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REGULARITY OF PAIRS OF POSITIVE OPERATORS

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0. Introduction

In this paper, we consider a pair (A, B) of closed operators on a Banach space X with domain $D(A)$ and $D(B)$. The pair (A, B) is called regular if for every $f \in X$, the problem $Au + Bu = f$ possesses one and only one solution.

Related to the notion of coercively positive pair of operators, introduced in [S], we also consider the existence of a solution to the problem $\lambda Au + Bu = f$ for all $\lambda > 0$, with some uniformity in λ. This stronger property is called λ-regularity.

These notions of regularity and λ-regularity naturally arise in vector-valued Cauchy problems; see [G], [DG], [S] and also [CD]. The uniformity in λ, given by the λ-regularity, is often useful in certain applications to partial differential equations.

In [G], under the hypothesis that $0 \in \rho(B)$ and in [DG], some sufficient conditions are given to ensure the regularity of a pair (A, B) on certain subspaces of X, related to the operator B. These subspaces, denoted by $D_B(\theta, p)$, are real interpolation spaces between $D(B)$ and X (Theorem 1.2).

It was observed in [S] that if $0 \in \rho(A) \cap \rho(B)$, then the pair is λ-regular on $D_B(\theta, p)$.

In this paper, we prove the λ-regularity of this pair (A, B), considered in [G], on $D_B(\theta, p)$ under the weaker assumption that $0 \in \rho(B)$ only (Theorem 2.1). Note that if B is bounded, then the pair is λ-regular on X.

We construct an example of a regular pair (A, B) of operators in a Hilbert space, with B bounded, satisfying the assumptions of the theorem of Grisvard [G], which is not λ-regular (Example 2.2).

1. Preliminaries

In this section we give precise definitions of regularity and λ-regularity of a pair of operators. Then, for the sake of completeness, we recall a result of Da Prato and Grisvard [DG] (see also [CD]), which is the starting point of our results.

Let X be a Banach space and A and B be two closed operators in X.

DEFINITION 1. The pair (A, B) is called *regular*, if for all $f \in X$, there exists a unique $u \in D(A) \cap D(B)$ such that $Au + Bu = f$.

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If the pair (A, B) is regular, it follows from the Banach theorem that

$$\|u\| + \|Au\| + \|Bu\| \leq M\|Au + Bu\| \tag{1.0}$$

for some $M \geq 1$ and for all $u \in D(A) \cap D(B)$.

It is easy to verify the following lemma.

LEMMA 1.0. *Let A and B be two closed operators in X. Then the pair (A, B) is regular if and only if*

(1) (1.0) *holds and*
(2) $R(A + B)$ *is dense in X.*

Moreover, if $0 \in \rho(A)$ or $\rho(B)$ (where $\rho(.)$ denotes the resolvent set of an operator), then (1.0) *is equivalent to*

$$\|Au\| + \|Bu\| \leq M\|Au + Bu\| \tag{1.1}$$

for some $M \geq 1$ and for all $u \in D(A) \cap D(B)$.

Remark 1. The operator $A + B$ is closed if and only if

$$\|u\| + \|Au\| + \|Bu\| \leq M\big(\|Au + Bu\| + \|u\|\big)$$

for some $M \geq 1$ and for all $u \in D(A) \cap D(B)$.

In particular, if the pair (A, B) is regular, $A + B$ has to be closed.

A regular pair of operators (A, B) is called *coercive* in [S].

Also, the stronger notion of *coercively positive* pair is introduced in [S], which motivates our Definition 2.

DEFINITION 2. The pair (A, B) is called λ*-regular* in X, if for all $f \in X$ and for all $\lambda > 0$, there exists a unique $u \in D(A) \cap D(B)$ such that $\lambda Au + Bu = f$ and moreover, for all $\lambda > 0$,

$$\|\lambda Au\| + \|Bu\| \leq M\|\lambda Au + Bu\| \tag{1.1$_\lambda$}$$

for some $M \geq 1$, *independent of* λ and for all $u \in D(A) \cap D(B)$.

Remark 2. Clearly if $(1.1)_\lambda$ holds, then the inequality

$$\lambda\|Au\| + \mu\|Bu\| \leq M\|\lambda Au + \mu Bu\|$$

holds for some $M \geq 1$, for all $\lambda, \mu > 0$ and $u \in D(A) \cap D(B)$, which shows that the definition of λ-regularity is symmetric in A and B.

