) = \int_{\partial\Omega^m} H_k\,\mu.$$

From (iii) and (6.10) it follows that

$$\text{volume}(\partial\Omega_t) = \sum_{k=0}^{n} M_k(\partial\Omega)\cosh^{n-k} t\,\sinh^k t, \tag{6.11}$$

and we define the total k-th mean curvature of $\partial\Omega$ as the coefficient $M_k(\partial\Omega)$ in (6.11). It is obvious that $M_k(\partial\Omega) = \lim_{m\to\infty} M_k(\partial\Omega^m)$ and $M_k(\partial\Omega)$ does not depend on the sequence $\partial\Omega^m$ converging to $\partial\Omega$.

Now, let us suppose that $\partial\Omega$ is h-convex. Although the $\partial\Omega^m$ may not be h-convex, since they converge to $\partial\Omega$, the infimum$\{\langle N, \partial_\ell\rangle(p_m),\ p_m \in \partial\Omega^m\}$ converges to the infimum$\{\langle N, \partial_\ell\rangle(p),\ p \in \partial\Omega\} = 2\sqrt{\tau}/(1+\tau)$, and the estimates of part (a) of Theorem 4 are still valid for $\partial\Omega^m$ if we substract ϵ_m from the lower bound and add ϵ_m to the upper bound, with $\epsilon_m \to 0$. This finishes the proof of Theorem 4 when $\partial\Omega$ is only an h-convex hypersurface with no regularity condition.

7. Relations with the intrinsic diameter

All the bounds given in Theorems 1–4 are stated as functions of the inradius of the h-convex domain Ω. Here we relate this inradius (and also the circumradius) with the intrinsic diameter of $\partial\Omega$.

PROPOSITION 7.1. *Let $\partial\Omega$ be a compact h-convex hypersurface of $\mathbb{H}^{n+1}$, and let Ω be the h-convex domain bounded by Ω. Let d be the intrinsic diameter of $\partial\Omega$, and r and τ be as in the preceeding sections. Then*

$$d \le \pi \sinh\left(r + \ln\frac{(1+\sqrt{\tau})^2}{1+\tau}\right).$$

Proof. Let o be the center of an inball of Ω. It follows from Theorem 1 that Ω is contained in the geodesic ball B with center o and radius $R_0 = r + \ln\frac{(1+\sqrt{\tau})^2}{1+\tau}$. The intrinsic diameter of the sphere ∂B, the boundary of B, is $\pi\sinh R_0$. For every point $x \in \partial B$, the orthogonal projection of x onto $\partial\Omega$ is the point $x' \in \partial\Omega$ such that $\text{dist}(x, \partial\Omega) = \text{dist}(x, x')$. Given any rectifiable curve α in ∂B and its orthogonal projection α' in $\partial\Omega$, it follows from the version for the hyperbolic space of the Busemann-Feller Lemma (cf. [MI]) that length$(\alpha') \le$ length(α), and it follows from this that $d \le \pi\sinh R_0$, as we wanted to prove. □

For the relation between d and the circumradius R, we shall use:

LEMMA ([De]). *Let S be a compact domain in $\mathbb{H}^{n+1}$ with extrinsic diameter D and circumradius R. Then*

$$\sqrt{\frac{2n}{n+1}}\sinh\frac{D}{2} \geq \sinh R.$$

With the help of this lemma we shall prove:

PROPOSITION 7.2. *Let $\partial\Omega$ be a compact h-convex hypersurface of $\mathbb{H}^{n+1}$, and let Ω be the h-convex domain bounded by Ω. Let d be the intrinsic diameter of $\partial\Omega$, and let R be the circumradius of Ω. Then*

$$\sqrt{\frac{2n}{n+1}}\frac{d}{2} \geq \sinh R.$$

Proof. Let D be the extrinsic diameter of $\partial\Omega$. Let p, $q \in \partial\Omega$ such that $\operatorname{dist}(p, q) = D$. From the h-convexity of Ω, it follows that $h(p, q) \subset \Omega$. From an argument using the version for the hyperbolic space of the Busemann-Feller Lemma as in Proposition 7.1, we get that the intrinsic diameter d_0 of $\partial h(p, q)$ satisfies $d_0 \leq d$. But d_0 is equal to the length of a horocycle segment from p to q. A computation using the model of the ball for $\mathbb{H}^{n+1}$, and parametrizing the horocycle segment pq by $((1+\tau)\sin t, -1+\tau+(1+\tau)\cos t)$ gives

$$d_0 = 2\int_0^{\arctan\frac{2\sqrt{\tau}}{1-\tau}} \frac{2(1+\tau)}{2-(\tau^2+1+(\tau^2-1)\cos t)}dt = \frac{4\sqrt{\tau}}{1-\tau}.$$

And, using the fact that $D = \operatorname{dist}(p, q) = 2\operatorname{dist}(o, p)$, which is given in the proof of Proposition 2.2, we have

$$\frac{D}{2} = \ln\frac{\sqrt{d_0^2+4}+d_0}{2} \leq \ln\frac{\sqrt{d^2+4}+d}{2},$$

and the result now follows from Dekster's Lemma. □

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