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SOME STRUCTURE THEOREMS FOR COMPLETE H-SURFACES IN HYPERBOLIC 3-SPACE $\mathbb{H}^3$

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1. Introduction

Let M be a properly embedded connected constant mean curvature surface (nonzero) in hyperbolic 3-space $\mathbb{H}^3$ with boundary a strictly convex curve C. We assume M is complete and C is contained in a geodesic plane P. Let $\mathbb{H}^3_+$ be one of the two half-spaces determined by P.

In [NR] it is shown that when M is compact and transverse to P along C, then M is entirely contained in a half-space of $\mathbb{H}^3$ determined by P. Then all the symmetries of C are also symmetries of M; in particular, M is spherical if C is a circle.

In Euclidean 3-space, some interesting results on complete noncompact H-surfaces are obtained in [RS.1]. Our main contribution is to extend this work to the hyperbolic case.

We study complete noncompact M of finite topology that are transverse to P along C.

In the first part of this paper, we further assume that M is contained in $\mathbb{H}^3_+$. We will prove that if M is contained in a solid half-cylinder (i.e., the integral curves of the Killing vector field associated with the hyperbolic translation along a geodesic at a bounded distance from this geodesic) orthogonal to P of $\mathbb{H}^3_+$ outside of some compact set of $\mathbb{H}^3$, then M inherits the symmetries of C.

Then we give a generalisation of this result; we allow M to have a finite number of cylindrically bounded ends orthogonal to P contained in $\mathbb{H}^3_+$. In this case, M also inherits the symmetries of C. In particular, M is equal to a Delaunay surface when C is a circle.

In the second part of this paper, we give conditions that ensure M is contained in $\mathbb{H}^3_+$ when M is contained in $\mathbb{H}^3_+$ only near C and when the ends of M are annular ends orthogonal to P.

2. Symmetries of complete noncompact H-surfaces in a half-space of $\mathbb{H}^3$

Let P be a geodesic plane in hyperbolic space $\mathbb{H}^3$ and let $\mathbb{H}^3_+$ be one of the two half-spaces determined by P.

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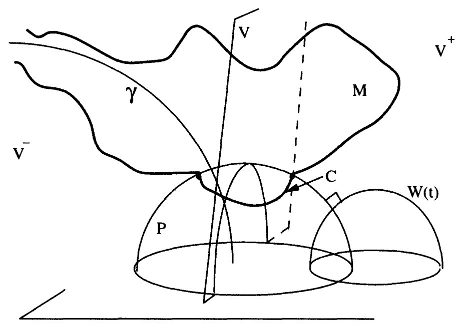

Figure 1

THEOREM 2.1. *Let M be a properly embedded complete noncompact constant mean curvature surface of finite topology in $\mathbb{H}^3$. Suppose $\partial M = C$ is a strictly convex curve in P, and M is transverse to P along C. If $M \subset \mathbb{H}^3_+$ and M is cylindrically bounded outside of some compact set of $\mathbb{H}^3$, then M inherits the symmetries of C.*

Remark 2.2. The hypothesis that M is contained in a solid half-cylinder Z, outside of some compact set, implies that M has mean curvature greater than one:

First, cylinders of hyperbolic radius R have mean curvature $H_Z = \coth(2R)$ and so $H_Z > 1$.

Second, the mean curvature of M is at least as big as the mean curvature of Z. The proof of this latter result is inspired by [RS.2]. The idea is to deform a compact annulus, say K, of height $4R$ of the half-cylinder Z (we choose K such that $\partial K \cap M$ is empty) along the one-parameter family of Delaunay surfaces with constant mean curvature H_Z that have maximum bulge at the height of ∂K . The family converges to one period of a chain of spheres, so there must be a Delaunay surface in the family that first makes one-sided tangential contact at an interior point of M. Hence the mean curvature of M is at least that of Z.

Remark 2.3. In [KKMS] it is proved that any noncompact properly embedded constant mean curvature surface of finite topology in $\mathbb{H}^3$, which is cylindrically bounded and has a compact boundary, must approach a Delaunay surface exponentially at infinity. So our first theorem works for one-end surfaces.

Proof of Theorem 2.1. We will prove that M is invariant by reflection in every plane V that is a plane of symmetry of C. The idea is to show that M is almost invariant by reflection in ε-tilted planes from V, for every $\varepsilon > 0$.

Let γ be the geodesic orthogonal to P such that M stays in the solid half-cylinder about γ in $\mathbb{H}^3_+$ outside of some compact set. Denote $\gamma \cap P$ by q. The end has asymptotically the direction of γ.

