ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 1, Spring 1999

BOUNDED POINT EVALUATIONS FOR CERTAIN $P^t(\mu)$ SPACES

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ABSTRACT. The set of bounded point evaluations for $P^t(\mu)$ is determined for certain measures μ supported in the closed unit disk in the complex plane. Here $P^t(\mu)$ denotes the closure in $L^t(\mu)$ of the polynomials in z.

1. Introduction

For a positive measure μ with compact support in the complex plane $\mathbb{C}$ and for $1 \leq t < \infty$, let $P^t(\mu)$ denote the closure in $L^t(\mu)$ of the polynomials in z. A point w in $\mathbb{C}$ is a bounded point evaluation for $P^t(\mu)$ if there exists a constant $M > 0$ such that

$$|p(w)| \leq M\|p\|_{L^t(\mu)}$$

for each polynomial p. We denote the set of bounded point evaluations for $P^t(\mu)$ by $\text{bpe}(P^t(\mu))$.

The existence of bounded point evaluations and mean polynomial approximation have received a great deal of attention, culminating in a result of Thomson [27]: either $P^t(\mu) = L^t(\mu)$ or $P^t(\mu)$ has bounded point evaluations. In the latter case, $P^t(\mu)$ admits a structure related to that of $\text{bpe}(P^t(\mu))$. If $P^t(\mu)$ is irreducible, i.e., if $P^t(\mu)$ does not split into the direct sum of nontrivial spaces $P^t(\mu_1)$ and $P^t(\mu_2)$, then the set of bounded point evaluations for $P^t(\mu)$ is a simply connected region whose closure contains the support of μ.

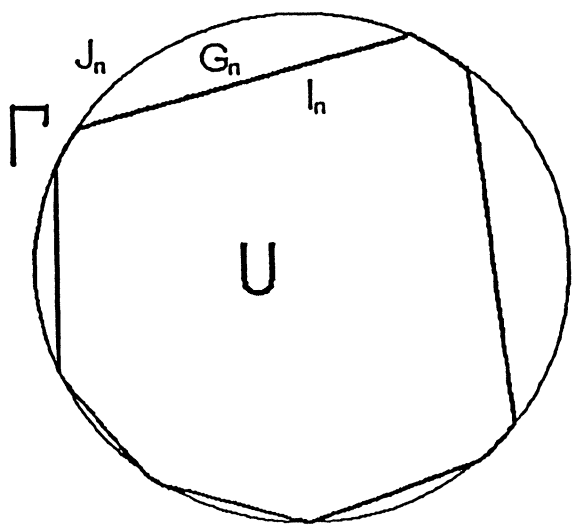
Thomson's dichotomy does not enable one however to determine the bounded point evaluations for an arbitrary measure μ; indeed, such a characterization seems generally out of reach. Nevertheless, for some natural classes of measures, it is possible to study the structure of the bounded point evaluations, and much interesting analysis has resulted. The canonical result in this vein is of course Szegö's Theorem, which characterizes the set of bounded point evaluations for a measure μ with support on the unit circle ∂D. In the case that μ is weighted area measure restricted to a bounded region, we mention work by Carleman [9], Keldyš [19], Džrbašjan and Šaginjan [13, p. 158], Havin [15], Havin and Maz'ja [16], [17], Šaginjan [24], Shapiro

Received September 17, 1997.
1991 Mathematics Subject Classification. Primary 46E30; Secondary 30H05, 30E10, 46E15
The third-named author was supported in part by a grant from the National Science Foundation and a seed-money grant from the University of Hawaii.

[25], and Brennan [6], [7]. Akeroyd [1], [2], [3] has dealt with harmonic measures for some crescent–like sets, and Hruscev [18], Kriete [20], Trent [28], Vol'berg [29], [30], and Kriete and MacCluer [21] are among those who have contributed in the case that μ consists of weighted Lebesgue measure on the unit circle and weighted area measure inside the unit disk. More recent work by Akeroyd [4], [5] is also relevant in this setting. The literature in this area is extensive, and our references above are by no means complete.

In the present paper, we continue the study of bounded point evaluations for μ of the form $d\mu = h\,dm + W\,dA|_D$, where m and $A|_D$ respectively denote Lebesgue measure on the unit circle and area measure on the unit disk D. Before stating our results, we need to introduce some notation.

Let K be a compact subset of the unit circle ∂D and let $\{J_n\}$ denote the components of $\partial D \setminus K$. For each n, let I_n denote the chord in the closed unit disk $\overline{D}$ with the same endpoints as J_n. Denote by G_n the region with boundary $J_n \cup I_n$; let Γ be the rectifiable Jordan curve $K \cup \bigcup_n \{I_n\}$ and U the interior of Γ.



Define the measure μ by

$$d\mu = h\,dm|_K + W\,dA|_{\cup G_n}$$

where h is a nonnegative integrable function satisfying $\int_K \log h\,dm > -\infty$ and W is a positive continuous function on each G_n such that $W \in L^1(A|_{\cup G_n})$. It follows that $G_n = \operatorname{bpe}(P^1(WA|_{G_n}))$. We should mention that if the function h is not log-integrable over K, an argument similar to that in [5, Theorem 2.2] yields

$$P^t(\mu) = P^t(WdA|_{\cup G_n}) \oplus L^t(hm|_K).$$

In this case, zero is not a bounded point evaluation for $P^t(\mu)$.

For simplicity, in this section we just give our characterization in the case that the weight W is identically 1. It will be stated and proved for more general weights in the next sections. Let $z_0 \in U$ and let ω be the harmonic measure for U at z_0. Denote by $|I_n|$ and $|J_n|$ the lengths of I_n and J_n, respectively.

THEOREM A. *Define the measure μ by $d\mu = h\,dm|_K + dA|_{\cup G_n}$, where h is a nonnegative integrable function satisfying $\int_K \log h\,dm > -\infty$. Then the following conditions are equivalent*:

(*a*) *z_0 is a bounded point evaluation for $P^t(\mu)$*;
(*b*) $\mathrm{bpe}(P^t(\mu)) = D$ *and $P^t(\mu)$ is irreducible*;
(*c*) *$P^t(\mu)$ does not split*;
(*d*) $\sum_{n=1}^{\infty} \omega(I_n) \log \frac{1}{|I_n|} < \infty$.

