MARKOV-TYPE INEQUALITIES FOR CONSTRAINED POLYNOMIALS WITH COMPLEX COEFFICIENTS

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Dedicated to the memory of Professor Paul Erdős

1. Introduction, notation

We introduce the following classes of polynomials. Let

$$\mathcal{P}_n := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i, \ a_i \in \mathbb{R} \right\}$$

denote the set of all algebraic polynomials of degree at most n with real coefficients. Let

$$\mathcal{P}_n^c := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i, \ a_i \in \mathbb{C} \right\}$$

denote the set of all algebraic polynomials of degree at most n with complex coefficients.

Let $\mathcal{P}_{n,k}$ denote the set of all polynomials of degree at most n with *real coefficients* and with at most k ($0 \le k \le n$) zeros in the open unit disk.

Let $\mathcal{P}_{n,k}^c$ denote the set of all polynomials of degree at most n with *complex coefficients* and with at most k ($0 \le k \le n$) zeros in the open unit disk.

Let $\mathcal{P}_n(r)$ denote the set of all polynomials of degree at most n with *real coefficients* and with no zeros in the union of open disks with diameters $[-1, -1 + 2r]$ and $[1 - 2r, 1]$, respectively ($0 < r \le 1$).

Let $\mathcal{P}_n^c(r)$ denote the set of all polynomials of degree at most n with *complex coefficients* and with no zeros in the union of open disks with diameters $[-1, -1+2r]$ and $[1 - 2r, 1]$, respectively ($0 < r \le 1$).

The following two inequalities are well known in approximation theory. See, for example, A.A. Markov [89], V.A. Markov [16], Duffin and Schaeffer [41], Bernstein [58], Cheney [66], Lorentz [86], DeVore and Lorentz [93], Natanson [64] (some of these references discuss only the case when the polynomial has real coefficients).

MARKOV INEQUALITY. *The inequality*

$$\|p'\|_{[-1,1]} \le n^2 \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n^c$.

BERNSTEIN INEQUALITY. *The inequality*

$$|p'(y)| \le \frac{n}{\sqrt{1 - y^2}} \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n^c$ *and* $y \in (-1, 1)$.

In the above two theorems and throughout the paper $\| \cdot \|_A$ denotes the supremum norm on $A \subset \mathbb{R}$.

Markov- and Bernstein-type inequalities in L_p norms are discussed, for example, in DeVore and Lorentz [93], Lorentz, Golitschek, and Makovoz [96], Golitschek and Lorentz [89], Nevai [79], Máté and Nevai [80], Rahman and Schmeisser [83], Milovanović, Mitrinović, and Rassias [94].

Throughout his life Erdős showed a particular fascination with inequalities for constrained polynomials. One of his favorite type of polynomial inequalities was Markov- and Bernstein-type inequalities. For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest. He liked to see what happened when the polynomials are restricted in certain ways. Markov- and Bernstein-type inequalities for classes of polynomials under various constraints have attracted a number of authors. For example, it has been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. He proved that if n is odd, then

$$\sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} = \left(\frac{n + 1}{2}\right)^2,$$

where the supremum is taken for all $0 \ne p \in \mathcal{P}_n$ that are monotone on $[-1, 1]$. This may look quite surprising, since one would expect that if a polynomial is this far away from the "equioscillating" property of the Chebyshev polynomial, then there should be a more significant improvement in the Markov inequality. In a short paper in 1940, Erdős [40] has found a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(y)| \le \min\left\{\frac{c\sqrt{n}}{\left(1 - y^2\right)^2}, \frac{en}{2}\right\} \|p\|_{[-1,1]}, \qquad y \in [-1, 1],$$

for every polynomial p of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1, 1)$. This result motivated a number of people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints.

Generalizations of the above Markov- and Bernstein-type inequalities of Erdős have been extended later in many directions.

After a number of less general results of Erdős [40], Lorentz [63], Scheick [72], Szabados and Varma [80], Szabados [81], Máté [81], the essentially sharp Markov-type estimate

$$c_3 n(k+1) \leq \sup_{0 \neq p \in \mathcal{P}_{n,k}} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_4 n(k+1), \tag{1.1}$$

with absolute constants $c_3 > 0$ and $c_4 > 0$, was proved by Borwein [85] (in a slightly less general formulation) and by Erdélyi [87a] (in the above form). A simpler proof is given by Erdélyi [91] that relates the upper bound in (1.1) to a beautiful Markov-type inequality of Newman [76] for Müntz polynomials. See also Borwein and Erdélyi [95a] and Lorentz, Golitschek, and Makovoz [96]. A sharp extension of (1.1) to L_p norms is also proved by Borwein and Erdélyi [95b]. The lower bound in (1.1) was proved and the upper bound was conjectured by Szabados [81] earlier. Another example that shows the lower bound in (1.1) is given by Erdélyi [87b].

The following essentially sharp Markov-type inequality of Erdélyi [89] for the class $\mathcal{P}_n(r)$, that was anticipated by Erdős, is discussed in the recent book of Lorentz, Golitschek, and Makovoz [96] in a more general setting. Namely there are absolute constants $c_3 > 0$ and $c_4 > 0$ such that

$$c_3 \min\left\{\frac{n}{\sqrt{r}}, n^2\right\} \leq \sup_{0 \neq p \in \mathcal{P}_n(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_4 \min\left\{\frac{n}{\sqrt{r}}, n^2\right\}. \tag{1.2}$$

In this paper we examine what happens if in (1.1) and (1.2) we allow polynomials with complex rather than real coefficients. The "right" analogous results of (1.1) and (1.2) for the complex classes $\mathcal{P}_{n,k}^c$ and $\mathcal{P}_n^c(r)$ are established.

2. New results

Our first theorem is the "right" analogue of (1.1) for polynomials with complex coefficients.

THEOREM 2.1.   *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 n \max\{k+1, \log n\} \leq \sup_{0 \neq p \in \mathcal{P}_{n,k}^c} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 n \max\{k+1, \log n\}.$$

Our second result is the "right" analogue of (1.2) for polynomials with complex coefficients.

THEOREM 2.2.   *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\frac{c_1 n \log\left(n\sqrt{r}\right)}{\sqrt{r}} \leq \sup_{0 \neq p \in \mathcal{P}_n^c(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq \frac{c_2 n \log\left(n\sqrt{r}\right)}{\sqrt{r}}$$

for every $(e/n)^2 \leq r \leq 1$, *and*

$$c_1 n^2 \leq \sup_{0 \neq p \in \mathcal{P}_n^c(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 n^2$$

for every $0 < r < (e/n)^2$.