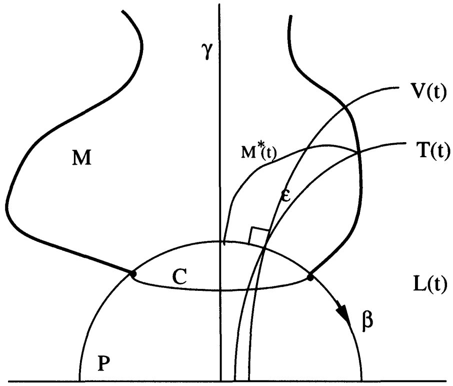

Figure 2

Now we will prove that γ must lie in V. Suppose, on the contrary, that γ is in one of the half-spaces determined by V, say V^-. Let $W(t)$ be a family of planes in V^+ orthogonal to P along a geodesic in P such that $W(0) = V$. The part of M in V^+ is compact. For t large, $W(t)$ is disjoint from M. We apply the Alexandrov reflection technique to M and the planes $W(t)$ (we will explain this technique in more detail later). By increasing t, no accident will occur (i.e., the symmetry of an interior point of M will not touch C) before reaching $t = 0$ and then, at the two points $C \cap W(0)$, the boundary maximum principle implies that M is invariant by reflection in V and therefore compact, which contradicts the assumption that γ is in V^- (Figure 1).

Let β be the geodesic in P orthogonal to $V \cap P$ at q. Let $\varepsilon > 0$ and let T be a plane that forms an angle ε with V at q and $T \cap P = V \cap P$. Since the end of M has asymptotically the direction of γ, one can translate T along γ (i.e., every geodesic uniquely determines a one-parameter group of hyperbolic translations) to a plane $\tilde{T}$ so that C and the end of M are in distinct half-spaces of $\mathbb{H}^3$ determined by $\tilde{T}$. Let L be the half-space of $\mathbb{H}^3 \backslash \tilde{T}$ that contains C.

Let $D \subset P$ be the simply connected domain bounded by C. Notice that $M \cup D$ is a properly embedded submanifold of $\mathbb{H}^3$ (with corner along C) hence separates $\mathbb{H}^3$ into two components. Parametrize β so that $\beta(0) = q$ and for $\beta(t) \subset L$, t will be positive.

Let $T(t)$ be the family of ε-tilted planes along $\beta(t)$, $0 \leq t < \infty$, such that $T(0) = T$. For t large, $T(t)$ is disjoint from M. We apply the Alexandrov reflection process to M and the planes $T(t)$.

Let $L(t)$ be the half-space of $\mathbb{H}^3 \setminus T(t)$ that contains $\beta(\tau)$ for τ larger than t. Let $M(t)^*$ denote the symmetry of $M \cap L(t)$ through $T(t)$. Notice that the symmetry of $P \cap L(t)$ through $T(t)$ intersects P only in $T(t) \cap P$ since $T(t)$ is ε-tilted (Figure 2).

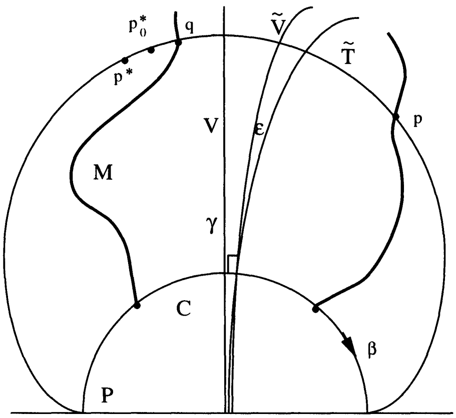

Figure 3

As the planes $T(t)$ approach T, consider the first point of contact of $M(t)^*$ with M. This point cannot be the image of an interior point of M since this implies by the maximum principle that M is invariant by this tilted plane, which is impossible (the end of M is orthogonal to P).

Another possibility is that at the first contact point M is orthogonal to $T(t)$. Then the boundary maximum principle implies that M is invariant by reflection in the tilted plane $T(t)$, which is not possible. Therefore, the first point of contact must be the image of a point of C or there is no contact point until $t = 0$. This is still true for ε as small as we want.

Now we will prove that M is invariant by reflection in V. Suppose, on the contrary, that M is not symmetric in V. Then there are points p, q on M such that the Killing segment $[p, q]$ (i.e., the integral curve of the Killing vector field associated with the hyperbolic translation along β) joining p to q is orthogonal to V, p and q are on opposite sides of V, and $d(p, V) > d(q, V)$. (Each orbit of the Killing field of β is invariant by symmetry in V.) Thus the symmetry p^* of p through V is on the other side of q, i.e., $[q, p] \subset [p^*, p]$ (Figure 3).

Now move V along β towards p to a plane $\tilde{V}$ that is orthogonal to P, so that the symmetry p_0^* of p through $\tilde{V}$ still satisfies $[q, p] \subset [p_0^*, p]$. If $\tilde{V}$ is close to V, then this is always the case.