We note that the convergence of the series (d) is independent of $z_0 \in U$, since ω is comparable to harmonic measure at any other point $z_1 \in U$. We remark that the convergence of this series is very sensitive. An example was shown to us by F. Nazarov for which the addition of a single point to K changes the series (d) from convergent to divergent.

Condition (d) is related to the well-known Carleson condition (introduced in [10]) on the set K, which is that

$$\sum_{n=1}^{\infty} |J_n| \log \frac{1}{|J_n|} < \infty.$$

Notice that $\omega(I_n) \le C|I_n|$ and thus the Carleson condition implies that $\mathrm{bpe}(P^t(\mu)) = D$ for the measure μ in Theorem A. We will present an example showing that the Carleson condition is in fact strictly stronger than this.

THEOREM B. *There is a compact subset K of the unit circle with $m(K) > 0$ such that K satisfies Theorem A (d) but*

$$\sum_{n=1}^{\infty} |J_n| \log \frac{1}{|J_n|} = \infty.$$

Sufficient conditions for $P^t(\mu)$ to be irreducible will be stated and proved in §2, and the case that $P^t(\mu)$ splits will be considered in §3. Theorem A will be an immediate consequence, as the hypotheses for these theorems will be satisfied when the weight W is identically 1. Theorem B will be proved in §4.

2. $P^t(\mu)$ is irreducible

Let I_n, J_n, G_n, K and U be defined as in the introduction. Define the measure μ by

$$d\mu = h\,dm|_K + W\,dA|_{\cup G_n}$$

where h is a nonnegative integrable function satisfying $\int_K \log h\,dm > -\infty$ and W is a positive continuous function on each G_n such that $W \in L^1(A|_{\cup G_n})$. Let $\alpha_{t,n}(\lambda)$ be the norm of the kernel function of $P^t(WA|_{G_n})$ corresponding to $\lambda \in G_n$; i.e.,

$$\alpha_{t,n}(\lambda) = \sup |p(\lambda)| \left(\int_{G_n} |p|^t W\,dA\right)^{-\frac{1}{t}},$$

where the supremum is taken over all nonzero polynomials. For an open set G, we use $\delta_G(z)$ to denote the distance of z from the boundary of G.

THEOREM 2.1. *Suppose that K satisfies*

$$\sum_{n=1}^{\infty} \omega(I_n) \log \frac{1}{|I_n|} < \infty \tag{2.1}$$

and that $1 \leq t < \infty$ is such that for some s, $C > 0$,

$$\alpha_{t,n}(\lambda) \leq C\delta_{G_n}(\lambda)^{-s}$$

for every $n \geq 1$ and $\lambda \in G_n$. Then $P^t(\mu)$ is irreducible and $\mathrm{bpe}(P^t(\mu)) = D$.

Recall that ω is harmonic measure for U at a point $z_0 \in U$, and that convergence of the series (2.1) is independent of the choice of z_0. A typical example of a weight W to which the theorem applies is given by $W(z) = (1 - |z|^2)^\alpha$, where α is greater than -1; see Corollary 2.3 below.

A region G is said to satisfy a θ-wedge condition if there exists $r > 0$ and $\theta \in (0, 1)$ such that, for every $w \in \partial G$, a closed circular sector of radius r and opening $\theta\pi$ lies in $\bar{G}$, with vertex at w. In particular, it is clear that U satisfies a θ-wedge condition for some $\theta > 0$. We now fix such a θ.

Let $J_n' \subset U \setminus G_n$ be the arc of the circle connecting the endpoints of I_n and at the angle $\theta/2$ to I_n at each endpoint. We now fix $z_0 \in U$ such that J_n' separates z_0 from I_n, for all n. Choose a Riemann mapping φ from the unit disk D onto U such that $\varphi(z_0) = 0$, and let ψ be the inverse of φ. We will denote by C or c absolute constants that may change from one step to the next. Similarly, $C(z_0)$ will denote a quantity that depends at most on z_0, etc.

LEMMA 2.2. *For each n, there exists a smooth curve γ_n in G_n that joins the endpoints of J_n such that*

$$\int_{\gamma_n} \log \frac{1}{\delta_{G_n}(z)} d\omega_V \leq C \cdot \theta^{-2} \omega(I_n) \log \frac{1}{|I_n|}.$$

Here V is the region bounded by γ, where $\gamma = \bigcup_n \gamma_n \cup K$ and ω_V is the harmonic measure for V at z_0.

Proof. Let $\alpha_n = \psi(I_n)$ and $\beta_n = \psi(J_n')$. Let a_n, b_n be the endpoints of J_n, and set $A_n = \psi(a_n)$ and $B_n = \psi(b_n)$.

Claim 1.

$$\delta_D(z) \geq c \min\{|z - A_n|^{2/\theta}, |z - B_n|^{2/\theta}\}, \qquad z \in \beta_n.$$

Since U satisfies a θ-wedge condition, from Theorem 1 in [22] we get

$$|\varphi(z_1) - \varphi(z_2)| \leq C|z_1 - z_2|^{\theta}. \tag{2.2}$$

On the other hand, since U is convex, $\psi'(z)$ is bounded [23, p 225], so

$$|z_1 - z_2| \leq C|\varphi(z_1) - \varphi(z_2)|. \tag{2.3}$$

Now let $z \in \beta_n$, and assume without loss of generality that $|z - B_n| \leq |z - A_n|$. Then, using (2.2) and (2.3), there exists $z_1 \in \partial D$ such that

$$\begin{aligned}\delta_D(z) &= |z - z_1| \geq c|\varphi(z) - \varphi(z_1)|^{1/\theta} \geq c\delta_U(\varphi(z))^{1/\theta} \\ &\geq c|\varphi(z) - b_n|^{2/\theta} \geq c|z - B_n|^{2/\theta}.\end{aligned}$$

Claim 2. There exists a C^2 curve $\Gamma_n \subset \psi(G_n^0)$, where G_n^0 is bounded by J_n' and I_n, that joins A_n and B_n such that $\Gamma = \bigcup \Gamma_n \cup \psi(K)$ is a C^2 curve and

$$\delta_{\psi(G_n^0)}(z) \geq c \min\{|z - A_n|^{2/\theta}, |z - B_n|^{2/\theta}\}, \qquad z \in \Gamma_n.$$