Remark 2.3. Theorems 2.1 and 2.2 should be compared with their cousins (1.1) and (1.2) in the real case. It may be surprising that if $k = o(\log n)$, we have an essentially different Markov-type inequality for $\mathcal{P}_{n,k}^c$ than for $\mathcal{P}_{n,k}$ (a similar comment can be made on comparing Theorem 2.2 and (1.2)). However, a closer look at the problem suggests that the real surprise should be the fact that if $\log n \leq k \leq n$, we have essentially the same Markov-type inequalities for $\mathcal{P}_{n,k}$ and $\mathcal{P}_{n,k}^c$. Indeed, the "standard" argument to derive Markov's inequality for $\mathcal{P}_n^c$ from Markov's inequality for $\mathcal{P}_n$ goes as follows. Suppose

$$\|q'\|_{[-1,1]} \leq n^2 \|q\|_{[-1,1]}$$

for every $q \in \mathcal{P}_n$. Now let $p \in \mathcal{P}_n^c$ be arbitrary. Fix an arbitrary point $a \in [-1, 1]$, and choose a constant $c \in \mathbb{C}$ with $|c| = 1$ so that $cp'(a)$ is real. We introduce $q \in \mathcal{P}_n$ defined by

$$q(x) := \mathrm{Re}(cp(x)), \qquad x \in \mathbb{R}.$$

Then

$$|p'(a)| = |cp'(a)| = |q'(a)| \leq n^2 \|q\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}.$$

Since this holds for every $p \in \mathcal{P}_n^c$ and $a \in [-1, 1]$, we have

$$\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$.

Observe that, while $p \in \mathcal{P}_n^c$ implies $q := \mathrm{Re}(cp) \in \mathcal{P}_n$, $p \in \mathcal{P}_{n,k}^c$ does not imply that $q := \mathrm{Re}(cp) \in \mathcal{P}_{n,k}$. This suggests that in order to establish the "right" Markov-type inequalities for $\mathcal{P}_{n,k}^c$, the arguments need to be more clever than the above standard extension.

Remark 2.4. The case $k = 0$ of Theorem 2.1 was first observed by Halász, who mentioned this to me in a private letter. See also Borwein and Erdélyi [95a], where a modified version of Halász' argument is presented. Halász also claims an independent proof of Theorem 2.1 using potential theoretic methods. After a personal discussion about the possibility of extending the case $k = 0$ to the general case $0 \leq k \leq n$, we worked on the problem simultaneously and we obtained our result at about the same time. Halász' approach may be presented in one of his later publications. Moreover, his methods give the $k = 0$ case of the conjectured Markov-type inequality

$$\|p'\|_{[-1,1]} \leq C_k n^{2-\alpha} \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$ that has at most k zeros in the diamond of the complex plane with diagonal $[-1, 1]$ and with angle $\alpha\pi \in [0, \pi]$ at -1 and 1 (C_k is a constant depending only on k).

Remark 2.5.   What do the extremal polynomials $p^* \in \mathcal{P}_n^c$ look like, say in Theorem 2.1? We cannot say much about the characterization of the extremal polynomials. Some properties may be suspected from those of the quasi-extremal polynomials $p \in \mathcal{P}_n^c$ showing the lower bound in Theorem 1.2. These polynomials are given explicitly in the proof. This phenomenon is in contrast to the real case where we can characterize the polynomials $p^* \in \mathcal{P}_n$ for which

$$\frac{|p^{*'}(1)|}{\|p^*\|_{[-1,1]}} = \sup_{p \in \mathcal{P}_n} \frac{|p'(1)|}{\|p\|_{[-1,1]}}.$$

It can be shown easily that such a $p^* \in \mathcal{P}_n$ must have only real zeros and at least $n - k - 1$ of these zeros must be at -1. In addition, roughly speaking, the extremal polynomial $p^* \in \mathcal{P}_n$ is "very close" to being an incomplete Chebyshev polynomial, that is, to being a polynomial that has $n - k$ zeros at -1 and "equioscillates" the maximal number of times (that is $k + 1$ times) on the interval $[-1, 1]$. See more about incomplete Chebyshev polynomials in Chapter 3 of Lorentz, Golitschek, and Makovoz [96].

Remark 2.6.   The crucial idea to prove both Theorem 2.1 and 2.2 is a combination of a Chebyshev-type inequality and Nevanlinna's inequality. The Chebyshev-type inequality gives an upper bound for the modulus of a polynomial $p \in \mathcal{P}_n^c$ on the real line assuming that $\|p\|_{[-1,1]} \leq M$. Combining this with Nevanlinna's inequality offers an upper bound for $|p|$ in complex neighborhoods of 1 and -1, assuming that the $|p|$ is bounded by M on the interval $[-1, 1]$. Finding the "right" neighborhoods of 1 and -1 where $|p(z)|$ is bounded by cM allows us to give an upper bound for $|p'(1)|$ and $|p'(-1)|$ by the Cauchy Integral Formula. The desired upper bound for $|p'(z)|$, $z \in [-1, 1]$, can now be obtained by a linear transformation.

Remark 2.7.   The inequality

$$\|p^{(m)}\|_{[-1,1]} \leq T_n^{(m)}(1) \cdot \|p\|_{[-1,1]}$$

for every polynomial p of degree at most n with complex coefficients was first proved by V.A. Markov [92] in 1892 (here T_n denotes the Chebyshev polynomial of degree n). He was the brother of the more famous A.A. Markov who proved the above inequality for $m = 1$ in 1889 by answering a question raised by the prominent Russian chemist, D. Mendeleev. S.N. Bernstein presented a shorter variational proof of V.A. Markov's inequality in 1938 (see the collected works of Bernstein [58]). The simplest known proof of Markov's inequality for higher derivatives are due to Duffin and Shaeffer [41], who gave various extensions as well.

Various analogues of the Markov and Bernstein inequalities are known in which the underlying intervals, the maximum norms, and the family of functions are replaced by more general sets, norms, and families of functions, respectively. These inequalities are called Markov- and Bernstein-type inequalities. If the norms are the same in both sides, the inequality is called Markov-type; otherwise it is called Bernstein-type (this distinction is not completely standard). Markov- and Bernstein-type inequalities·are known on various regions of the complex plane and the n-dimensional Euclidean space, for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints, exponential sums of n terms, just to mention a few. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, they play a fundamental role in proving so-called inverse theorems of approximation.

There are many books discussing Markov- and Bernstein-type inequalities in detail. See for example Cheney [66], Lorentz [86], DeVore and Lorentz [93], and Lorentz, Golitschek, and Makovoz [96].