It follows that the symmetry of M through $\tilde{V}$ (the side of M containing p) intersects M in more than just $M \cap \tilde{V}$, i.e., in interior points of $M \setminus \tilde{V}$. If $\tilde{T}$ denotes the planes $\tilde{V}$ tilted an angle ε at $\beta \cap \tilde{V}$, then for ε sufficiently small, the symmetry of M through $\tilde{T}$ still intersects M in more than $M \cap \tilde{V}$. ($\star$) (There are two ways to tilt $\tilde{V}$. We do this so that the symmetry C^* of the shorter arc of $C \setminus \tilde{T}$ through $\tilde{T}$ is in $\mathbb{H}^3_+$ and does not touch P.)

Now M is transverse to P along C, so for ε sufficiently small, C^* intersects M only at the endpoints of C^*; i.e., $C^* \cap \tilde{T}$. Also, one can choose $\varepsilon > 0$ so that this last condition holds for symmetry in all planes sufficiently close to $\tilde{T}$ and ε-tilted along β.

Notice that C^* is not tangent to M at its endpoints because C is strictly convex.

Let $V(t)$ be a plane orthogonal to P along β and on the same side of $\tilde{V}$ as p, intersecting C in two points, or one point. Since M is transverse to P along C and V is a plane of symmetry of C, there is an $\varepsilon > 0$ such that if $T(t)$ denotes $V(t)$ tilted by ε along β, then the symmetry of the short arc of $C \setminus T(t)$ through $T(t)$ intersects M only at its endpoints. Moreover, ε can be chosen to work for all planes sufficiently close to $T(t)$ and ε-tilted along β.

Now by compactness of the shorter arc of $C \setminus \tilde{V}$, there is an $\varepsilon > 0$ such that this property holds for all planes $T(t)$ on the same side as p.

For t sufficiently large, $T(t)$ is disjoint from M. As the planes $T(t)$ approach $\tilde{T}$, consider the first point of contact of $M(t)^*$ with M.

As we have shown before, this first point of contact must be the image of a point of C. However, by our choice of $\tilde{T}$, there is no such point of C.

This contradicts ($\star$) and, therefore, M is symmetric in V and we have proved Theorem 2.1. □

THEOREM 2.4. *Let M be a properly embedded complete noncompact constant mean curvature surface of finite topology in $\mathbb{H}^3$. Suppose $\partial M = C$ is a strictly convex curve in P and M is transverse to P along C. If $M \subset \mathbb{H}^3_+$ and M has only cylindrically bounded ends with asymptotic axes all orthogonal to P, then M inherits the symmetries of C. In particular, M is equal to a Delaunay surface when C is a circle.*

Proof. Let V be a plane of symmetry of C . We have only to show that the axes of all the ends are contained in V, then by a similar reasoning as in the proof of *Theorem* 2.1. we can prove Theorem 2.4.

Consider one of the half-spaces determined by V in $\mathbb{H}^3$, say V^-. Let $\beta(t)$ be a geodesic in $V^- \cap P$ orthogonal to $V \cap P$, t positive. Let $\varepsilon > 0$ and let T be a plane passing through $V \cap P$ that forms an angle ε with V at $\beta(0)$ such that $T \cap \mathbb{H}^3_+ \subset V^-$. Let $T(t)$ be the family of ε-tilted planes along β such that $T(0) = T$ and denoted by $T(t)^-$ the half-space of $\mathbb{H}^3$ determined by $T(t)$ contained in V^-.

Notice that M is transverse to P along C, so for ε sufficiently small, the symmetry of $C \cap T(t)^-$ does not touch M, for all t.

M has a finite number of cylindrically bounded ends, so for t large $T(t)$ is disjoint from M. We apply the Alexandrov reflection process to M and the planes $T(t)$. As the planes $T(t)$ approach T, consider the first point of contact of the symmetries of $M \cap T(t)^-$ through $T(t)$ with M. This point cannot be the image of an interior point of M by the maximum principle. Also, by our choice of ε, this point is not the image of a point of C.

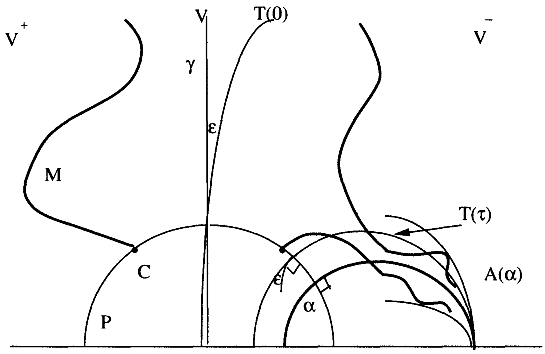

Figure 4

However, another possibility is that $T(t)$ meets M at infinity and we cannot apply the maximum principle. This means that M has an end in V^-; let α denote its axis. The end stays in the Killing cylinder about α and approaches a Delaunay surface $D(\alpha)$ exponentially at infinity. By our way of tilting T, if $\alpha \cap \partial\mathbb{H}^3 \subset T(\tau)$ so $\alpha \subset T(\tau)^-$. Let $A(\alpha)$ be the Killing cylinder tangent to $D(\alpha)$ and containing $D(\alpha)$. For $\tilde{\tau} > \tau$ and close to τ, the symmetry of $T(\tilde{\tau})^- \cap A(\alpha)$ through $T(\tilde{\tau})$ is not contained in $A(\alpha)$. Hence the symmetry of $M \subset T(\tilde{\tau})^-$ intersects M (Figure 4).