To see the claim, let F_1 be a Riemann map from D to the upper half plane R_+^2 with $F_1(0) = i$ and F_1 mapping the middle point of α_1 to ∞. For $n \geq 2$, let $\alpha_n' = F_1(\alpha_n)$ and $\beta_n' = F_1(\beta_n)$. Let $A_n' = F_1(A_n)$ and $B_n' = F_1(B_n)$. Then it follows from Claim 1 that

$$\delta_{R_+^2}(z) \geq c \min\{|z - A_n'|^{2/\theta}, |z - B_n'|^{2/\theta}\}, \qquad z \in \beta_n'.$$

Let

$$\Gamma_n' = \{z = x + iy : \ y = c\frac{(x - A_n')^{2/\theta}(B_n' - x)^{2/\theta}}{(A_n' - B_n')^{2/\theta}}, \ B_n' \leq x \leq C_n'\}.$$

It is easy to check that $\Gamma' = \bigcup_{i=2}^{\infty} \Gamma_n' \cup F_1(\psi(K)) \cup \alpha_1$ is a C^2 curve and $\Gamma_n' \subset F_1(\psi(G_n^0))$. Let $\Gamma_n = F_1^{-1}(\Gamma_n')$. Let F_2 be another Riemann map from D to R_+^2 with $F_2(0) = i$ and F_2 mapping the middle point of $\partial D \setminus \alpha_1$ to ∞. Using the same method as above, we can construct Γ_1' as above such that $\Gamma_1' \cup F_2(\partial D \setminus \alpha_1)$ is smooth. Let

$\Gamma_1 = F_2^{-1}(\Gamma_1')$ and let $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n \cup \psi(K)$. Clearly, Γ satisfies the conditions of the claim.

Let $M_n = \varphi(\Gamma_n)$. Let γ_n be the reflection of M_n with respect to I_n and $\gamma = \bigcup \gamma_n \cup K$. From the construction, $\gamma \subset D \cup K$. Let V be the region bounded by γ. By the Schwarz reflection principle, we see that ψ extends to a Riemann map from V to $\psi(V)$ where the boundary of $\psi(V)$ is the reflection of Γ with respect to the unit circle. Hence the boundary of $\psi(V)$ is a C^2 curve.

Claim 3. For $z \in D$ we have

$$c \cdot \theta \log \frac{1}{\delta_D(z)} \le \log \frac{1}{\delta_U(\varphi(z))} \le C \log \frac{1}{\delta_D(z)}.$$

To see this, let $z_1 \in \partial D$ such that $\delta_D(z) = |z - z_1|$. Using (2.2) and (2.3), we have

$$\delta_D(z) \ge c|\varphi(z) - \varphi(z_1)|^{1/\theta} \ge c\delta_U(w)^{1/\theta},$$

and similarly, $\delta_U(w) \ge c\delta_D(z)$. Taking logarithms gives the claim.

Claim 4.

$$c \cdot \theta \log \frac{1}{\omega(I_n)} \le \log \frac{1}{|I_n|} \le C \log \frac{1}{\omega(I_n)}.$$

In fact, using (2.2) and (2.3), we get

$$c \log \frac{1}{|I_n|} \le \log \frac{1}{\omega(I_n)} = \log \frac{1}{|\alpha_n|} \le C \cdot \theta^{-1} \log \frac{1}{|I_n|}.$$

Let ω_0 be the harmonic measure for $\Omega = \psi(V)$ at zero. Let z^* be the reflection of the point z with respect to I_n and Γ_n^0 be the reflection of Γ_n with respect to the unit circle. Now

$$\int_{\gamma_n} \log \frac{1}{\delta_{G_n}(z)} d\omega_V \le C \int_{\gamma_n} \log \frac{1}{\delta_U(z^*)} d\omega_V \le C \int_{\gamma_n} \log \frac{1}{\delta_D(\psi(z^*))} d\omega_V,$$

where the last step is from Claim 3. Using Claim 2, we see that

$$\int_{\gamma_n} \log \frac{1}{\delta_D(\psi(z^*))} d\omega_V \le C\cdot\theta^{-1} \int_{\gamma_n} \left(\log \frac{1}{|\psi(z^*) - A_n|} + \log \frac{1}{|\psi(z^*) - B_n|}\right) d\omega_V. \tag{2.4}$$

Now, working with just the first term of this last integral, we estimate

$$\int_{\gamma_n} \log \frac{1}{|\psi(z^*) - A_n|} d\omega_V \le C \int_{\gamma_n} \log \frac{1}{|\psi(z) - A_n|} d\omega_V = C \int_{\Gamma_n} \log \frac{1}{|w - A_n|} d\omega_0,$$

where the change of variable $w = \psi(z)$ was used. Since the boundary of Ω is C^2, we see that $|w - A_n|$ is comparable to the arc length $s(w)$ from A_n to w on Γ_n, and $d\omega_0$ is comparable to ds. Therefore,

$$\int_{\Gamma_n} \log \frac{1}{|w - A_n|} d\omega_0 \leq C \int_{\Gamma_n} \log \frac{1}{s(w)} ds(w) \leq C|A_n - B_n| \log \frac{1}{|A_n - B_n|}$$
$$\leq C|\alpha_n| \log \frac{1}{|\alpha_n|}.$$

But $|\alpha_n| = \omega(I_n)$ and so from Claim 4 and the last two displays we get

$$\int_{\gamma_n} \log \frac{1}{|\psi(z^*) - A_n|} d\omega_V \leq C\omega(I_n) \log \frac{1}{\omega(I_n)} \leq C \cdot \theta^{-1} \omega(I_n) \log \frac{1}{|I_n|}.$$

Clearly the same estimate applies to the term in (2.4) involving B_n, and hence

$$\int_{\gamma_n} \log \frac{1}{\delta_{G_n}(z)} d\omega_V \leq C \cdot \theta^{-2} \omega(I_n) \log \frac{1}{|I_n|},$$

as required.