Remark 2.8.   It is not that hard to see that our proof of Theorem 2.1 can be extended to higher derivatives. That is, there are constants $C_m > 0$ and $C'_m > 0$ such that for every integer $0 \leq m \leq n$, we have

$$C_m \left(\max\{k+1, \log n\}\right)^m \leq \sup_{0 \neq p \in \mathcal{P}^c_{n,k}} \frac{\|p^{(m)}\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq C'_m \left(n \max\{k+1, \log n\}\right)^m .$$

This extension, that cannot be done by a simple induction, is left to the reader.

Remark 2.9.   Note that the case $k = n$ in Theorem 2.1 is the case when there are no restrictions on the zeros. Hence, up to the best possible constant, our Theorem 2.1 contains the inequality of the Markov brothers.

3. Lemmas for Theorem 2.1

We need a few lemmas.

LEMMA 3.1.   *Let $0 \leq k \leq n$ be integers and let $s \in [0, 1]$. We have*

$$\|p\|_{[-1-s,1+s]} \leq \exp\left(18\left(\sqrt{nks} + ns\right)\right) \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}^c_{n,k}$.

Observe that the above lemma follows immediately from its "real case" when $\mathcal{P}^c_{n,k}$ is replaced by $\mathcal{P}_{n,k}$. To see this apply Lemma 3.2 below with $p \in \mathcal{P}^c_{n,k}$ replaced by $p\overline{p} \in \mathcal{P}_{2n,2k}$ and obtain the conclusion of Lemma 3.1.

LEMMA 3.2. *Let* $0 \le k \le n$ *be integers and let* $s \in [0, 1]$. *We have*

$$\|p\|_{[-1-s,1+s]} \le \exp\left(18\left(\sqrt{nks} + ns\right)\right) \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$.

Because of symmetry, Lemma 3.2 reduces to:

LEMMA 3.3. *Let* $0 \le k \le n$ *be integers and let* $s \in [0, 1]$. *We have*

$$\|p\|_{[-1,1+s]} \le \exp\left(18\left(\sqrt{nks} + ns\right)\right) \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$.

The following lemma shows that it is sufficient to prove Lemma 3.3 only for some special elements of $\mathcal{P}_{n,k}$ with some additional nice properties.

LEMMA 3.4. *Let* $0 \le k \le n$ *be fixed integers and let* $0 < a < s < 2$ *be fixed real numbers. There exists a* $p^* \in \mathcal{P}_{n,k}$ *for which*

$$\sup_{p \in \mathcal{P}_{n,k}} \frac{|p(1+a)|}{\|p\|_{[-1,1-s]}}$$

is attained. This p^* *is of the form*

$$p^*(x) = (x+1)^{n-k} q^*(x), \qquad q^* \in \mathcal{P}_k.$$

To finish the proof of Lemma 3.1, it is now sufficient to refer to the result below proved by Borwein and Erdélyi [92]. More precisely the lemma below follows from Theorem 2 of Borwein and Erdélyi [92]. Then Lemma 3.3 follows from Lemma 3.5 with the help of Lemma 3.4, and as we have already remarked Lemma 3.1 follows from Lemma 3.3.

LEMMA 3.5. *Let* $0 \le k \le n$ *be integers and let* $s \in [0, 1]$. *We have*

$$\|p\|_{[-1,1+s]} \le \exp\left(18\left(\sqrt{nks} + ns\right)\right) \|p\|_{[-1,1-s]}$$

for every polynomial p *of the form*

$$p(x) = (x+1)^{n-k} q(x), \qquad q \in \mathcal{P}_k.$$

Now we examine the growth of a $p \in \mathcal{P}_{n,k}^c$ near to 1 subject to $\|p\|_{[-1,1]} = 1$.

LEMMA 3.6. *There is an absolute constant c_5 such that*

$$\log |p(z)| \le c_5$$

for every $p \in \mathcal{P}_{n,k}^c$ with $\|p\|_{[-1,1]} \le 1$, and for every $z \in \mathbb{C}$ satisfying

$$|z - 1| \le \frac{1}{n \max\{k + 1, \log n\}} .$$

This will follow by a combination of Lemma 3.1 and our next lemma. The proof of Lemma 3.7 below (in fact a more general result) may be found in Boas [54] (pages 92 and 93).

LEMMA 3.7 (NEVANLINNA'S INEQUALITY). *Let x, $y \in \mathbb{R}$. The inequality*

$$\log |p(x + iy)| \le \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log |p(t)|}{(t - x)^2 + y^2} \, dt$$

holds for every polynomial p with complex coefficients.

The upper bound of Theorem 2.1 will be obtained by a combination of the Cauchy integral formula, Lemma 3.6, and a linear transformation.

4. Lemmas for Theorem 2.2

The line of proof is similar to that of Theorem 2.1. We need a few lemmas. For technical reasons we need to introduce the following classes of polynomials. Let $\mathcal{P}_n(r)^+$ denote the set of all polynomials of degree at most n with *real coefficients* and with no zeros in the open disk with diameter $[1 - 2r, 1]$ $(0 < r \le 1)$.
Recall that the classes $\mathcal{P}_n(r)$ and $\mathcal{P}_n^c(r)$ are defined in the Introduction.

LEMMA 4.1. *Let $0 < s \le r \le 1$. We have*

$$\|p\|_{[-1-s,1+s]} \le \exp\left(8nr^{-1/2}s\right) \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c(r)$.

Observe that the above lemma follows immediately from its "real case" when $p \in \mathcal{P}_n^c(r)$ is replaced by $p \in \mathcal{P}_n(r)$. To see this apply Lemma 4.2 below with $p \in \mathcal{P}_n^c(r)$ replaced by $p\overline{p} \in \mathcal{P}_{2n}(r)$ and obtain the conclusion of Lemma 4.1.

LEMMA 4.2. *Let $0 < s \le r \le 1$. We have*

$$\|p\|_{[-1-s,1+s]} \le \exp\left(8nr^{-1/2}s\right) \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n(r)$.

Because of symmetry, Lemma 4.2 reduces to

LEMMA 4.3.   *Let* $0 < s \le r \le 1$. *We have*

$$\|p\|_{[-1,1+s]} \le \exp\left(8nr^{-1/2}s\right)\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n(r)^+$.

The following lemma shows that it is sufficient to prove Lemma 4.3 only for some special elements of $\mathcal{P}_n(r)^+$ with some additional nice properties.

LEMMA 4.4.   *Let* $0 < a < s \le r \le 1$ *be fixed. There exists a* $p^* \in \mathcal{P}_n(r)^+$ *for which*

$$\sup_{p \in \mathcal{P}_n(r)^+} \frac{|p(1+a)|}{\|p\|_{[-1,1-s]}}$$

is attained. This p^* *has all its zeros in* $[-1, 1-2r]$ *and* $|p(1)| = \|p\|_{[-1,1]}$.