This implies that there occurred a first contact point before reaching $t = \tau$. Therefore there is no end in V^-. We can repeat this process in V^+.

So the result follows. □

3. Conditions on complete noncompact H-surfaces to be in a half-space

Now we study surfaces M satisfying the hypothesis of Theorem 2.4 except we do not assume M is globally contained in $\mathbb{H}^3_+$. Our interest is to obtain natural geometric conditions that force such a surface to be in a half-space. In Corollary 3.2 of this section we will see that Theorem 3.1 gives such conditions.

With each annular end of M we associate its axis α_i (i.e., a half-geodesic in $\mathbb{H}^3_+$ orthogonal to P) and write $\mathcal{A}(\alpha_i)$, $1 \le i \le n$. Let $\mathcal{D}(\alpha_i)$ denote the limited Delaunay surface which we parametrize by τ_i, the radius of a smallest circle orthogonal to α_i. For mean curvature H, one has $\tanh(2\tau_i) \le \frac{1}{H}$. Let γ be any geodesic orthogonal to P and let $Y(\gamma)$ be the Killing vector field associated with the hyperbolic translation along γ . We choose the orientation of the vector field such that $Y(\gamma)$ on P is pointing in $\mathbb{H}^3_+$.

Let D be the domain in P bounded by C.

THEOREM 3.1. *Let M be a properly embedded complete noncompact constant mean curvature surface in $\mathbb{H}^3$. Suppose $\partial M = C$ is a strictly convex curve in*

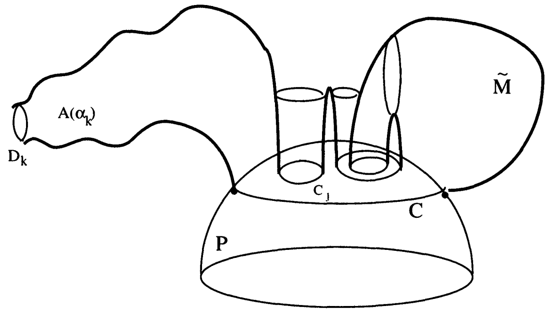

Figure 5

P and M is transverse to P. If M has a finite number of cylindrically bounded ends, topologically an annulus with asymptotic axes all orthogonal to P, in $\mathbb{H}^3_+$ *and* $M \subset \mathbb{H}^3_+$ *near C, then either* $M \subset \mathbb{H}^3_+$ *or there is a simple closed curve in* $M \cap \operatorname{ext} D \cap P$ *that generates* $\pi_1(\operatorname{ext} D)$*, and*

$$\int_D |Y(\gamma)| < \sum_{i=1}^{n} \pi \cosh\{\operatorname{dist}(\alpha_i, \gamma)\} \sinh(\tau_i) \left(\frac{\cosh(\tau_i)}{H} - \sinh(\tau_i) \right).$$

COROLLARY 3.2. *Assume M satisfies the hypothesis of Theorem* 3.1 *and* ∂M *is a circle of radius R with center at* $\gamma \cap P$*. If* $\sinh^2 R \geq nd\frac{H-\sqrt{H^2-1}}{2H}$*, then M equals a Delaunay surface.* (Here n is the number of ends and $d = \sup_i \cosh\{\operatorname{dist}(\alpha_i, \gamma)\}$.)

Proof of Theorem 3.1. Let D_i be a disk orthogonal to α_i whose boundary is a simple closed curve of $\mathcal{A}(\alpha_i)$, a generator of $\pi_1(\mathcal{A}(\alpha_i))$. Let ν_i denote the conormal to M along ∂D_i , oriented such that $\langle \nu_i, Y(\alpha_i) \rangle$ is negative, and let ν denote the conormal to M along C oriented to point in $\mathbb{H}^3_+$.

Notice that each Killing vector field associated with a geodesic orthogonal to P is orthogonal at each point in P.

First we prove that $M \cap \operatorname{ext} D = \emptyset$ and $M \cap \operatorname{int} D \neq \emptyset$ is impossible. So we assume the contrary and arrive at a contradiction.

Let $\tilde{M}$ be the connected component of $M \cap \mathbb{H}^3_+$ that contains C, $\partial \tilde{M} = C \cup C_1 \cup \cdots \cup C_m$ where the C_j, $1 \leq j \leq m$, are the simple closed curves of M in $\operatorname{int} D$. $\tilde{M}$ together with a proper subdomain D_0 of $\operatorname{int} D$ bound a 3-dimensional domain noncompact in $\mathbb{H}^3_+$.