It is also clear that this inequality is equivalent to the following ones:

$$\|Au\| \leq M\|Au + \lambda Bu\|,$$

for some $M \geq 1$ and all $\lambda > 0$ and $u \in D(A) \cap D(B)$, and

$$\lambda\|Bu\| \leq M\|Au + \lambda Bu\|$$

for some $M \geq 1$ and all $\lambda > 0$ and $u \in D(A) \cap D(B)$.

LEMMA 1.0.λ. *Let A and B be two closed operators in X (not necessarily densely defined). If $0 \in \rho(A)$, then the pair (A, B) is λ-regular if and only if:*

(1) $(1.1)_\lambda$ *holds for all* $\lambda > 0$;
(2) *There exists* $\lambda_0 > 0$ *such that* $R(\lambda_0 A + B)$ *is dense in* X.

Proof. Clearly, it is enough to prove that conditions (1) and (2) imply that the pair (A, B) is λ-regular.

First observe that conditions (1) and (2) together with Lemma 1.0, where A is replaced by $\lambda_0 A$, and the fact that $0 \in \rho(A)$, imply that the pair $(\lambda_0 A, B)$ is regular. Thus, in particular, $0 \in \rho(\lambda_0 A + B)$.

Next we show that if $0 \in \rho(\lambda_1 A + B)$ for some $\lambda_1 > 0$, then $0 \in \rho(\lambda A + B)$ for all $\lambda > 0$ such that

$$(*) \qquad \frac{\lambda}{\lambda_1} \in \left(\frac{M}{M+1}, \frac{M}{M-1}\right) \text{ if } M > 1 \text{ and } \left(\frac{M}{M+1}, \infty\right) \text{ if } M = 1.$$

Indeed, problem $\lambda Au + Bu = f$ is equivalent to

$$\lambda_1 Au + Bu = \left(1 - \frac{\lambda_1}{\lambda}\right) Bu + \frac{\lambda_1}{\lambda} f.$$

Setting $v = \lambda_1 Au + Bu$, we have

$$(**) \qquad v = \left(1 - \frac{\lambda_1}{\lambda}\right) B(\lambda_1 A + B)^{-1} v + \frac{\lambda_1}{\lambda} f.$$

From $(1.1)_\lambda$, it follows that

$$\|B(\lambda_1 A + B)^{-1}\| \leq M.$$

Under assumption $(*)$, by the Banach fixed point theorem, it is clear that there exists one and only one $v \in X$ satisfying $(**)$ and hence $(\lambda A, B)$ is a regular pair for such λ. Noting that $\|B(\lambda Au + B)^{-1}\| \leq M$ also holds for λ in this interval, we can repeat this argument and, since $\frac{M}{M+1} < 1$ and $\frac{M}{M-1} > 1$, show by induction that the pair $(\lambda A, B)$ is regular for all $\lambda > 0$, which together with $(1.1)_\lambda$ implies that the pair (A, B) is λ-regular. This finishes the proof of Lemma 1.0.λ. □

Let us recall classical definitions on closed operators: A closed linear operator $A : D(A) \subset X \to X$ (not necessarily densely defined) is called *positive* in $(X, \|\cdot\|)$ [Tr] if there exists $C > 0$ such that

$$\|u\| \leq C\|u + \lambda Au\|, \text{ for every } \lambda > 0 \text{ and } u \in D(A), \tag{1.2}$$

and if $R(I + \lambda A) = X$ for some $\lambda > 0$, equivalently for all $\lambda > 0$.

Remark 3. In [Tr], an operator A is called positive if it is positive and satisfies the additional assumption that $0 \in \rho(A)$. In this paper, it is convenient to relax this extra condition.

Observe also that A is positive if and only if the pair (A, I) is λ-regular.

If A is positive, injective and densely defined, it is easy to prove that A^{-1} is also positive.

If X is reflexive and A is positive, then A is densely defined [K].

Let $\Sigma_\sigma := \{\lambda \in \mathbb{C}\backslash\{0\}; |\arg \lambda| \leq \sigma\} \cup \{0\}$, for $\sigma \in [0, \pi)$. If A is positive, there exists $\theta \in [0, \pi)$ such that (1.3) holds, [K p. 288]:

(1.3) (i) $\sigma(A) \subseteq \Sigma_\theta$ and
(ii) for each $\theta' \in (\theta, \pi]$, there exists $M(\theta') \geq 1$ such that $\|\lambda(\lambda I - A)^{-1}\| \leq M(\theta')$, for every $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg \lambda| \geq \theta'$

where $\sigma(A)$ denotes the spectrum of A.

The number $\omega_A := \inf\{\theta \in [0, \pi); (1.3)\text{holds}\}$ is called the *spectral angle* of the operator A. Clearly $\omega_A \in [0, \pi)$.