To finish the proof of Lemma 4.1, it is now sufficient to prove the lemma below. More precisely Lemma 4.3 follows from Lemma 4.5 with the help of Lemma 4.4, and as we have already remarked Lemma 4.1 follows from Lemma 4.3.

LEMMA 4.5.   *Let* $0 < s \le r \le 1$. *We have*

$$\|p\|_{[-1,1+s]} \le \exp\left(8nr^{-1/2}s\right)\|p\|_{[-1,1]}$$

for every polynomial $p \in \mathcal{P}_n$ *having all its zeros in* $[-1, 1-2r]$.

To prove Lemma 4.5 we need the following result from Erdélyi [89] (see also Borwein and Erdélyi [95a, p. 237].

LEMMA 4.6.   *Every polynomial* $p \in \mathcal{P}_n$ *has at most*

$$2n\sqrt{\delta}\,\frac{\|p\|_{[-1,1]}}{|p(1)|}$$

zeros (counting multiplicities) in $[1 - \delta, 1]$, $\delta > 0$.

Now we examine the growth of a $p \in \mathcal{P}_n^c(r)$ near to 1 subject to $\|p\|_{[-1,1]} = 1$.

LEMMA 4.7.   *Suppose* $(e/n)^2 < r \le 1$. *There is an absolute constant* c_5 *such that*

$$\log|p(z)| \le c_5$$

for every $p \in \mathcal{P}_n^c(r)$ *with* $\|p\|_{[-1,1]} \le 1$, *and for every* $z \in \mathbb{C}$ *satisfying*

$$|z - 1| \le \frac{\sqrt{r}}{n\log(n\sqrt{r})}.$$

To prove Lemma 4.7 we also need the following Chebyshev-type inequality valid for all $p \in \mathcal{P}^c$. See, for example, Borwein and Erdélyi [95a]. The lemma below can also be viewed as the case $k = n$ of Lemma 3.1 with a better constant.

LEMMA 4.8.   Let $s \in [0, 1]$. We have

$$\|p\|_{[-1-s,1+s]} \leq \exp\left(5ns^{1/2}\right) \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$.

5. Proof of Theorem 2.1

As discussed in Section 3, the proof of Lemma 3.1 is reduced to that of Lemma 3.4. So we start this section with the proof of Lemma 3.4.

Proof of Lemma 3.4.   The existence of $p^* \in \mathcal{P}_{n,k}$ is a standard compactness argument combined with Rouché's theorem. We omit the details of this part.

Now we show that p^* has only real zeros. Suppose that p^* has a non-real zero z_0. Consider the polynomial

$$p_\varepsilon^*(z) := p^*(z) \left(1 - \varepsilon \, \frac{(z - (1 + a))(z - (1 - a))}{(z - z_0)(z - \overline{z}_0)} \right) .$$

It is easy to check that for a sufficiently small $\epsilon > 0$, $p_\varepsilon^* \in \mathcal{P}_{n,k}$. To this end one needs to verify only that if z_0 is non-real and $|z_0| \geq 1$, then for sufficiently small $\varepsilon > 0$, the two zeros of the quadratic polynomial

$$(z - z_0)(z - \overline{z}_0) - \varepsilon(z - (1 + a))(z - (1 - a))$$

are outside the open unit disk. This follows from the fact that for sufficiently small $\epsilon > 0$, the above quadratic polynomial has two non-real zeros with modulus r, where

$$r^2 = \frac{|z_0|^2 - \varepsilon(1 - a^2)}{1 - \epsilon} \geq 1, \qquad |z_0| \geq 1, \quad a \in (0, 1) .$$

Observe now that for sufficiently small $\varepsilon > 0$, $p_\varepsilon^* \in \mathcal{P}_{n,k}$ contradicts the extremality of p^*. This contradiction shows that p^* has only real zeros, indeed.

It remains to prove that if $z_0 \in \mathbb{R} \setminus (-1, 1)$ is a zero of p^*, then $z_0 = -1$. Indeed, if $z_0 \in [1, \infty)$ is a zero of p^*, then

$$q^*(z) := \frac{p^*(z)}{z - z_0}$$

contradicts the extremality of p^*. If $z_0 \in (-\infty, -1)$ is a zero of p^*, then for sufficiently small $\varepsilon > 0$,

$$p_\varepsilon^*(z) := p^*(z) \left(1 - \varepsilon \, \frac{(1+a) - z}{z - z_0} \right),$$

contradicting the extremality of p^*. $\square$

Proof of Lemma 3.6.   Let $p \in \mathcal{P}_{n,k}^c$. We normalize so that

$$\max_{-1 \le t \le 1 + \frac{2}{n(k+1)}} |p(t)| = 1; \tag{5.1}$$

that is,

$$\log |p(t)| \le 0, \qquad -1 \le t \le 1 + \frac{2}{n(k+1)}. \tag{5.2}$$

In the rest of the proof let

$$z = x + iy, \qquad x, \; y \in \mathbb{R}, \quad |x-1|, \; |y| \le \frac{1}{n \max\{k+1, \log n\}}. \tag{5.3}$$

We have

$$\frac{|y|}{\pi} \int_{-\infty}^{-1} \frac{\log |p(t)|}{(t-x)^2 + y^2} \, dt \le \frac{|y|}{\pi} \int_{-\infty}^{-1} \frac{n \log(2|t|)}{(t-x)^2 + y^2} \, dt$$

$$\le \frac{n}{\pi n \max\{k+1, \log n\}} \int_{-\infty}^{-1} \frac{\log(2|t|)}{t^2} \, dt$$

$$\le \frac{c}{\max\{k+1, \log n\}} \le c \tag{5.4}$$

with an absolute constant c. Here we used the well-known inequality $|p(t)| \le |2t|^n$ valid for all $p \in \mathcal{P}_n^c$ with $\|p\|_{[-1,1]} \le 1$ and for all $t \in \mathbb{R} \setminus (-1, 1)$. Obviously

$$\frac{|y|}{\pi} \int_{-1}^{1+\frac{2}{n(k+1)}} \frac{\log |p(t)|}{(t-x)^2 + y^2} \, dt \le 0. \tag{5.5}$$