Proof of Theorem 2.1. Define

$$h_0 = \begin{cases} \delta_{G_n}^{ts}(z), & \text{for } z \in \gamma_n \\ \min(1, h), & \text{for } z \in K. \end{cases}$$

Since $V \subset D$, for each $\Delta \subset K$ we have

$$\omega_V(\Delta) = \omega_V(z_0, \Delta) \leq \omega_D(z_0, \Delta) \leq c \cdot m(\Delta)/(1 - |z_0|).$$

Then using Lemma 2.2 we get

$$\begin{aligned} \int \log h_0 d\omega_V &= ts \sum \int_{\gamma_n} \log \delta_{G_n} d\omega_V + \int_{K \cap \{h \leq 1\}} \log h \, d\omega_V \\ &\geq Cts \sum \omega(I_n) \log |I_n| + \frac{c}{1 - |z_0|} \int_{K \cap \{h \leq 1\}} \log h \, dm \\ &\geq Cts \sum \omega(I_n) \log |I_n| + \frac{c}{1 - |z_0|} \int_K \log h \, dm - \frac{c}{1 - |z_0|} \int h \, dm \\ &> -\infty, \end{aligned}$$

By Szegö's Theorem [14, p. 136], there is a constant $C > 0$ such that for each polynomial p,

$$|p(z_0)|^t \leq C \int |p|^t h_0 d\omega_V.$$

On the other hand, we are assuming that

$$\delta^{ts}(w)|p(w)|^t \leq C\int_{G_n} |p(z)|^t W dA(z) \leq C\int_D |p(z)|^t d\mu$$

for $w \in \gamma_n$. Therefore,

$$\begin{aligned}\int |p|^t h_0 d\omega_V &\leq C\sum\int_{\gamma_n} |p|^t \delta_{G_n}^{ts} d\omega_V + C\int |p|^t h\, d\omega_V \\ &\leq C\int_D |p|^t d\mu + C\int |p|^t h\, d\omega_V \\ &\leq C\int |p|^t d\mu. \end{aligned} \tag{2.5}$$

So z_0 is a bounded point evaluation for $P^t(\mu)$.

Since, for every $\lambda \in V$, harmonic measure for V at λ is comparable to ω_V, by Szegö's Theorem we have

$$|p(\lambda)|^t \leq C(\lambda)\int |p|^t h_0 d\omega_V.$$

Thus from (2.5) we see that $V \subset \text{bpe}(P^t(\mu))$. Since $G_n = \text{bpe}(P^t(WdA|_{G_n}) \subset \text{bpe}(P^t(\mu))$, we conclude that

$$D \subset \text{bpe}(P^t(\mu)).$$

Now suppose that $P^t(\mu)$ is not irreducible. By Thomson's theorem [27], there exists $E \subset K$ with $m(E) > 0$ such that $L^t(\mu|_E)$ is a summand of $P^t(\mu)$. Since $\omega_V(E) \geq \omega(E) > 0$, from (2.5) we see that the characteristic function of E is a nonzero element of $P^t(h_0 d\omega_V)$. Hence,

$$P^t(h_0 d\omega_V) = P^t(h_0 d\omega_V|_E) \oplus P^t(h_0 d\omega_V|_{E^c}).$$

This is a contradiction since, from Szegö's Theorem,

$$\text{bpe}(P^t(h_0 d\omega_V|_E)) = \text{bpe}(P^t(h_0 d\omega_V|_{E^c})) = \emptyset.$$

COROLLARY 2.3. *If K satisfies the condition* (2.1) *and $W(z) = (1-|z|^2)^\alpha$ for some $\alpha > -1$, then $P^t(\mu)$ is irreducible and* $\text{bpe}\big(P^t(\mu)\big) = D$.

Proof. Assume without loss of generality that $\alpha \geq 0$. Since $|p|^t$ is subharmonic for each polynomial p, for $\lambda \in \gamma_n$ we have

$$\begin{aligned}|p(\lambda)|^t &\leq \frac{C}{\delta_{G_n}^2(\lambda)}\int_{|z-\lambda|\leq\frac{3}{4}\delta_{G_n}(\lambda)} |p(z)|^t\, dA(z) \\ &\leq \frac{C}{\delta_{G_n}^{2+\alpha}(\lambda)}\int_{G_n} |p(z)|^t(1-|z|^2)^\alpha\, dA(z).\end{aligned}$$

Thus $\alpha_{t,n}(\lambda) \leq C\delta_{G_n}^{-s}$, with $s = \frac{2+\alpha}{t}$.

3. $P^t(\mu)$ splits

THEOREM 3.1. *Suppose that $W \in L^p(A|_{\cup G_n})$ for some $p > 1$. If K satisfies*

$$\sum_{n=1}^{\infty} \omega(I_n) \log \frac{1}{|I_n|} = \infty,$$

then

$$P^t(\mu) = L^t(\mu|_K) \oplus (\oplus P^t(\mu|_{G_n})).$$

If $m(K) = 0$, the proof of Theorem 3.1 will be similar to that of Theorem 5.7 in [7]. However if $m(K) \neq 0$ the Cauchy transform that we will use is no longer in a Sobolev space, and therefore some other ideas will have to be used.

Recall that φ is a Riemann map from D to U such that $\varphi(0) = z_0$. Define

$$\sigma_{z_0}(z) = \frac{z_0 - z}{1 - \bar{z}_0 z}.$$

LEMMA 3.2. *There exists a constant $0 < c_0 < \frac{1}{2}$ such that for each r sufficiently close to $1, r < 1$, there exits a smooth function τ_r, $0 \le \tau_r \le 1$ on $\mathbb{C}$ satisfying*

$$\tau_r(z) = \begin{cases} 0 & \text{if } |z| > 1, \\ 1 & \text{if } z \in \varphi(rD) \setminus \{|\sigma_{z_0}(z)| \le 2c_0\}, \\ 0 & \text{if } |\sigma_{z_0}(z)| < c_0, \end{cases}$$

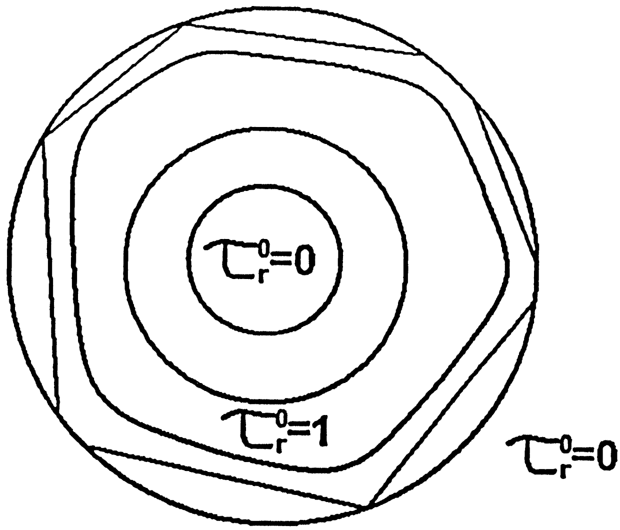
and such that

$$\left| \frac{\partial \tau_r}{\partial \bar{z}}(z) \right| \le C(z_0) \frac{1}{1 - |z|},$$

where $C(z_0)$ is independent of r.