An operator A is said to be of *type* (ω, M) [Tan], if A is positive, ω is the spectral angle of A and

$$M := \inf\{C \geq 0; (1.2) \text{ holds }\} = \min\{C \geq 0; (1.2) \text{ holds}\}.$$

Note that M is also the smallest constant in (1.3) ii) for $\theta' = \pi$.

Two positive operators A and B in X are said to be (*resolvent*) *commuting* if the bounded operators $(I + \lambda A)^{-1}$ and $(I + \mu B)^{-1}$ commute for some $\lambda, \mu > 0$, equivalently for all $\lambda, \mu > 0$.

If A and B are commuting positive operators then $A + B$ (with domain $D(A) \cap D(B)$) is closable [DG].

The following theorem, which is a consequence of a theorem of Da Prato-Grisvard [DG] and of Grisvard [G] will be essential in the sequel.

THEOREM 1.1. *Let A and B be two commuting positive operators in X such that*

(*i*) $D(A) + D(B)$ *is dense in* X,
(*ii*) $\omega_A + \omega_B < \pi$.

Then the closure of $A + B$ *is of type* (ω, M) *with* $\omega \leq \max(\omega_A, \omega_B)$.
If moreover

(*iii*) $0 \in \rho(A)$ *or* $\rho(B)$ (*resolvent set of* A *or* B), *then*
(*a*) *there exists* $M \geq 1$ *such that*

$$\|u\| \leq M\|Au + Bu\|, \quad \textit{for all } u \in D(A) \cap D(B), \tag{1.4}$$

and $0 \in \rho(\overline{A + B})$,
(*b*) $R(A + B) \supseteq D(A) + D(B)$,
(*c*) $A + B$ *is closed if and only if* $R(A + B) = X$ *if and only if* (1.1) *holds*,
(*d*) *the inverse of* $\overline{A + B}$ *is given by*

$$(\overline{A + B})^{-1}x = \frac{1}{2\pi i}\int_\gamma (A + z)^{-1}(B - z)^{-1}x\,dz, \tag{$*$}$$

where γ *is any simple curve in* $\rho(B) \cap \rho\big(-A\big)$ *from* $\infty e^{-i\theta_0}$ *to* $\infty e^{i\theta_0}$, *with* $\omega_B < \theta_0 < \pi - \omega_A$.

Remark 4. (1) Under hypotheses (i)–(iii) of Theorem 1.1, assumption 2) of Lemma 1.0 is always satisfied. Therefore, in order to prove the regularity of a pair (A, B), it is sufficient to verify inequality (1.1), which means that $A(\overline{A + B})^{-1}$ is a bounded operator.

(2) Similarly, under hypotheses (i)–(iii) of Theorem 1.1, assumption (2) of Lemma 1.0.λ is always satisfied. Therefore, in order to prove the λ-regularity of a pair (A, B), it is sufficient to verify inequality $(1.1)_\lambda$, which means that $\lambda A(\overline{\lambda A + B})^{-1}$ is a uniformly bounded operator for all $\lambda > 0$.

In this paper, we shall always be in the situation of (i)–(ii) of Theorem 1.1, which means that we will consider the following three hypotheses for a pair of positive operators A and B in X of type respectively (ω_A, M_A) and (ω_B, M_B):

H_0: $D(A) + D(B)$ is dense in X.
H_1: A and B are resolvent commuting.
H_2: $\omega_A + \omega_B < \pi$.

In order to obtain results on the regularity and the λ-regularity of a pair of operators, we need to introduce the *interpolation spaces* $D_A(\theta, p)$, associated with a closed operator A, for $\theta \in (0, 1)$ and $p \in [1, +\infty]$. These spaces are subspaces of X which are dense in X for the norm $\|.\|$ whenever A is densely defined.

For $\theta \in (0, 1)$ and $p \in [1, +\infty)$, $D_A(\theta, p)$ is the subspace of X consisting of all x such that

$$\|t^\theta A(A + t)^{-1}x\| \in L^p_*,$$

where L_*^p is the space of p-integrable Borel functions on $(0, +\infty)$ equipped with its invariant measure dt/t.

For $\theta \in]0, 1[$, $D_A(\theta, \infty)$ is the subspace of X consisting of all $x \in X$ such that

$$\sup\{\|t^\theta A(A+t)^{-1}x\| \mid t \in (0, +\infty)\} < +\infty.$$

When 0 belongs to $\rho(A)$, $D_A(\theta, p)$ equipped with the norm

$$\|x\|_{D_A(\theta,p)} = \|t^\theta A(A+t)^{-1}x\|_{L_*^p}$$

becomes a Banach space.