Now we use Lemma 3.1 and (5.3) to obtain

$$\frac{|y|}{\pi} \int_{1+\frac{2}{n(k+1)}}^{2} \frac{\log |p(t)|}{(t-x)^2 + y^2} \, dt$$

$$\le \frac{|y|}{\pi} \int_{1+\frac{2}{n(k+1)}}^{2} \frac{18 \left(\sqrt{nk(t-1)} + n(t-1) \right)}{(t-x)^2 + y^2} \, dt$$

$$\le \frac{|y|}{\pi} \int_{1+\frac{2}{n(k+1)}}^{2} \frac{72 \left(\sqrt{nk(t-1)} + n(t-1) \right)}{(t-1)^2} \, dt$$

$$\leq \frac{72\sqrt{nk}}{\pi n(k+1)} \int_{1+\frac{2}{n(k+1)}}^{2} (t-1)^{-3/2} \, dt$$

$$+ \frac{72n}{\pi n \max\{k+1, \log n\})} \int_{1+\frac{2}{n(k+1)}}^{2} (t-1)^{-1} \, dt$$

$$\leq \frac{144\sqrt{nk} \left(\frac{2}{n(k+1)}\right)^{-1/2}}{\pi n \max\{k+1, \log n\}} + \frac{72n \log(n(k+1)/2)}{\pi n \max\{k+1, \log n\}}$$

$$\leq c \tag{5.6}$$

with an absolute constant c. Finally, similarly to (5.4), for $n \geq 2$, we have

$$\frac{|y|}{\pi} \int_{2}^{\infty} \frac{\log|p(t)|}{(t-x)^2 + y^2} \, dt$$

$$\leq \frac{|y|}{\pi} \int_{2}^{\infty} \frac{n \log(2|t|)}{(t-x)^2 + y^2} \, dt$$

$$\leq \frac{n}{\pi n \max\{k+1, \log n\}} \int_{2}^{\infty} \frac{4 \log(2|t|)}{(t-1)^2} \, dt$$

$$\leq \frac{c}{\max\{k+1, \log n\}} \tag{5.7}$$

with an absolute constant c. Now from (5.1)–(5.7) and Lemma 3.7 (Nevanlinna's inequality), if $z \in \mathbb{C}$ satisfies (5.3), then

$$|p(z)| \leq \exp\left(\frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log|p(t)|}{(t-x)^2 + y^2} \, dt\right) \leq c \max_{-1 \leq t \leq 1+\frac{2}{n(k+1)}} |p(t)| \tag{5.8}$$

with an absolute constant c. Finally observe that Lemma 3.1 implies

$$\max_{-1 \leq t \leq 1+\frac{2}{n(k+1)}} |p(t)| \leq c_6 \|p\|_{[-1,1]} \tag{5.9}$$

with an absolute constant c_6. The lemma now follows from (5.8) and (5.9). $\square$

Proof of Theorem 2.1. It follows from Lemma 3.6 and Cauchy's integral formula in a standard fashion that

$$|q'(1)| \leq c_7 n \max\{k+1, \log n\} \|q\|_{[-1,1]} \tag{5.10}$$

for every $q \in \mathcal{P}_{n,k}^c$. Now let α be a linear transformation that maps $[-1, 1]$ onto

$[-1, y]$ so that $1 \mapsto y$ if $y \in [0, 1]$, or onto $[y, 1]$ so that $1 \mapsto y$ if $y \in [-1, 0]$. Let $p \in \mathcal{P}_{n,k}^c$. Then $q := p \circ \alpha \in \mathcal{P}_{n,k}^c$. Applying (5.10) to $q \in \mathcal{P}_{n,k}^c$, we obtain

$$
\begin{aligned}
|p'(y)| &= \frac{2}{1+y} |q'(1)| \\
&\leq \frac{2}{1+y} c_7 n \max\{k+1, \log n\} \|q\|_{[-1,1]} \\
&= \frac{2}{1+y} c_7 n \max\{k+1, \log n\} \|p\|_{[-1,y]} \\
&\leq 2c_7 n \max\{k+1, \log n\} \|p\|_{[-1,1]}
\end{aligned}
$$

if $y \in [0, 1]$, and

$$
\begin{aligned}
|p'(y)| &= \frac{2}{1-y} |q'(1)| \\
&\leq \frac{2}{1-y} c_7 n \max\{k+1, \log n\} \|q\|_{[-1,1]} \\
&= \frac{2}{1-y} c_7 n \max\{k+1, \log n\} \|p\|_{[y,1]} \\
&\leq 2c_7 n \max\{k+1, \log n\} \|p\|_{[-1,1]}
\end{aligned}
$$

if $y \in [-1, 0]$. This proves the upper bound of the theorem.

When $\log n \leq k \leq n$ the lower bound of the theorem follows from an example given by Szabados [81, Example 1], see also Erdélyi [87b]. These examples are in fact polynomials with real coefficients. Szabados' example is given by defining

$$
p(x) = \left(\frac{1-x}{2} \right)^{n-k} P_k^{(2n-2k-1/2, 0)}(x) ,
$$

where $P_k^{(\alpha, \beta)}$ denotes the kth Jacobi polynomial with parameters α and β. Erdélyi [87b] offers a more elementary but more technical example.

As the upper bound in (1.1) shows, when $k = o(\log n)$ the polynomials showing the lower bound of the theorem cannot be real. For the case $0 \leq k \leq \log n$, we offer the following example. Let

$$
z_m := \exp\left(\frac{2m\pi i}{2n+1} \right), \qquad m = 1, 2, \ldots, n,
$$

be those $(2n+1)$th roots of unity that lie in the open upper half-plane. Let

$$
p_{2n+1}(z) := p_{2n+2}(z) := (z-1) \prod_{m=1}^{n} (z - z_m)^2 .
$$

Then $p_{2n+1} \in \mathcal{P}_{2n+1,0}^c$ and $|p_{2n+1}(x)| = |x^{2n+1} - 1|$ for every $x \in \mathbb{R}$. Note that this implies

$$
|p_{2n+1}(-1)| = \|p_{2n+1}\|_{[-1,1]} = 2 .
$$

Also

$$\left| \frac{p'_{2n+1}(-1)}{p_{2n+1}(-1)} \right| = \left| -\frac{1}{2} + 2\sum_{m=1}^{n} \frac{1}{-1-z_m} \right| \geq \left| 2\operatorname{Im}\left(\sum_{m=\lfloor n/2 \rfloor+1}^{n} \frac{1}{-1-z_m} \right) \right|$$

$$\geq 2\frac{1}{\sqrt{2}} \sum_{m=\lfloor n/2 \rfloor+1}^{n} \frac{1}{|-1-z_m|} \geq 2\frac{1}{\sqrt{2}} \frac{2}{\pi} \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\frac{(2k+1)\pi}{2n+1} \right)^{-1}$$

$$\geq c_8 n \log n$$

with an absolute constant $c_8 > 0$. $\quad\square$

6. Proof of Theorem 2.2

As discussed in Section 4, with the help of Lemma 4.4, the proof of Lemma 4.1 is reduced to that of Lemma 4.5. The proof of Lemma 4.4 is very similar to that of Lemma 3.4, and it is left to the reader. So we start this section with the proof of Lemma 4.5.