For each annular end $\mathcal{A}(\alpha_k)$ in $\tilde{M}$, we consider the disk D_k. We form a compact embedded cycle $\bar{M}$ by removing from $\tilde{M} \cup D_0$ the part of each $\mathcal{A}(\alpha_k)$ that is above ∂D_k and attaching D_k (Figure 5).

Let $Y(\gamma)$ be the Killing vector field of $\mathbb{H}^3_+$ associated with the hyperbolic translation along γ, a half-geodesic orthogonal to P. The flux of $Y(\gamma)$ across $\bar{M}$ is zero and this yields the balancing formula [KKMS]:

$$\frac{1}{2H}\int_{\partial D_0}\langle Y(\gamma),\nu\rangle=\int_{D_0}\langle Y(\gamma),n_{D_0}\rangle+\sum_k m_k \qquad (\star)$$

where m_k is the mass of the end in direction α_k:

$$m_k=\pi\cosh\{\operatorname{dist}(\alpha_k,\gamma)\}\sinh(\tau_k)\left(\frac{\cosh(\tau_k)}{H}-\sinh(\tau_k)\right)$$

(m_k is positive) and n_{D_0} is the normal orienting D_0.

We indicate how $(\star)$ is derived: since the flux is zero across $\bar{M}$,

$$\int_{D_0}\langle Y(\gamma),n_{D_0}\rangle+\int_{\hat{M}}\langle Y(\gamma),n_{\hat{M}}\rangle+\sum_k\int_{D_k}\langle Y(\gamma),n_{D_k}\rangle=0 \qquad \text{(i)}$$

where $\hat{M}=\bar{M}\setminus(D_0\cup\bigcup_k D_k)$, $n_{\hat{M}}=\frac{\vec{H}}{|\vec{H}|}$ and $n_{D_k}=-\frac{Y(\alpha_k)}{|Y(\alpha_k)|}$.

Here n_{D_0} is a unit vector field of $\mathbb{H}^3$ orthogonal to P and pointing in $\mathbb{H}^3_+$. The reason for this is that $\vec{H}$ points towards D_0 along C because there are no exterior intersection curves of M with P by assumption.

Furthermore one has

$$\int_{\hat{M}}\langle Y(\gamma),n_{\hat{M}}\rangle=-\frac{1}{2H}\int_{\partial\hat{M}}\langle Y(\gamma),\nu\rangle, \qquad \text{(ii)}$$

ν the inward pointing conormal to $\hat{M}$.

Each annular end $\mathcal{A}(\alpha_k)$ converges geometrically to a Delaunay end $\mathcal{D}(\alpha_k)$. Thus

$$\int_{D_k}\langle Y(\gamma),n_{D_k}\rangle-\frac{1}{2H}\int_{\partial D_k}\langle Y(\gamma),\nu_k\rangle$$
$$=\pi\cosh\{\operatorname{dist}(\alpha_k,\gamma)\}\sinh(\tau_k)\left(\frac{\cosh(\tau_k)}{H}-\sinh(\tau_k)\right) \qquad \text{(iii)}$$

Here τ_k is the radius of a smallest circle orthogonal to α_k of $\mathcal{D}(\alpha_k)$. The hyperbolic distance between the geodesics α_k and γ is realized by a geodesic segment in P since both are orthogonal to P. To prove this last formula, one constructs a compact cycle and applies the divergence theorem. Choose a planar disk $\tilde{D}_k$, whose boundary approximates a "shortest parallel" circle of $\mathcal{D}(\alpha_k)$ and do this so that $\partial\tilde{D}_k\cup\partial D_k$ bounds an embedded annulus on $\mathcal{A}(\alpha_k)$. Then, using (ii) also,

$$\int_{D_k}\langle Y(\gamma),n_{D_k}\rangle-\frac{1}{2H}\int_{\partial D_k}\langle Y(\gamma),\nu_k\rangle=-\int_{\tilde{D}_k}\langle Y(\gamma),n_{\tilde{D}_k}\rangle+\frac{1}{2H}\int_{\partial\tilde{D}_k}\langle Y(\gamma),\tilde{\nu}_k\rangle$$

where $n_{\tilde{D}_k}=\frac{Y(\alpha_k)}{|Y(\alpha_k)|}$ and $\langle Y(\alpha_k),\tilde{\nu}_k\rangle>0$.

As the $\tilde{D}_k$ get higher, $\partial \tilde{D}_k$ converges geometrically to a shortest circle of $\mathcal{D}(\alpha_k)$ and hence $\tilde{\nu}_k$ converges to $\frac{Y(\alpha_k)}{|Y(\alpha_k)|}$. Therefore the right side of the above equation is constant and equals to the flux of the Delaunay surface evaluated with respect to γ (we explain this calculation in more details in the appendix). This quantity m_k is positive since $\tanh(2\tau_i) \leq \frac{1}{H}$.