When $0 \in \rho(A)$ and A is bounded, $\|.\|_{D_A(\theta,p)}$ is equivalent to the norm of X.

The following fundamental result, due to Grisvard (Theorem 2.7 of [G]) is the starting point of this paper.

THEOREM 1.2. *Let X be a complex Banach space, and let A and B be two positive operators in X, of type (ω_A, M_A) and (ω_B, M_B) respectively, satisfying hypotheses H_0, H_1, H_2.*

If $0 \in \rho(B)$, the pair (A, B) is regular in $D_B(\theta, p)$.

2. Results

The first result of this paper is the following theorem which is an extension of Theorem 1.2 to the case of λ-regularity.

THEOREM 2.1. *Let X be a complex Banach space, and let A and B be two positive operators in X, of type (ω_A, M_A) and (ω_B, M_B) respectively, satisfying hypotheses H_0, H_1, H_2. If $0 \in \rho(B)$, the pair (A, B) is λ-regular in $D_B(\theta, p)$ for every $0 < \theta < 1$ and $1 \leq p \leq \infty$.*

Remark 5. If moreover B is bounded, it is clear that the pair (A, B) is λ-regular in X.

The next example shows that in particular, even if X is a Hilbert space, the hypothesis $0 \in \rho(B)$ cannot be omitted in Theorem 2.1.

Example 2.2. There exists a Hilbert space G and there exist two positive operators A and B in G satisfying hypotheses H_0, H_1 and H_2, with B bounded, such that the pair (A, B) is regular, but not λ-regular in G.

Remark 6. In [L, Theorem 2.4] (see also [CD]), another example is given, where A is the derivative acting on $L^p([0, T]; Y)$ for some non reflexive space Y, such that the pair (A, B) is not λ-regular in $D_A(\theta, p)$.

Proof of Theorem 2.1. Fix $\lambda > 0$. By Theorem 1.2, we know that the pair $(A, \lambda B)$ is regular in $D_B(\theta, p)$. In particular, for all $x \in D_B(\theta, p)$,

$$y_\lambda = (\overline{A + \lambda B})^{-1} x \in D(A) \cap D(B)$$

and we have $By_\lambda \in D_B(\theta, p)$ together with the inequality

$$\|\lambda B y_\lambda\|_{D_B(\theta,p)} \leq C\|x\|_{D_B(\theta,p)}.$$

We shall show that C is independent of λ. For this, we are going to use equality $(*)$ of Theorem 1.1, applied to A and λB. Without loss of generality, since $0 \in \rho(B)$, we can suppose that γ consists of the half line $(\infty e^{-i\theta_0}, \varepsilon e^{-i\theta_0}]$, the arc of the circle $C_\varepsilon = \{z : |z| = \varepsilon, |arg(z)| \leq \theta_0\}$ and the half line $[\varepsilon e^{i\theta_0}, \infty e^{i\theta_0})$, for some fixed θ_0, $\omega_B < \theta_0 < \pi - \omega_A$ and for sufficiently small ε in order to insure that γ is in $\rho(-A) \cap \rho(\lambda B)$. Since A is of type (ω_A, M_A), by (1.3) there exists M'_A such that for all z such that $|\arg z| \leq \theta_0$,

$$\|(A + z)^{-1}\| \leq \frac{M'_A}{|z|}.$$

As in the proof of Theorem 3.11 of [DG], for every $t > 0$ we can write

$$\begin{aligned}
(\lambda B + t)^{-1} y_\lambda &= (\lambda B + t)^{-1} (\overline{A + \lambda B})^{-1} x \\
&= \frac{1}{2\pi i} \int_\gamma (A + z)^{-1} (\lambda B + t)^{-1} (\lambda B - z)^{-1} x \, dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by } (*) \text{ and } H_1 \\
&= \frac{1}{2\pi i} \int_\gamma (A + z)^{-1} (\lambda B - z)^{-1} x \, \frac{dz}{t + z} \\
&\quad - \frac{1}{2\pi i} \int_\gamma (A + z)^{-1} (\lambda B + t)^{-1} x \, \frac{dz}{t + z} \\
&= \frac{1}{2\pi i} \int_\gamma (A + z)^{-1} (\lambda B - z)^{-1} x \, \frac{dz}{t + z} \\
&\quad - (\lambda B + t)^{-1} \frac{1}{2\pi i} \int_\gamma (A + z)^{-1} x \, \frac{dz}{t + z} \\
&= \frac{1}{2\pi i} \int_\gamma (A + z)^{-1} (\lambda B - z)^{-1} x \, \frac{dz}{t + z}
\end{aligned}$$

by analyticity of the function $\frac{(A+z)^{-1}}{t+z}$ and the fact that $\|\frac{(A+z)^{-1}}{t+z}\| \leq \frac{M'_A}{|z(z+t)|}$ for $|\arg z| \leq \theta_0$.