Proof. Let r be close enough to 1 so that $\sigma_{z_0} \circ \varphi(rD)$ contains $2c_0 D$. From Schwarz's lemma, we see that $\sigma_{z_0} \circ \varphi(rD) \subset rD$. Let τ be a smooth function on R^1 such that $0 \le \tau \le 1$, $\tau(x) = 1$ for $x > 1$, $\tau(x) = 0$ for $x < 0$, and $0 \le \tau'(x) \le C$. Define

$$\tau_r^0(z) = \begin{cases} \tau\left(\frac{1-|z|}{1-r}\right) & \text{if } |z| > r \\ 1 & \text{if } 2c_0 < |z| \le r \\ \tau(2|z| - 2c_0) & \text{if } |z| \le 2c_0. \end{cases}$$



It is easy to check the function $\tau_r = \tau_r^0 \circ \sigma_{z_0}$ has the required properties.

For $q > 1$ and $f \in L^q(A)$, recall that the maximum function of f is defined by

$$M_f(\lambda) = \sup_{r>0} \frac{1}{A(O(\lambda, r))} \int_{O(\lambda,r)} |f(z)|\, dA(z),$$

where $O(\lambda, r)$ is the disk with center λ and radius r. It is well known that $f \mapsto M_f$ is bounded on $L^q(A)$; see for example [26, p. 5]. For $t \geq 1$, let t' be the conjugate exponent of t, so that $\frac{1}{t} + \frac{1}{t}' = 1$.

LEMMA 3.3. *Suppose that $1 \leq t$ and that $g \in L^{t'}(\mu)$ annihilates $P^t(\mu)$. Define k on $\mathbb{C}$ by $k|_{\cup G_n} = gW\chi_{\cup G_n}$. Let $H\colon U \to \mathbb{C}$ be given by*

$$H(w) = \int \left(\frac{w - \frac{1}{\bar{w}}}{z - \frac{1}{\bar{w}}}\right)^{N+1} \frac{k(z)}{z - w}\, dA(z),$$

where N is a positive integer. Then, there is a constant $C(N)$ so that for all $w \in U$,

$$|H(w)|, \left|\int \frac{(w - \frac{1}{\bar{w}})^{j+1}}{(z - \frac{1}{\bar{w}})^{j+2}} k(z)\, dA(z)\right| \leq C(N) M_k(w)(1 - |w|),$$

for all $j = 0, 1, 2, \ldots, N$.

Proof. Since the proofs of the two inequalities are similar, we show only that $|H(w)| \leq C(N)M_k(w)(1-|w|)$. For $w \in U$, let $\delta_w = |w - \frac{1}{\bar{w}}|$ and write

$$\begin{aligned} H(w) &= \int_{\{|z-w|<3\delta_w\}} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}}\right)^{N+1} \frac{k(z)}{z-w}\,dA(z) \\ &\quad + \int_{\{|z-w|\geq 3\delta_w\}} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}}\right)^{N+1} \frac{k(z)}{z-w}\,dA(z) \\ &= I_1 + I_2. \end{aligned}$$

The lemma is established by the following estimates for I_1 and I_2.

$$\begin{aligned} |I_1| &\leq \sum_{k=1}^{\infty} \left| \int_{\left\{\frac{1}{2^k}3\delta_w \leq |z-w| < \frac{1}{2^{k-1}}3\delta_w\right\}} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}}\right)^{N+1} \frac{k(z)}{z-w}\,dA(z) \right| \\ &\leq C(N) \sum_{k=1}^{\infty} \frac{2^k}{\delta_w} \int_{\left\{\frac{1}{2^k}3\delta_w \leq |z-w| < \frac{1}{2^{k-1}}3\delta_w\right\}} |k(z)|\,dA(z) \\ &\leq C(N)M_k(w)(1-|w|), \end{aligned}$$

and

$$\begin{aligned} |I_2| &\leq \sum_{k=1}^{\infty} \left| \int_{\{2^{k-1}3\delta_w \leq |z-w| < 2^k 3\delta_w\}} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}}\right)^{N+1} \frac{k(z)}{z-w}\,dA(z) \right| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^{(N+2)k}\delta_w} \int_{\{2^{k-1}3\delta_w \leq |z-w| < 2^k 3\delta_w\}} |k(z)|\,dA(z) \\ &\leq CM_k(w)(1-|w|). \end{aligned}$$

Suppose that g and H are as in the previous lemma. Let q satisfy

$$1 \leq q \leq \frac{t'p}{t'+p-1} \ (< t')$$

and let t'' satisfy $\frac{1}{t'} + \frac{1}{t''} = \frac{1}{q}$. Then

$$\frac{(q-1)t''}{q} + 1 \leq p.$$

Since $W \in L^p(A|_{\cup G_n})$ and $g \in L^{t'}(\mu)$, it follows from Hölder's inequality that

$$\|gW\|_{L^q(A|_{\cup G_n})} \leq \|g\|_{L^{t'}(\mu)} \left(\int |W|^{\frac{(q-1)t''}{q}+1} dA\right)^{\frac{1}{t''}}$$

Hence, $gW \in L^q(A|_{\cup G_n})$. From now on, we fix such a q with $1 < q < 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

For $0 < r < 1$, let ω_r be the harmonic measure on $U_r = \varphi(rD)$ evaluated at z_0, i.e., $\omega_r \circ \varphi = \frac{1}{2\pi r}\, d\theta$. By a change of variables,

$$\int_{\partial U_r} f\, d\omega_r = \frac{1}{2\pi i r} \int_{\partial U_r} f(z) \frac{(\varphi^{-1}(z))'}{\varphi^{-1}(z)}\, dz,$$

for all f continuous on ∂U_r.