Proof of Lemma 4.5. Let $0 < a < s \leq r \leq 1$. Suppose

$$p(x) = c \prod_{j=1}^{n} (x - x_j), \qquad x_j \in [-1, 1-2r],$$

and assume that $|p(1)| = \|p\|_{[-1,1]}$. Let

$$I_\nu := (1 - 2(\nu+1)^4 r, 1 - 2\nu^4 r], \qquad \nu = 1, 2, \ldots.$$

It follows from Lemma 4.6 that

$$\frac{|p(1+a)|}{|p(1)|} = \prod_{j=1}^{n} \frac{1+a-x_j}{1-x_j} \leq \prod_{j=1}^{n} \left(1 + \frac{a}{1-x_j}\right) \leq \prod_{j=1}^{n} \exp\left(\frac{a}{1-x_j} \right)$$

$$\leq \exp\left(\sum_{j=1}^{n} \frac{a}{1-x_j} \right) \leq \exp\left(a \sum_{\nu=1}^{\infty} \sum_{x_j \in I_\nu} \frac{1}{1-x_j} \right)$$

$$\leq \exp\left(a \sum_{\nu=1}^{\infty} 2n\sqrt{2(\nu+1)^4 r} \, \frac{1}{2\nu^4 r} \right)$$

$$\leq \exp\left(2^{1/2}a \sum_{\nu=1}^{\infty} \frac{(\nu+1)^2}{\nu^4} \frac{n}{\sqrt{r}} \right) \leq \exp\left(\frac{16na}{\sqrt{r}} \right)$$

$$\leq \exp\left(\frac{16ns}{\sqrt{r}} \right)$$

and the lemma is proved. $\quad\square$

Proof of Lemma 4.7.   Let $p \in \mathcal{P}_n^c$. We normalize so that

$$\max_{-1 \leq t \leq 1 + \frac{2\sqrt{r}}{n}} |p(t)| = 1; \tag{6.1}$$

that is,

$$\log |p(t)| \leq 0, \qquad -1 \leq t \leq 1 + \frac{2\sqrt{r}}{n}. \tag{6.2}$$

In the rest of the proof let

$$z = x + iy, \qquad x, \ y \in \mathbb{R}, \quad |x - 1|, \ |y| \leq \frac{\sqrt{r}}{n \log(n\sqrt{r})}. \tag{6.3}$$

Note that the assumption $(e/n)^2 < r \leq 1$ implies $\log(n\sqrt{r}) \geq 1$. We have

$$
\begin{aligned}
\frac{|y|}{\pi} \int_{-\infty}^{-1} \frac{\log |p(t)|}{(t - x)^2 + y^2} \, dt &\leq \frac{|y|}{\pi} \int_{-\infty}^{-1} \frac{n \log(2|t|)}{(t - x)^2 + y^2} \, dt \\
&\leq \frac{\sqrt{r}}{\pi n} \int_{-\infty}^{-1} \frac{n \log(2|t|)}{t^2} \, dt \\
&\leq \frac{cn\sqrt{r}}{n} \\
&\leq c
\end{aligned}
\tag{6.4}
$$

with an absolute constant c. Here we used the well-known inequality $|p(t)| \leq |2t|^n$ valid for all $p \in \mathcal{P}_n^c$ with $\|p\|_{[-1,1]} \leq 1$ and for all $t \in \mathbb{R} \setminus (-1, 1)$. Obviously

$$\frac{|y|}{\pi} \int_{-1}^{1 + \frac{2\sqrt{r}}{n}} \frac{\log |p(t)|}{(t - x)^2 + y^2} \, dt \leq 0. \tag{6.5}$$

Observe that $(e/n)^2 < r \leq 1$ implies $2\sqrt{r}/n < r$. Now we use Lemma 4.1 and (6.1) to obtain

$$
\begin{aligned}
\frac{|y|}{\pi} \int_{1 + \frac{2\sqrt{r}}{n}}^{1+r} \frac{\log |p(t)|}{(t - x)^2 + y^2} \, dt &\leq \frac{|y|}{\pi} \int_{1 + \frac{2\sqrt{r}}{n}}^{1+r} \frac{16nr^{-1/2}(t - 1)}{(t - x)^2 + y^2} \, dt \\
&\leq \frac{|y|}{\pi} \int_{1 + \frac{2\sqrt{r}}{n}}^{1+r} \frac{64nr^{-1/2}(t - 1)}{(t - 1)^2} \, dt \\
&\leq \frac{|y|}{\pi} \frac{64n}{\sqrt{r}} \int_{1 + \frac{2\sqrt{r}}{n}}^{1+r} (t - 1)^{-1} \, dt \\
&\leq \frac{64n}{\sqrt{r}} \frac{\sqrt{r}}{\pi n \log(n\sqrt{r})} \log\left(\frac{rn}{2\sqrt{r}}\right) \\
&\leq c
\end{aligned}
\tag{6.6}
$$

with an absolute constant c. Also, Lemma 4.8 yields

$$
\begin{aligned}
\frac{|y|}{\pi} \int_{1+r}^{2} \frac{\log|p(t)|}{(t-x)^2 + y^2}\, dt
&\leq \frac{|y|}{\pi} \int_{1+r}^{2} \frac{5n(t-1)^{1/2}}{(t-x)^2 + y^2}\, dt \\
&\leq \frac{|y|}{\pi} \int_{1+r}^{2} \frac{20n(t-1)^{1/2}}{(t-1)^2}\, dt \\
&\leq \frac{|y|}{\pi} \int_{1+r}^{2} 20n(t-1)^{-3/2}\, dt \\
&\leq \frac{\sqrt{r}}{\pi n \log(n\sqrt{r})}\, \frac{20n}{\sqrt{r}} \\
&\leq c
\end{aligned}
\tag{6.7}
$$

with an absolute constant c. Here we used the fact that the assumption $(e/n)^2 < r \leq 1$ implies

$$
\frac{2\sqrt{r}}{n \log(n\sqrt{r})} \leq \frac{2\sqrt{r}}{n} \leq r.
$$

Similarly to (6.4), we obtain

$$
\begin{aligned}
\frac{|y|}{\pi} \int_{2}^{\infty} \frac{\log|p(t)|}{(t-x)^2 + y^2}\, dt
&\leq \int_{2}^{\infty} \frac{n \log(2|t|)}{(t-x)^2 + y^2}\, dt \\
&\leq n \frac{\sqrt{r}}{\pi n \log(n\sqrt{r})} \int_{2}^{\infty} \frac{4 \log(2|t|)}{(t-1)^2}\, dt \\
&\leq \frac{c\sqrt{r}}{\log(n\sqrt{r})} \leq c
\end{aligned}
\tag{6.8}
$$

with an absolute constant c. Here we used the fact that the assumption $(e/n)^2 < r \leq 1$ implies

$$
\frac{2\sqrt{r}}{n \log(n\sqrt{r})} \leq \frac{2\sqrt{r}}{n} \leq r \leq 1.
$$