Hence

$$\begin{aligned}
\lambda B(\lambda B + t)^{-1} y_\lambda &= y_\lambda - t(\lambda B + t)^{-1} y_\lambda \\
&= (\overline{A + \lambda B})^{-1} x - t(\lambda B + t)^{-1} (\overline{A + \lambda B})^{-1} x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_\gamma (A+z)^{-1}(\lambda B - z)^{-1}x\,dz \\
&\quad - \frac{1}{2\pi i}\int_\gamma \frac{t}{t+z}(A+z)^{-1}(\lambda B - z)^{-1}x\,dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{z}{t+z}(A+z)^{-1}(\lambda B - z)^{-1}x\,dz
\end{aligned}
$$

Then

$$\lambda B(\lambda B + t)^{-1}y_\lambda = \frac{1}{2\pi i}\int_\gamma \frac{z}{z+t}(A+z)^{-1}(\lambda B - z)^{-1}xdz.$$

First, we claim that

$$\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon} \frac{z}{z+t}(A+z)^{-1}(\lambda B - z)^{-1}xdz = 0.$$

Since B is invertible, $\|(\lambda B - z)^{-1}\|$ is uniformly bounded with respect to z in a neighborhood of the origin. So there exists ε_0 such that $\|(\lambda B - z)^{-1}\| \leq 2\|(\lambda B)^{-1}\|$ for $|z| \leq \varepsilon_0$. We can suppose that $\varepsilon_0 \leq \frac{t}{2}$. Then, for $\varepsilon \leq \varepsilon_0$ we have

$$
\begin{aligned}
&\left\|\int_{C_\varepsilon} \frac{z}{z+t}(A+z)^{-1}(\lambda B - z)^{-1}x\,dz\right\| \\
&\leq \int_{C_\varepsilon} \frac{|z|}{|z+t|}\|(A+z)^{-1}\|\;\|(\lambda B - z)^{-1}\|\;\|x\|\;|dz| \\
&\leq 2M'_A\|(\lambda B)^{-1}\|\|x\|\varepsilon\int_{-\theta_0}^{\theta_0}\frac{d\theta}{t+\varepsilon cos\theta} \leq \frac{8M'_A\|(\lambda B)^{-1}\|\|x\|\varepsilon\theta_0}{t}
\end{aligned}
$$

which tends to zero when $\varepsilon \to 0^+$. The claim is proved; hence we have

$$\lambda B(\lambda B + t)^{-1}y_\lambda = \frac{1}{2\pi i}\int_{\gamma_0} \frac{z}{z+t}(A+z)^{-1}(\lambda B - z)^{-1}x\,dz$$

where γ_0 consists of the half-lines $\{z : arg(z) = -\theta_0\}$ and $\{z : arg(z) = \theta_0\}$.

By hypotheses H_1 and H_2,

$$\lambda B(\lambda B + t)^{-1}\lambda B y_\lambda = \frac{1}{2\pi i}\int_{\gamma_0} \frac{z}{z+t}(A+z)^{-1}\lambda B(\lambda B - z)^{-1}x\,dz$$

and so

$$
\begin{aligned}
&\|\lambda B(\lambda B + t)^{-1}\lambda B y_\lambda\| \\
&\leq \frac{1}{2\pi}\int_{\gamma_0} \frac{|z|}{|z+t|}\|(A+z)^{-1}\|\;\|\lambda B(\lambda B - z)^{-1}x\|\;|dz| \\
&\leq K\int_0^{+\infty} \frac{r}{\sqrt{t^2+r^2+2trcos\theta_0}}\phi_\lambda(r)\frac{dr}{r}
\end{aligned}
$$

where K is a constant depending only on A and B, and

$$\phi_\lambda(r) = \max\{\|\lambda B(\lambda B - re^{i\theta_0})^{-1}x\|,\ \|\lambda B(\lambda B - re^{-i\theta_0})^{-1}x\|\} = \phi_1\left(\frac{r}{\lambda}\right).$$