LEMMA 3.4. *With the notation above, for each ϵ, $0 < \epsilon < \frac{1}{q'}$, there exists a constant $C(N, z_0, \mu) > 0$ (depending on N, z_0 and μ) so that*

$$\sup_{0<r<1} \int \frac{|H(z)|}{(1-|z|)^\epsilon}\, d\omega_r(z) \le C(N, z_0, \mu) \|g\|_{L^{t'}(\mu)}.$$

Proof. Let $0 < \epsilon < \frac{1}{q'}$ and let τ_r be the smooth positive function constructed in Lemma 3.2. Using Green's formula [14, p 26], we have

$$\begin{aligned} \int \frac{|H(z)|}{(1-|z|)^\epsilon} d\omega_r(z) &= \frac{1}{2\pi i r} \int_{\partial U_r} \frac{|\tau_r(z)H(z)|}{(1-|z|)^\epsilon} \frac{(\varphi^{-1}(z))'}{\varphi^{-1}(z)} dz \\ &= \frac{1}{\pi r} \int_{U_r} \bar\partial \left(\frac{|\tau_r(z)H(z)|}{(1-|z|)^\epsilon} \right) \frac{(\varphi^{-1}(z))'}{\varphi^{-1}(z)} dA(z). \end{aligned}$$

Since the function $\frac{(\varphi^{-1}(w))'}{\varphi^{-1}(w)}$ is bounded on the set $\{\tau_r > 0\} \cap U$, we conclude that

$$\int \frac{|H(z)|}{(1-|z|)^\epsilon}\, d\omega_r(z) \le C \int_{U_r} \left| \bar\partial \frac{\tau_r(z)|H(z)|}{(1-|z|)^\epsilon} \right| dA(z)$$

for some constant C.

For a compactly supported finite Borel measure ν, the Cauchy transform of ν is defined by

$$\hat\nu(\lambda) = \int \frac{1}{z-\lambda}\, d\nu(z).$$

The function $\hat\nu$ is locally integrable with respect to area measure. Let

$$k_r(w) = \tau_r(w)k(w) - \frac{1}{\pi}\widehat{kA}(w)\frac{\partial \tau_r}{\partial \bar w}.$$

It is well known that $\widehat{k_r A}(w) = \tau_r(w)\widehat{kA}(w)$; see for example [11] or [14, p. 50 Lemma 10.1]. Hence, the functions k_r and $\widehat{k_r A}$ each have compact support and are each members of $L^q(A)$. Therefore, by the Calderon–Zygmund theorem ([8] or [26, p. 35])

$$\|\mathrm{grad}(\widehat{k_r A})\|_q \le C\|k_r\|_q,$$

where $\operatorname{grad} f$ denotes the weak gradient of f. Thus $\widehat{k_r A}$ is in the Sobolev space $W_1^q = \{u \in L^q(A) : |\operatorname{grad}(u)| \in L^q(A)\}$. Notice that for $w \in D$,

$$\frac{1}{z-w} = \frac{1}{z-\frac{1}{\bar{w}}} \sum_{j=0}^{N} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}} \right)^j + \frac{(w-\frac{1}{\bar{w}})^{N+1}}{(z-\frac{1}{\bar{w}})^{N+1}} \frac{1}{z-w}. \tag{3.1}$$

We write

$$\begin{aligned} \tau_r(w)H(w) &= \tau_r(w)\widehat{kA}(w) - \tau_r(w) \int \frac{k(z)}{z-\frac{1}{\bar{w}}}\, dA(z) \\ &\quad - \tau_r(w) \int \frac{k(z)}{z-\frac{1}{\bar{w}}} \sum_{j=1}^{N} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}} \right)^j dA(z) \\ &= \widehat{k_r A}(w) + F(w)\tau_r(w). \end{aligned}$$

From the construction of τ_r, it follows that $\tau_r F$ is in W_1^q, and therefore so is $\tau_r H$. On the other hand, by Lemma 3.3, we get

$$|\bar{\partial} F(w)| \le C(N) M_k(w) \qquad \text{for } |w| \ge \frac{1}{2}.$$

It is easy to check that

$$\bar{\partial}(\tau_r H) = -\pi k \tau_r + \widehat{kA}(w) \frac{\partial \tau_r}{\partial \bar{w}} + \bar{\partial} F \tau_r + F(w) \bar{\partial} \tau_r = -\pi k \tau_r + \bar{\partial} F \tau_r + H(w) \bar{\partial} \tau_r.$$

Using a theorem from [26, p. 77] , we have

$$\begin{aligned} \|\bar{\partial}(\tau_r |H|)\|_q &\le \left\| \frac{\partial}{\partial x}(|\tau_r H|) \right\|_q + \left\| \frac{\partial}{\partial y}(|\tau_r H|) \right\|_q \\ &\le \left\| \frac{\partial}{\partial x}(\tau_r H) \right\|_q + \left\| \frac{\partial}{\partial y}(\tau_r H) \right\|_q \\ &\le C \|\bar{\partial}(\tau_r H)\|_q. \end{aligned}$$

Now using Lemmas 3.2 and 3.3, we see that

$$\begin{aligned} \int \frac{|H(z)|}{(1-|z|)^\epsilon}\, d\omega_r(z) &\le C \int_{U_r} \frac{|\bar{\partial}(\tau_r |H|)|}{(1-|z|)^\epsilon}\, dA(z) + C \int_{U_r} \frac{\tau_r |H|}{(1-|z|)^{1+\epsilon}}\, dA(z) \\ &\le C \|\bar{\partial}(\tau_r |H|)\|_q \left\| \frac{1}{(1-|w|)^\epsilon} \right\|_{q'} + C \int_{U_r} \frac{\tau_r |H|}{(1-|z|)^{1+\epsilon}}\, dA(z) \\ &\le C \left(\int |\bar{\partial}(\tau_r H)|^q\, dA \right)^{\frac{1}{q}} + C(N) \int \frac{\tau_r(z) M_k(z)}{(1-|z|)^\epsilon}\, dA(z) \end{aligned}$$

$$
\begin{aligned}
&\le C(N)\|k\|_q + C\|\bar{\partial}\tau_r H\|_q \\
&\quad + C(N)\|\tau_r\bar{\partial}F\|_q + C(N)\|\tau_r M_k\|_q \\
&\le C(N, z_0)\|k\|_q \\
&\le C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)}.
\end{aligned}
$$

Here we used the fact that the Maximum Operator is bounded on $L^q(A)$. The lemma is now established.