Now, from (6.1)–(6.8) and Lemma 3.7 (Nevanlinna's inequality), if $z \in \mathbb{C}$ satisfies (6.3) then

$$
|p(z)| \leq \exp\left(\frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log|p(t)|}{(t-x)^2 + y^2}\, dt \right) \leq c \max_{-1 \leq t \leq 1 + \frac{2\sqrt{r}}{n}} |p(t)|
\tag{6.9}
$$

with an absolute constant c. Finally observe that Lemma 4.1 implies

$$
\max_{-1 \leq t \leq 1 + \frac{2\sqrt{r}}{n}} |p(t)| \leq c\|p\|_{[-1,1]}
\tag{6.10}
$$

with an absolute constant c. The lemma now follows from (6.9) and (6.10). $\square$

Proof of Theorem 2.2.   First assume that $(e/n)^2 \leq r \leq 1$.   It follows from Lemma 4.7 and Cauchy's integral formula in a standard fashion that

$$|p'(1)| \leq \frac{c_7 n \log\left(n\sqrt{r}\right)}{\sqrt{r}} \, \|p\|_{[-1,1]} \tag{6.11}$$

for every $p \in \mathcal{P}_n^c(r)$, where $c_7 > 0$ is an absolute constant.   Now, by a linear transformation, we obtain

$$|p'(y)| \leq \frac{2c_7 n \log\left(n\sqrt{r}\right)}{\sqrt{r}} \, \|p\|_{[-1,1]} \tag{6.12}$$

for every $p \in \mathcal{P}_n^c(r)$ and $y \in [1 - r, 1]$. By symmetry, we have

$$|p'(y)| \leq \frac{2c_7 n \log\left(n\sqrt{r}\right)}{\sqrt{r}} \, \|p\|_{[-1,1]} \tag{6.13}$$

for every $p \in \mathcal{P}_n^c(r)$ and $y \in [-1, -1+r]$. It is an obvious consequence of Bernstein's inequality (see the Introduction) that

$$|p'(y)| \leq \frac{n}{\sqrt{r}} \, \|p\|_{[-1,1]} \tag{6.14}$$

for every $p \in \mathcal{P}_n^c$ (here we do not need to exploit the information about the zeros). Inequalities (6.12)–(6.14) yield the upper bound of the theorem under the assumption $(e/n)^2 \leq r \leq 1$. When $0 < r < (e/n)^2$ the upper bound of the theorem follows from Markov's Inequality (here we do not need the information about the zeros again). By this the upper bound of the theorem is completely proved.

When $1/8 \leq r \leq 1$, the lower bound of the theorem follows from the case $k = 0$ of the lower bound in Theorem 2.1 When $0 < r \leq n^{-2}$, the Chebyshev polynomial T_n defined by

$$T_n(x) := \cos(n \arccos x), \qquad x \in [-1, 1],$$

shows the lower bound of the theorem. For the case $n^{-2} < r \leq 1/8$, we offer the following example. We define

$$k := \left\lfloor r^{-1/2} \right\rfloor \leq n, \qquad \text{and} \qquad m := \left\lfloor \frac{n}{k} \right\rfloor \geq 1. \tag{6.15}$$

Let

$$z_j := (1 - 2r) + 2r \exp\left(\frac{(2j-1)\pi i}{2m+1}\right), \qquad j = 1, 2, \ldots, m, \tag{6.16}$$

be the zeros of $(z - (1 - 2r))^{2m+1} + (2r)^{2m+1}$ in the open upper half-plane. Let

$$P_m(z) := (z - (1 - 4r)) \prod_{j=1}^{m} (z - z_j)^2. \tag{6.17}$$

Let

$$\tilde{z}_j := -z_j, \qquad j = 1, 2, \ldots, m, \tag{6.18}$$

be the zeros of $(z - (-1 + 2r))^{2m+1} + (2r)^{2m+1}$ in the open upper half-plane. Let

$$\tilde{P}_m(z) := (z - (-1 + 4r)) \prod_{j=1}^{m} (z - \tilde{z}_j)^2 . \tag{6.19}$$

We introduce

$$Q_k(z) := U_k \left(\frac{z}{1 - 2r} \right) ,$$

where U_k is the kth Chebyshev polynomial of the second kind defined by

$$U_k(z) = \frac{\sin((k + 1)\theta)}{\sin \theta} , \qquad z = \cos \theta , \quad \theta \in (0, \pi) .$$

Let

$$p_{n,r} := P_m \tilde{P}_m Q_k^{4m+2} .$$

Obviously $p_{n,r} \in \mathcal{P}_{(4m+2)(k+1)}^c(r) \subset \mathcal{P}_{10n+2}^c(r)$. We show that

$$\max_{-1 \le z \le 1} |p_{n,r}(z)| = |p_{n,r}(1)| . \tag{6.20}$$

Observe that

$$|(P_m \tilde{P}_m)(z)| \le 4((1 - 2r)^2 - z^2)^{2m+1} , \qquad -1 + 4r \le z \le 1 - 4r ,$$

and

$$|Q_k^2(z)((1 - 2r)^2 - z^2)|^{2m+1} \le (1 - 2r)^{4m+2} \le 1 , \qquad -1 + 2r \le z \le 1 - 2r ,$$

hence

$$|p_{n,r}(z)| = \left| \left(P_m \tilde{P}_m Q_k^{4m+2} \right)(z) \right| \le 4 , \qquad -1 + 4r \le z \le 1 - 4r . \tag{6.21}$$

Also

$$|(P_m \tilde{P}_m)(1)| = \max_{z \in [-1,1] \setminus [-1+4r, 1-4r]} |(P_m \tilde{P}_m)(z)| ,$$

hence

$$|p_{n,r}(1)| = \max_{z \in [-1,1] \setminus [-1+4r, 1-4r]} |p_{n,r}(z)| . \tag{6.22}$$