The hypothesis $x \in D_B(\theta, p)$ means that $r^\theta\phi_1(r) \in L^p_*(\mathbf{R}^+)$ (see [DG]); thus we have

$$\begin{aligned}
&t^\theta\|\lambda B(\lambda B + t)^{-1}\lambda B y_\lambda\| \\
&\le K\int_0^{+\infty}\frac{rt^\theta}{\sqrt{t^2 + r^2 + 2trcos\theta_0}}\phi_\lambda(r)\frac{dr}{r} \\
&= K\int_0^{+\infty}\frac{(rt^{-1})^{1-\theta}}{\sqrt{1 + (rt^{-1})^2 + 2rt^{-1}cos\theta_0}}r^\theta\phi_\lambda(r)\frac{dr}{r} \\
&= Kf * g(t)
\end{aligned}$$

where

$$\begin{aligned}
f(t) &= \frac{t^{1-\theta}}{\sqrt{1 + t^2 + 2tcos\theta_0}} \in L^1_*(\mathbf{R}^+) \\
g(t) &= t^\theta\phi_\lambda(t) \in L^p_*(\mathbf{R}^+)
\end{aligned}$$

By Young's theorem, we can write

$$\begin{aligned}
&\|t^\theta\lambda B(\lambda B + t)^{-1}\lambda B y_\lambda\|_{L^p_*(\mathbf{R}^+)} \\
&\le K\|f\|_{L^1_*(\mathbf{R}^+)}\|g\|_{L^p_*(\mathbf{R}^+)} \\
&\le K'\left(\int_0^{+\infty}(r^\theta\phi_\lambda(r))^p\frac{dr}{r}\right)^{1/p} \\
&= K'\lambda^\theta\left(\int_0^{+\infty}(r^\theta\phi_1(r))^p\frac{dr}{r}\right)^{1/p} \\
&\le K''\lambda^\theta\|x\|_{D_B(\theta,p)}.
\end{aligned}$$

where K'' is a constant depending only on A and B, see [DG]. On the other hand,

$$\begin{aligned}
&\|t^\theta\lambda B(\lambda B + t)^{-1}\lambda B y_\lambda\|_{L^p_*(\mathbf{R}^+)} \\
&= \left(\int_0^{+\infty}\left(t^\theta\|\lambda B(\lambda B + t)^{-1}\lambda B y_\lambda\|\right)^p\frac{dt}{t}\right)^{1/p} \\
&= \lambda^{1+\theta}\left(\int_0^{+\infty}\left(t^\theta\|B(B + t)^{-1}B y_\lambda\|\right)^p\frac{dt}{t}\right)^{1/p} \\
&= \lambda^{1+\theta}\|By_\lambda\|_{D_B(\theta,p)};
\end{aligned}$$

hence

$$\lambda^\theta \|\lambda B y_\lambda\|_{D_B(\theta,p)} \leq K''\lambda^\theta \|x\|_{D_B(\theta,p)}$$

or

$$\|\lambda B(\overline{A+\lambda B})^{-1}x\|_{D_B(\theta,p)} \leq K''\|x\|_{D_B(\theta,p)}.$$

This is the inequality that we wanted. It implies that

$$\|\lambda B(\overline{A+\lambda B})^{-1}\|_{D_B(\theta,p)} \leq K'',$$

which shows the λ-regularity of the pair (A, B) on $D_B(\theta, p)$ by Remark 4.2. □

Let us mention another case of λ-regularity which is a consequence of Theorem 1.2 applied in the context of [DV], namely when B^{is} is bounded for all $s \in [-1, +1]$:

COROLLARY 2.3. *Let H be a Hilbert space and let A and B be two positive operators in H satisfying H_0, H_1 and H_2. If* $0 \in \rho(B)$ *and* $\sup\{\|B^{is}\| \mid |s| \leq 1\} < +\infty$, *then the pair (A, B) is λ-regular in H.*

Proof of Corollary 2.3. As mentioned in [DV], under the hypothesis that $\sup\{\|B^{is}\| \mid |s| \leq 1\} < +\infty$, $D_B(\theta, 2) = D(B^\theta)$. Thus Theorem 2.1 implies that (A, B) is a λ-regular pair in $D(B^\theta)$. Then Dore and Venni show that, under the hypothesis of Corollary 2.3, (A, B) is a regular pair in H. An adaptation of their proof can be done to prove that in fact, the pair is λ-regular. Indeed, for $x \in H$, by Theorem 2.1, observing that $B^{-\theta}x \in D_B(\theta, 2)$, we have

$$\begin{aligned}\|\lambda B(A+\lambda B)^{-1}x\| &= \|B^\theta \lambda B(A+\lambda B)^{-1}B^{-\theta}x\| \\ &\leq C\|B^\theta B^{-\theta}x\| = C\|x\|\end{aligned}$$

where $C > 0$ is independent of $\lambda > 0$. □

Construction of Example 2.2. Let G be a complex Hilbert space and let A and B be two positive operators with B bounded, satisfying hypotheses H_1 and H_2. Observe that since B is bounded, H_0 is also satisfied. If moreover $0 \in \rho(A)$, then by Theorem 1.1, the pair (A, B) is regular and $G = D_B(\theta, p)$ for every $\theta \in (0, 1)$ and $p \in [1, \infty]$. Hence if the pair (A, B) is not λ-regular, we are done.