LEMMA 3.5. *Suppose that $1 \le t < \infty$ and $g \in L^{t'}(\mu)$. If $g \perp P^t(\mu)$, then there is a constant C so that*

$$\sup_{0<r<1} \int |\widehat{g\mu}(z)|\, d\omega_r(z) \le C(N, z_0)\|g\|_{L^{t'}(\mu)}.$$

Proof. Using (3.1), for $g \perp P^t(\mu)$ we get

$$
\begin{aligned}
\widehat{g\mu}(w) &= \int \left(\frac{1}{z-w} - \frac{1}{z-\frac{1}{\bar{w}}} \sum_{j=0}^{N} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}} \right)^j \right) g(z)d\mu \\
&= H(w) + \int_K \left(\frac{1}{z-w} - \frac{1}{z-\frac{1}{\bar{w}}} \sum_{j=0}^{N} \left(\frac{w-\frac{1}{\bar{w}}}{z-\frac{1}{\bar{w}}} \right)^j \right) g(z)d\mu \\
&= H(w) + G(w).
\end{aligned}
$$

First, notice that

$$\frac{|w-\frac{1}{\bar{w}}|^j}{|z-\frac{1}{\bar{w}}|^j |z-w|} \le 2^{j-1} P_w, \qquad 1 \le j \le N,$$

where $P_w(z)$ is the Poisson kernel for w. We bound G as follows.

$$
\begin{aligned}
|G(w)| &\le \int_K P_w(z)|g(z)|h(z)\, dm(z) + \int_K \sum_{j=1}^{N} \frac{|w-\frac{1}{\bar{w}}|^j}{|z-\frac{1}{\bar{w}}|^{j+1}} |g(z)|h(z)\, dm(z) \\
&\le \int_K P_w(z)|g(z)|h(z)\, dm(z) \\
&\quad + \int_K \sum_{j=1}^{N} \frac{|w-\frac{1}{\bar{w}}|^j}{|z-\frac{1}{\bar{w}}|^j |z-w|} |w| \left| \frac{z-w}{1-\bar{w}z} \right| |g(z)|h(z)\, dm(z) \\
&\le 2^N \int_K P_w|g|h\, dm \\
&= 2^N u(w),
\end{aligned}
$$

where $u(w)$ is a positive harmonic function. By Lemma 3.4, we may conclude that

$$\begin{aligned}\int |\widehat{g\mu}(w)|d\omega_r &\leq \int |H(w)|d\omega_r + 2^N \int u(w)d\omega_r \\ &\leq C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)} + 2^N u(0) \\ &\leq C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)}.\end{aligned}$$

Now we fix N such that $\frac{1}{N} < \theta$. Let $e^{i\theta_n}$ be the midpoint of J_n and let

$$\Sigma_n = \left\{z \in D, -\frac{|J_n|}{2} + \theta_n + |I_n|^N \leq \arg(z) \leq \frac{|J_n|}{2} + \theta_n - |I_n|^N\right\}$$

and let

$$\Gamma_{nr} = \Sigma_n \cap \partial\varphi(rD).$$

LEMMA 3.6. *Let G be as in the proof of the previous corollary. Then there exists an absolute constant $C(N) > 0$ such that for $w \in \Gamma_{n1}$ we have*

$$|G(w)| \leq C(N)|J_n|\|g\|_{L^{t'}(\mu)}.$$

Proof. From the definition of G,

$$|G(w)| \leq \int_K \left|\frac{w - \frac{1}{\bar{w}}}{z - \frac{1}{\bar{w}}}\right|^{N+1} \frac{|g(z)|h(z)}{|z - w|}|dz|$$

where $g \in L^{t'}(\mu)$. Since for $w \in \Gamma_{n1}$ and $z \in K$ we have $|w - \frac{1}{\bar{w}}| \leq C|J_n||z - \frac{1}{\bar{w}}|$, and $|z - w| \geq c|J_n|^N$, the desired estimate is obtained.

Proof of Theorem 3.1. Let $g \in L^{t'}(\mu)$ be an annihilator of $P^t(\mu)$ and $\epsilon < \min(\frac{1}{q'}, \frac{1}{N})$. Fixing an M, we have

$$\begin{aligned}\int \log|\widehat{g\mu}|\, d\omega_r &\leq \int_{\cup_{i=1}^M \Gamma_{ir}} \log|\widehat{g\mu}|\, d\omega_r + \int |\widehat{g\mu}|\, d\omega_r \\ &\leq \int_{\cup_{i=1}^M \Gamma_{ir}} \log\frac{|\widehat{g\mu}|}{(1 - |z|)^\epsilon}\, d\omega_r \\ &\quad + \epsilon\int_{\cup_{i=1}^M \Gamma_{ir}} \log(1 - |z|)\, d\omega_r + \int |\widehat{g\mu}|\, d\omega_r.\end{aligned}$$

Hence, by Lemma 3.5 and the subharmonicity of $\log|\widehat{g\mu}|$, we get

$$\begin{aligned}\log|\widehat{g\mu}(z_0)| \leq{}& \int_{\cup_{i=1}^M \Gamma_{ir}} \log\frac{|\widehat{g\mu}|}{(1 - |z|)^\epsilon}\, d\omega_r + \epsilon\int_{\cup_{i=1}^M \Gamma_{ir}} \log(1 - |z|)\, d\omega_r \\ &+ C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)}.\end{aligned}$$

Thus, from Lemma 3.4 and letting $r \to 1$, we see that

$$\log|\widehat{g\mu}(z_0)| \le \int_{\cup_{i=1}^M \Gamma_{i1}} \frac{|G(z)|}{(1-|z|)^\epsilon}\,d\omega + \epsilon \int_{\cup_{i=1}^M \Gamma_{i1}} \log(1-|z|)\,d\omega + C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)}.$$

On the other hand, using Lemma 3.6 and the inequality $N\epsilon < 1$, we have

$$\int_{\cup_{i=1}^M \Gamma_{i1}} \frac{|G(z)|}{(1-|z|)^\epsilon}\,d\omega \le \sum_{i=1}^M |J_n|^{1-N\epsilon}\omega(\Gamma_{i1})\|g\|_{L^{t'}(\mu)} \le C(N)\|g\|_{L^{t'}(\mu)}.$$

Hence,

$$\begin{aligned}\log|\widehat{g\mu}(z_0)| &\le \epsilon \int_{\cup_{i=1}^M \Gamma_{i1}} \log(1-|z|)\,d\omega + C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)} \\ &\le C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)} + C(N)\sum_1^M \omega(\Gamma_{i1})\log|I_i|.\end{aligned}$$

From our construction, we see that

$$\omega(I_i \setminus \Gamma_{i1}) \le C(z_0)|I_i \setminus \Gamma_{i1}| \le C(z_0)|I_i|^N.$$