Taking (i), (ii), (iii) together into account, one obtains ($\star$).

Now $\langle \nu, Y(\gamma)\rangle$ is positive along ∂D_0, so ($\star$) implies

$$\frac{1}{2H}\int_C \langle Y(\gamma), \nu\rangle < \int_{D_0} \langle Y(\gamma), n_{D_0}\rangle + \sum_k m_k. \qquad (\star\star)$$

Next consider the cycle $M \cup D$. The flux of $Y(\gamma)$ across this cycle is also zero; more precisely, one obtains a compact cycle from $M \cup D$ as before, attaching D_i to each annular end $\mathcal{A}(\alpha_i)$ and removing the part of $\mathcal{A}(\alpha_i)$ that is above ∂D_i. The same type of calculation as above yields

$$\frac{1}{2H}\int_{\partial D} \langle Y(\gamma), \nu\rangle = \int_{D} \langle Y(\gamma), n_{D}\rangle + \sum_{i=1}^{n} m_i. \qquad (\star\star\star)$$

Clearly ($\star\star$) and ($\star\star\star$) both together are impossible since $\sum_{i=1}^n m_i \geq \sum_k m_k$ and $\int_{D_0} \langle Y(\gamma), n_{D_0}\rangle < \int_D \langle Y(\gamma), n_D\rangle$. This proves that not all components of $M \cap P$ can be in int D. So we may assume $M \cap \operatorname{ext} D \neq \emptyset$.

Next we will show that there cannot be more then one Jordan curve in ext D that generates $\pi_1(\operatorname{ext} D)$.

The idea is to apply the Alexandrov reflection principle using ε-tilted planes. We must do some cutting and pasting along the cycle of $M \cap \operatorname{int} D$, to obtain a manifold that separates $\mathbb{H}^3$. This enables us to be sure that the mean curvature vectors are pointing in the same direction when we do Alexandrov reflection.

Let σ be a geodesic orthogonal to P at a point of int D and let $P(t)$ be the family of planes orthogonal to σ such that $P(0) = P$ and $-\infty < t < \infty$. Let $C_1, \ldots, C_m$ be the Jordan curves of $M \cap \operatorname{int} D$. For each C_j, let $C_j^+(\epsilon)$ be the planar curve on M, near C_j, obtained by intersecting M with the plane $P(\epsilon)$. Similary, let $C_j^-(\epsilon)$ be the curve $M \cap P(-\epsilon)$ that is near C_j. We form an embedded surface $\tilde{M}$ by removing from M the annuli bounded by $C_j^+(\epsilon) \cup C_j^-(\epsilon)$ and attaching the planar domains $D_j^+(\epsilon) \cup D_j^-(\epsilon)$ bounded by $C_j^+(\epsilon) \cup C_j^-(\epsilon)$. Also, we attach D to M along C.

To ensure that $\tilde{M}$ is embedded, one uses different values of ϵ, when several C_j are concentric (Figure 6). $\tilde{M}$ is a properly embedded submanifold (with corners) of $\mathbb{H}^3$, $\partial \tilde{M} = \emptyset$, hence each connected component of $\tilde{M}$ separates $\mathbb{H}^3$ into two connected components.

Let $\bar{M}$ be the component of $\tilde{M}$ that contains C. We orient $\bar{M}$ by the mean curvature vector $\vec{H}$ of M. Notice that this makes sense since, abstractly, $\bar{M}$ is a submanifold of M (hence $\vec{H}$ is defined over it) to which one has attached D and the disks $D_j^{\pm}(\epsilon)$.

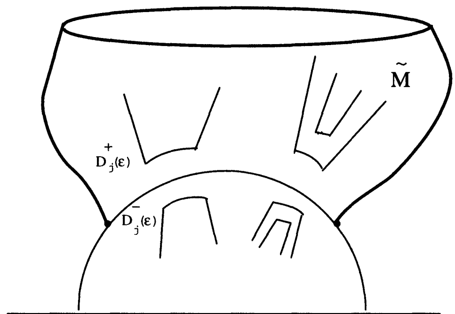

Figure 6

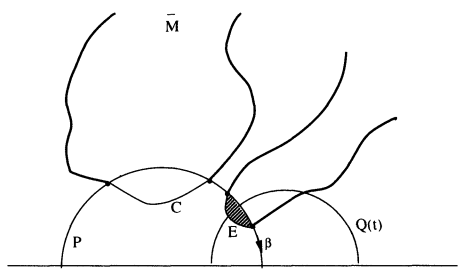

Figure 7

Clearly $\vec{H}$ extends across the disks to define a normal field to $\bar{M}$. The corners of $\bar{M}$ along the boundaries of the disks do not affect this.