In order to construct such a pair, we consider, as in [BC], the space

$$G = \ell_2(H) = \left\{ x = (x_k)_{k\in\mathbf{N}}\ ,\ x_k \in H \text{ and } \|x\|^2 = \sum_{k=1}^{+\infty} \|x_k\|^2 < +\infty \right\}$$

where $(H, \|.\|)$ is a complex Hilbert space. A family $(A_k)_{k\in\mathbf{N}}$ of bounded operators on H defines the following closed densely defined operator A on G:

$$\text{(2.1)} \qquad \begin{cases} D(A) := \{x = (x_k)_{k\in\mathbf{N}} \ , \ x_k \in H \ , \ \sum_{k\in\mathbf{N}} \|A_k x_k\|^2 < \infty\} \\ (Ax)_k := A_k x_k \ , \ k \in \mathbf{N} \text{ for } x = (x_k)_{k\in\mathbf{N}} \in D(A). \end{cases}$$

Moreover A is bounded if and only if $\sup_{k\in\mathbf{N}} \|A_k\| < \infty$ and if this is the case, we have $\|A\| = \sup_{k\in\mathbf{N}} \|A_k\|$.

If $0 \in \rho(A_k)$ for all $k \in \mathbf{N}$ and $\sup_{k\in\mathbf{N}} \|A_k^{-1}\| < \infty$, then $0 \in \rho(A)$. As in [BC], we shall say that the family of positive operators $(A_k)_{k\in\mathbf{N}}$ of type $(0, M_k)$ satisfies property (P) if for every $k \in \mathbf{N}$,

(i) $\sigma(A_k) \subset [0, \infty)$ and

(ii) for every $\theta \in [0, \pi[$, there is $M(\theta)$, independent of k, such that $\|(I+zA_k)^{-1}\| \leq M(\theta)$, for every $z \in \Sigma_\theta$.

We will need the following slight extension of Lemma 4.1 of [BC], which we state without proof.

LEMMA 2.4. *Let $(A_k)_{k\in\mathbf{N}}$, $(B_k)_{k\in\mathbf{N}}$ be two families of bounded positive operators on H, satisfying property (P) and such that $A_kB_k = B_kA_k$ for all $k \in \mathbf{N}$. Then the operators A and B defined by* (2.1) *are densely defined and of type* $(0, M_A)$ *and* $(0, M_B)$ *respectively. Moreover, the pair (A, B) satisfies hypotheses H_0, H_1, H_2.*

Now suppose that $(A_k)_{k\in\mathbf{N}}$ and $(\tilde{B}_k)_{k\in\mathbf{N}}$ are two families of operators in H as in Lemma 2.4 satisfying (2.2) and (2.3):

$$\text{(2.2)} \qquad 0 \in \rho(A_k) \text{ for every } k \in \mathbf{N} \text{ and } \sup_{k\in\mathbf{N}} \|A_k^{-1}\| < \infty$$

$$\text{(2.3)} \qquad \forall l \geq 1 \ , \ \exists x_l \in H, \ \|x_l\| = 1, \ \text{ such that } l\|A_l x_l + \tilde{B}_l x_l\| \leq \|A_l x_l\|.$$

Set $B_k = \mu_k \tilde{B}_k$, with $\mu_k > 0$, $k \in \mathbf{N}$ such that $\|B_k\| \leq 1$ for all $k \in \mathbf{N}$. Then the families $(A_k)_{k\in\mathbf{N}}$ and $(B_k)_{k\in\mathbf{N}}$ also satisfy the assumptions of Lemma 2.4. The pair (A, B) defined by (2.1) satisfies H_0, H_1, H_2. Moreover $0 \in \rho(A)$ by (2.2) and B is bounded with $\|B\| \leq 1$.