On the other hand, U satisfies the θ-wedge condition. Using Theorem 1 in [22] and (2.2), since $N > \frac{1}{\theta}$, we conclude that

$$\omega(I_i) \ge c(z_0)|I_i|^{\frac{1}{\theta}} \ge 2\omega(I_i \setminus \Gamma_{i1}),$$

for i sufficiently large. Hence, for such i,

$$\omega(I_i) \le 2\omega(\Gamma_{i1}),$$

and therefore,

$$\log|\widehat{g\mu}(z_0)| \le C(N, z_0, \mu)\|g\|_{L^{t'}(\mu)} + C(N)\sum_1^M \omega(I_i)\log|I_i|.$$

Now, letting $M \to \infty$, we see that

$$\widehat{g\mu}(z_0) = 0.$$

The same method shows that $\widehat{g\mu}(\lambda) = 0$ for each $\lambda \in U$. Thus $\frac{1}{z-\lambda} \in P^t(\mu)$. Therefore, no point of U is a bounded point evaluation for $P^t(\mu)$. Now the theorem follows from Thomson's theorem in [27].

Theorem A is a direct conclusion of Corollary 2.3 and Theorem 3.1, with the weight W identically 1.

4. Proof of Theorem B

In this section, we will construct an example for which the Carleson condition fails but

$$\sum_{n=1}^{\infty} \omega(I_n) \log \frac{1}{|I_n|} < \infty.$$

Let $\{E_n\}$ with $|E_n| = \frac{1}{3^n}$ be a sequence of arcs in the unit circle and $\{E'_n\}$ be the corresponding chords whose positions will be chosen. Let $\{F_n\}$ be another sequence of arcs such that F_n has an endpoint in common with E_n. Let $\{F'_n\}$ be the corresponding chords with

$$|F'_n| = |E'_n|^{2\pi|E_n|^{-1}}.$$

Now we can choose the positions of $\{E_n\}$ and $\{F_n\}$ such that

$$(E_n \cup F_n) \cap (E_m \cup F_m) = \emptyset \qquad n \neq m.$$

Choose an integer N_n such that

$$1 \leq |F'_n| \log \frac{N_n}{|F'_n|} \leq 2.$$

We divide F_n into N_n equal pieces denoted by $\{F_{nj}\}_{1\leq j\leq N_n}$. Let $\{F'_{nj}\}_{1\leq j\leq N_n}$ be the corresponding chords. Let U be the region bounded by $\cup E'_n \cup F'_{nj} \cup (\partial D \setminus \cup E_n)$.

LEMMA 4.1. *Let U be as above and ω be the harmonic measure for U at zero. Then the following conditions hold*:
(*i*)

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} |F'_{nj}| \log \frac{1}{|F'_{nj}|} = \infty,$$

that is, $\partial D \setminus (\cup E_n \cup F_n)$ does not satisfies the Carleson condition, and
(*ii*)

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N_n} \omega(F'_{nj}) \log \frac{1}{|F'_{nj}|} + \sum_{n=1}^{\infty} \omega(E'_n) \log \frac{1}{|E'_n|} < \infty.$$

Proof. Since

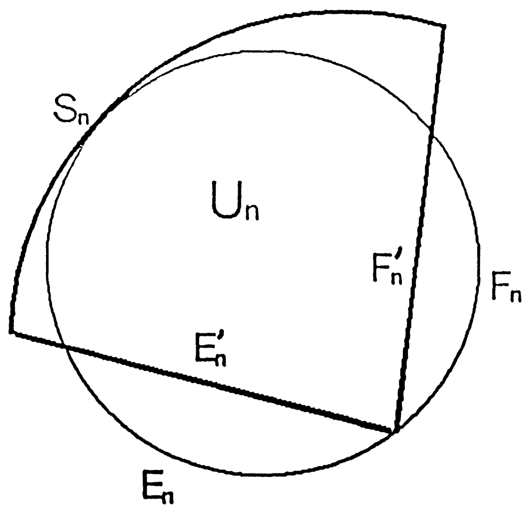
$$\sum_{j=1}^{N_n} |F'_{nj}| \log \frac{1}{|F'_{nj}|} \geq c|F'_n| \log \frac{N_n}{|F'_n|} \geq c,$$

it follows that (i) holds.

Let G_n be the region bounded by F_n' and F_n. Let $U_n = U \setminus \bar{G}_n$. Let ω_n be the harmonic measure for U_n at zero. We claim that

$$\omega_n(F_n') \leq C|F_n'||E_n|.$$

Let S_n be the sector with center at the common endpoint of E_n and F_n and radius 2 such that $F_n' \cup E_n' \subset \partial S_n$.



Let ω_{S_n} be the harmonic measure for S_n at zero. Since $U_n \subset S_n, \omega_n(F_n') \leq \omega_{S_n}(F_n')$. Let d_n be the common point of E_n and F_n. Then

$$g(z) = (z - d_n)^{\frac{\pi}{\pi - (|E_n| + |F_n|)/2}}$$

is a Riemann map from S_n to a half disk with radius between 2 and 3. Hence,

$$\omega_{S_n}(F_n') \leq C|F_n'|^{\frac{\pi}{\pi-(|E_n|+|F_n|)/2}} \leq C|F_n'||F_n'|^{\frac{|E_n|}{2\pi}} \leq C|F_n'||E_n|,$$

and the claim is established.

Clearly,

$$\omega(\cup_{j=1}^{N_n} F_{nj}') \leq \omega_n(F_n') \leq C|F_n'||E_n|.$$

On the other hand,

$$\sum_{j=1}^{N_n} \omega(F_{nj}') \log \frac{1}{|F_{nj}'|} \leq \omega(\cup_{j=1}^{N_n} F_{nj}') \log \frac{N_n}{|F_n'|} \leq C|E_n||F_n'| \log \frac{N_n}{|F_n'|} \leq C|E_n|.$$

Therefore, condition (ii) holds.

The following theorem is an immediate consequence of Theorem A and Lemma 4.1.

THEOREM 4.2. *Let U be as in Lemma 4.1 and let the measure μ satisfy the hypotheses of Theorem A. Then*

$$\sum_{n=1}^{\infty} |I_n| log \frac{1}{|I_n|} = \infty$$

and bpe $(P^t(\mu)) = D$, *for* $1 \leq t < \infty$.

Acknowledgment. We would like to thank James Brennan for pointing out an error in an earlier version of this paper. It was in correcting this error that we constructed the counterexample in Section 4 and we came to the final formulation of Theorem 3.1.

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