Now we start to prove that $\bar{M} \cap \text{ext}\, D$ is at most one cycle, that generates $\pi_1(\text{ext}\, D)$.

First we show that $\bar{M} \cap \text{ext}\, D$ has no components that are null homotopic in ext D. To see this, suppose that E were such a component. Let $\beta(t)$, $0 \leq t \leq \infty$, be a geodesic in P starting at a point of int D and intersecting E in at least two points. Let $Q(t)$ be the family of ε-tilted planes along $\beta(t)$ and denote by $Q(t)^-$ the half-space of $\mathbb{H}^3_+ \setminus Q(t)$ that contains $\beta(\tau)$ for τ larger then t (Figure 7).

There are two ways to tilt the planes. We do this so that the symmetry of $P \cap Q(t)^-$ through $Q(t)$ is contained in $\mathbb{H}^3_+$.

We apply the Alexandrov reflection process to $\bar{M}$ and the planes $Q(t)$. For t large, $Q(t)$ is disjoint from $\bar{M}$. Now if we approach $\bar{M}$ by $Q(t)$, there will be a first contact point of some $Q(t)$ with $\bar{M}$. One continues to decrease t and considers the symmetries of $\bar{M} \cap Q(t)^-$ through $Q(t)$. Since β intersects E in at least two points, there will be a $Q(\tau)$ where the symmetry of E touches $\bar{M}$ at an interior point of $\bar{M}$ near E. This occurs before reaching C since C is convex. Thus, $\bar{M}$ has a plane of symmetry , with C on one side of this plane which is impossible.

We remark that our manner of tilting the planes ensures that no $Q(t)$ touches $\bar{M}$ at infinity for $t > \tau$ (we refer to the argument in the proof of Theorem 2.4).

Second we show that $\bar{M} \cap \operatorname{ext} D$ has no more than one component that is a generator of $\pi_1(\operatorname{ext} D)$. Suppose that E_1, E_2 were two such components. Each of them bounds a domain in P that contains D, so β meets both of the cycles in at least two points. As before, Alexandrov reflection gives a plane of symmetry; more precisely, there is one position of $Q(t)$ before reaching C where a symmetry of $\bar{E}_1$ touches $\bar{M}$ at a interior point near E_2 (assuming E_1 is the exterior cycle). Thus $\bar{M}$ has a plane of symmetry before reaching C which is impossible.

This proves that $\bar{M} \cap \operatorname{ext} D$ is at most one cycle E and E generates $\pi_1(\operatorname{ext} D)$. The mean curvature vector $\vec{H}$ points towards C along E hence $\vec{H}$ points towards $\operatorname{ext} D$ along C.

Now we use the balancing formula

$$\frac{1}{2H}\int_{\partial D}\langle Y(\gamma), \nu\rangle = \int_D \langle Y(\gamma), n_D\rangle + \sum_{i=1}^{n} m_i$$

where $\langle Y(\gamma), \nu\rangle$ is positive along C. Since $\vec{H}$ points towards $\operatorname{ext} D$ along C, $\langle Y(\gamma), n_D\rangle$ is negative. Hence

$$\int_D |Y(\gamma)| < \sum_{i=1}^{n} \pi \cosh\{\operatorname{dist}(\alpha_i, \gamma)\} \sinh(\tau_i)\left(\frac{\cosh(\tau_i)}{H} - \sinh(\tau_i)\right). \qquad \square$$

Proof of Corollary 3.2. Consider the flux function $m(\tau)$ of the one-parameter family of Delaunay surfaces with constant mean curvature H:

$$m(\tau) = \pi \sinh(\tau)\left(\frac{\cosh(\tau)}{H} - \sinh(\tau)\right).$$

This function is zero at $\tau = 0$ corresponding to a chain of spheres and takes its maximum at τ_C where $\coth(2\tau_C) = H$ corresponds to the cylinder of radius τ_C. It is straightforward to check that $m(\tau_C) = \pi \frac{H-\sqrt{H^2-1}}{2H}$ and so

$$m_i = \pi \cosh\{\operatorname{dist}(\alpha_i, \gamma)\} \sinh(\tau_i)\left(\frac{\cosh(\tau_i)}{H} - \sinh(\tau_i)\right) < \cosh\{\operatorname{dist}(\alpha_i, \gamma)\} m(\tau_C)$$

for $1 \le i \le n$. Since $\int_D |Y(\gamma)| = \pi \sinh^2 R$ we get by assumption

$$\int_D |Y(\gamma)| \ge \pi n d \frac{H - \sqrt{H^2 - 1}}{2H}$$

where $d = \sup_i \cosh\{\operatorname{dist}(\alpha_i, \gamma)\}$.

So Theorem 3.1 implies that M is contained in the half-space and therefore M inherits the symmetries of its boundary (by Theorem 2.4). $\square$

4. Appendix

Let γ be any geodesic orthogonal to a fixed plane P and let $Y(\gamma)$ be the Killing vector field associated with the hyperbolic translation along γ . We choose the orientation of the vector field such that $Y(\gamma)$ on P is pointing in $\mathbb{H}^3_+$.