We claim that the regular pair (A, B) is not λ-regular. Clearly for every $\lambda > 0$, the pair $(A, \lambda B)$ is regular and if (A, B) is λ-regular, then there exists $M \geq 1$, independent of λ such that for all $y \in G$,

$$\text{(2.4)} \qquad \|A(A + \lambda B)^{-1} y\| \leq M\|y\|$$

Choose $y = y^{(l)} = (y_k^{(l)})_{k\in\mathbf{N}}$ with

$$\begin{aligned} y_k^{(l)} &= 0 \text{ for } k \neq l \\ y_l^{(l)} &= (A_l + \tilde{B}_l)x_l, \ l \in \mathbf{N}. \end{aligned}$$

Hence with $\lambda = \mu_l^{-1}$, from (2.4) we obtain

$$M\|(A_l + \tilde{B}_l)x_l\| \geq \|A_l x_l\| \geq l\|(A_l + \tilde{B}_l)x_l\| \tag{2.5}$$

for every $l \in \mathbf{N}$, a contradiction since $\|(A_l + \tilde{B}_l)x_l\| \neq 0$.

It remains to construct the operators A_l and $\tilde{B}_l$. For this purpose, we shall need the following lemma, which can be essentially found in [BC].

LEMMA 2.5. *Let H be a complex separable Hilbert space with a Schauder basis $(e_n)_{n\in\mathbf{N}}$ and let $(e_n^*)_{n\in\mathbf{N}}$ be the corresponding coordinate functionals. Let $(c_n)_{n\in\mathbf{N}}$ be a nondecreasing sequence of positive real numbers and let C_k be the linear operators defined by*

$$C_k x := \sum_{l=0}^{N_k} c_l e_l^*(x) e_k \tag{2.6}$$

where $N_k \in \mathbf{N}$ for all $k \in \mathbf{N}$.

Then the operators C_k are bounded positive operators of type $(0, M_k)$ satisfying property (P). Moreover, $0 \in \rho(C_k)$ for all $k \in \mathbf{N}$ and $\sup_{k\in\mathbf{N}} \|C_k^{-1}\| < \infty$.

In view of this lemma, if $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are two nondecreasing sequences of positive numbers and A_k, $\tilde{B}_k$ are defined by (2.6) where $(N_k)_{k\in\mathbf{N}}$ is an arbitrary sequence of natural numbers, then the operators A_k, $\tilde{B}_k$ satisfy all required properties except (2.3). In order to satisfy this condition, we choose for $(e_n)_{n\in\mathbf{N}}$ a conditional basis of ℓ_2 as in [BC] and we choose for $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ the sequences denoted by $f(n)$ and $g(n)$ in [BC], having the property that

$$\sup_{x\in G_0\,,\ \|x\|=1} \left\| \sum_{k=0}^{\infty} \frac{a_k}{a_k + b_k} e_k^*(x) e_k \right\| = \infty$$

where $G_0 = \operatorname{span}\{e_n\ ,\ n \in \mathbf{N}\}$. It follows that for every $l \in \mathbf{N}$, there exists $N_l \in \mathbf{N}$ and $\alpha_{k,l} \in \mathbf{C}$ for $0 \leq k \leq l$ such that

$$\left\| \sum_{k=0}^{N_l} \frac{a_k}{a_k + b_k} e_k^*(y^{(l)}) e_k \right\| \geq l$$

where $y^{(l)} = \sum_{k=0}^{N_l} \alpha_{k,l} e_l$, $0 < \|y^{(l)}\| \leq 1$. Setting

$$\begin{cases} A_k x = \sum_{m=0}^{N_k} a_m e_m^*(x) e_m \\ \tilde{B}_k x = \sum_{m=0}^{N_k} b_m e_m^*(x) e_m \end{cases}$$

we obtain

$$\|A_l(A_l + \tilde{B}_l)^{-1} y^{(l)}\| \geq l\|y^{(l)}\|$$

or equivalently

$$\|A_l\tilde{x}^{(l)}\| \geq l\|(A_l + \tilde{B}_l)\tilde{x}^{(l)}\|$$

where $\tilde{x}^{(l)} = (A_l + \tilde{B}_l)^{-1}y^{(l)} \neq 0$. Setting

$$x^{(l)} = \frac{\tilde{x}^{(l)}}{\|\tilde{x}^{(l)}\|}$$

we obtain (2.3). This concludes the construction of Example 2.2. □

Remark 7. In this construction, we can obtain a bounded operator A' by defining

$$A'_k = \nu_k A_k \text{ with } \nu_k > 0\ ,\ k \in \mathbf{N}$$

in order to ensure that $\|A'_k\| \leq 1$. Then, similar arguments show that the pair (A', B) does not satisfy $(1.1)_\lambda$ although it satisfies (1.1).

It follows from Theorem 2.1 that $0 \notin \rho(A') \cup \rho(B)$. Hence one cannot assert as in Example 2.2 that the pair (A', B) is regular.

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