Using (6.21), (6.15), and

$$|Q_k(z)| \le |Q_k(1 - 2r)| = |U_k(1)| = k + 1 , \qquad z \in [-1 + 2r, 1 - 2r] ,$$

we obtain

$$\max_{z \in [-1+4r, 1-4r]} |p_{n,r}(z)| \leq 4 \leq 2^{2m+1}$$

$$= r^{-(2m+1)}(2r)^{2m+1} \leq (k+1)^{4m+2}(2r)^{2m+1}$$

$$\leq |p_{n,r}(1-2r)| \leq |p_{n,r}(1)|. \tag{6.23}$$

Now (6.22) and (6.23) yield (6.20). Using (6.20) and (6.15), we obtain

$$\frac{|p'_{n,r}(1)|}{\|p_{n,r}\|_{[-1,1]}} = \frac{|p'_{n,r}(1)|}{|p_{n,r}(1)|} \geq \left| 2\,\mathrm{Im}\left(\sum_{j=1}^{m} \frac{1}{1-z_j}\right)\right| \geq \left| 2\,\mathrm{Im}\left(\sum_{j=1}^{\lfloor m/2 \rfloor} \frac{1}{1-z_j}\right)\right|$$

$$\geq 2\frac{1}{\sqrt{2}} \sum_{j=1}^{\lfloor m/2 \rfloor} \frac{1}{|1-z_j|} \geq 2\frac{1}{\sqrt{2}}\frac{2}{\pi} \sum_{j=1}^{\lfloor m/2 \rfloor} \left(r \frac{(2j-1)\pi}{2m+1}\right)^{-1}$$

$$\geq \frac{cm \log m}{r} \geq \frac{c'n\sqrt{r}\log(n\sqrt{r})}{r}$$

$$= \frac{c'n \log(n\sqrt{r})}{\sqrt{r}},$$

where $c > 0$ and $c' > 0$ are absolute constants. This finishes the proof.   □

REFERENCES

Bernstein, S.N., *Collected works: Vol. I, Constr. theory of functions* (1905–1930), *English Translation*, Atomic Energy Commision, Springfield, VA, 1958.

Boas, R.P., *Entire functions*, Academic Press, New York, NY, 1954.

Borwein, P.B., *Markov's inequality for polynomials with real zeros*, Proc. Amer. Math. Soc. **93** (1985), 43–48.

Borwein, P.B. and T. Erdélyi, *Remez-, Nikolskii- and Markov-type inequalities for generalized nonnegative polynomials with restricted zeros*, Constr. Approx. **8** 1992, 343–362.

_____, *Sharp Markov-Bernstein type inequalities for classes of polynomials with restricted zeros*, Constr. Approx. **10** (1994), 411–425.

_____, *Polynomials and polynomial inequalities*, Springer-Verlag, Graduate Texts in Mathematics, New York, NY, 1995a.

_____, *Markov and Bernstein type inequalities in L_p for classes of polynomials with constraints*, J. London Math. Soc. **51** (1995b), 573–588.

Cheney, E.W., *Introduction to approximation theory*, McGraw-Hill, New York, NY, 1966.

DeVore, R.A. and G.G. Lorentz, *Constructive approximation*, Springer-Verlag, Berlin, 1993.

Duffin, R.J. and A.C. Scheaffer, *A refinement of an inequality of the brothers Markoff*, Trans. Amer. Math. Soc. **50** (1941), 517–528.

Erdélyi, T., "Pointwise estimates for derivatives of polynomials with restricted zeros" in *Haar Memorial Conference*, J. Szabados & K. Tandori, Eds., North-Holland, Amsterdam, 1987a, pp. 329–343.

_____, *Pointwise estimates for the derivatives of a polynomial with real zeros*, Acta Math. Hung. **49** (1987b), 219–235.

_____, *Markov type estimates for derivatives of polynomials of special type*, Acta Math. Hung. **51** (1988), 421–436.

_____, *Markov-type estimates for certain classes of constrained polynomials*, Constr. Approx. **5** (1989), 347–356.

_____, *Bernstein-type inequalities for the derivative of constrained polynomials*, Proc. Amer. Math. Soc. **112** (1991), 829–838.

Erdélyi, T. and J. Szabados, *Bernstein-type inequalities for a class of polynomials*, Acta Math. Hungar. **52** (1989), 237–251.

Erdős, P., *On extremal properties of the derivatives of polynomials*, Ann. of Math. **2** (1940), 310–313.

von Golitschek, M. and G.G. Lorentz, *Bernstein inequalities in L_p, $0 < p \leq \infty$*, Rocky Mountain J. Math. **19** (1989), 145–156.

Lorentz, G.G., *The degree of approximation by polynomials with positive coefficients*, Math. Ann. **151** (1963), 239–251.

_____, *Approximation of functions*, 2nd ed., Chelsea, New York, NY, 1986.

Lorentz, G.G., M. von Golitschek and Y. Makovoz, *Constructive approximation: Advanced problems*, Springer-Verlag, New York, NY, 1996.

Markov, A.A., *Sur une question posée par D.I. Mendeleieff*, Izv. Acad. Nauk St. Petersburg **62** (1889), 1–24.

Markov, V.A., *Über Polynome die in einen gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. **77** (1916), 213–258; the original appeared in Russian in 1892.

Máté, A., *Inequalities for derivatives of polynomials with restricted zeros*, Proc. Amer. Math. Soc. **82** (1981), 221–224.

Máté, A. and P. Nevai, *Bernstein inequality in L_p for $0 < p < 1$, and $(C, 1)$ bounds of orthogonal polynomials*, Ann. of Math. **111** (1980), 145–154.

Milovanović, G.V., D.S. Mitrinović and Th.M. Rassias, *Topics in polynomials: Extremal problems, inequalities, zeros*, World Scientific, Singapore, 1994.

Natanson, I.P., *Constructive function theory, vol. 1*, Frederick Ungar, New York, NY, 1964.

Nevai, P., *Bernstein's inequality in L_p, $0 < p < 1$*, J. Approx. Theory **27** (1979), 239–243.

Newman, D.J., *Derivative bounds for Müntz polynomials*, J. Approx. Theory **18** (1976), 360–362.

Rahman, Q.I. and G. Schmeisser, *Les Inégalités de Markoff et de Bernstein*, Les Presses de L'Université de Montreal, 1983.

Schaeffer, A.C. and R.J. Duffin, *On some inequalities of S. Bernstein and V. Markoff for derivatives of polynomials*, Bull. Amer. Math. Soc. **44** (1938), 289–297.

Scheick, J.T., *Inequalities for derivatives of polynomials of special type*, J. Approx. Theory **6** (1972), 354–358.

Szabados, J., "Bernstein and Markov type estimates for the derivative of a polynomial with real zeros" in *Functional analysis and approximation*, Birkhäuser, Basel, 1981, pp. 177–188.

Szabados, J. and A.K. Varma, "Inequalities for derivatives of polynomials having real zeros" in *Approximation theory III* (E.W. Cheney, Ed.), Academic Press, New York, NY, 1980, pp. 881–888.

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