We calculate now the flux of a Delaunay end $\mathcal{D}(\alpha)$ in $\mathbb{H}^3_+$ with respect to a Killing vector field $Y(\gamma)$ where the axe α is a geodesic orthogonal to the plane P. Let τ be the radius of a smallest circle of $\mathcal{D}(\alpha)$ orthogonal to α.

We work in the upper half-space model of hyperbolic space, that is,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$$

with the hyperbolic metric, *i.e.* the Euclidean metric divided by x_3. After an ambient isometry, we can assume that P is $\{x_1^2 + x_2^2 + x_3^2 = 1\}$. Let γ be $\{x_1 = x_2 = 0\}$, so $Y(\gamma)$ is the radial vector field in $\mathbb{H}^3$: $Y(\mathbf{x}) = (x_1, x_2, x_3)$. We translate $\mathcal{D}(\alpha)$ along α such that $\mathcal{D}(\alpha) \cap P$ is a planar disk denoted by D of radius τ.

We want evaluate

$$(\diamond) \qquad m = -\int_D \langle Y(\gamma), n_D\rangle + \frac{1}{2H}\int_{\partial D} \langle Y(\gamma), \nu_D\rangle$$

where $n_D = \frac{Y(\alpha)}{|Y(\alpha)|}$ and $\nu_D = \frac{Y(\alpha)}{|Y(\alpha)|}$.

Notice that by the divergence theorem the choice of the planar disk orthogonal to α of radius τ where its intersection with $\mathcal{D}(\alpha)$ is a smallest circle, does not affect m.

At each point $\mathbf{x}$ in P, $Y_{\mathbf{x}}(\alpha)$ and $Y_{\mathbf{x}}(\gamma)$ are both orthogonal to P so we must only find an expression of the norm of $Y_{\mathbf{x}}(\gamma)$ for the points in $D \subset P$. This norm depends on the hyperbolic distance from $\mathbf{x}$ to $(0, 0, 1)$, i.e., on the x_3 coordinate of $\mathbf{x}$.

Let d be the distance between α and γ. This distance is realized by a geodesic segment in P, denote it by β. The points $\mathbf{x} \in D$ are parametrized by (ϕ, t) where ϕ is the angle between the geodesic segment in P joining $\mathbf{x}$ to the center c of D (i.e., $c = \alpha \cap P$) and β, and t is the hyperbolic distance of this segment, $\phi \in [0, 2\pi]$ and $t \in [0, \tau]$.

Using hyperbolic trigonometry formulas (cf.[B]), we have

$$|Y_{(\phi,t)}(\gamma)| = \cosh t \cosh d - \sinh t \sinh d \cos\phi$$

and so

$$\int_{\partial D} \langle Y(\gamma), \nu_D\rangle = \int_{\phi=0}^{2\pi} |Y_{(\phi,\tau)}(\gamma)| \sinh \tau d\phi$$

$$\int_D \langle Y(\gamma), n_D\rangle = \int_{t=0}^{\tau}\int_{\phi=0}^{2\pi} |Y_{(\phi,t)}(\gamma)| \sinh t d\phi dt.$$

This equations together with $(\diamond)$ imply

$$m = -\pi \sinh^2 \tau \cosh d + \frac{\pi}{H} \sinh\tau \cosh\tau \cosh d$$

The mean curvature of $\mathcal{D}(\alpha)$ is at least as small as the mean curvature of the cylinder of radius of the smallest radius of $\mathcal{D}(\alpha)$. Therefore $\tanh(2\tau) \leq \frac{1}{H}$ and this implies that

$$m = \pi \cosh d \sinh \tau \left(\frac{\cosh \tau}{H} - \sinh \tau \right) > 0.$$

REFERENCES

[B] W. BALLMANN, *Manifolds of nonpositive curvature*, Birkhäuser.

[RS.1] R. SA EARP and H. ROSENBERG, *Some structure theorems for complete Constant mean curvature surfaces with boundary a convex curve*, Proc. Amer. Math. Soc. **113**, (1991), 1045–1053.

[RS.2] ———, *Some remarks on surfaces of prescribed mean curvature*, Differential Geometry, Pitman Monographs and Surveys in Pure and Applied Mathematics, no. 52, (1991), pp. 123–145.

[KKMS] N. KOREVAAR, R. KUSNER, W. H. MEEKS III, and B. SOLOMON, *Constant mean curvature surfaces in hyperbolic space*, Amer. J. Math. **114**, (1992), 1–43.

[NR] B. NELLI and H. ROSENBERG, *Some remarks on embedded hypersurfaces in hyperbolic space of constant curvature and spherical boundary*, Annals of Global Analysis and Geometry **13**, (1995), 23–30.

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