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PLURIHARMONIC SYMBOLS OF ESSENTIALLY COMMUTING TOEPLITZ OPERATORS

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1. Introduction and result

Let V denote the normalized volume measure on the unit ball B of the complex n-space $\mathbb{C}^n$. The Bergman space A^p $(1 \leq p < \infty)$ is the closed subspace of the usual Lebesgue space $L^p = L^p(B, V)$ consisting of holomorphic functions. We let P be the Hilbert space orthogonal projection—called the Bergman projection—from L^2 onto A^2. As is well known, the Bergman projection P is given by the integral formula as follows:

$$P(\psi)(z) = \int_B \frac{\psi(w)}{(1 - z \cdot \bar{w})^{n+1}} \, dV(w) \qquad (z \in B) \tag{1}$$

for functions $\psi \in L^2$. Here and elsewhere, the notation $z \cdot \bar{w} = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ denotes the ordinary Hermitian inner product for points $z, w \in \mathbb{C}^n$. Note that the Bergman projection P naturally extends via the above integral formula to an integral operator from L^1 into the space of all functions holomorphic on B.

For a function $u \in L^2$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

for functions $f \in A^2$. The operator T_u: $A^2 \to A^2$ is densely defined and not bounded in general. However, T_u is always bounded on A^2 for bounded symbols u which we are concerned about in this paper.

We say that two bounded linear operators S, T on a Hilbert space H are *essentially commuting* on H if the commutator $ST - TS$ is compact on H. In the present paper, we consider a characterization problem of essentially commuting Toeplitz operators. In the one dimensional case, K. Stroethoff [S] has obtained a complete description of two harmonic symbols for essentially commuting Toeplitz operators: For bounded harmonic symbols u and v, T_u and T_v are essentially commuting if and only if, for each Hoffman's map L_m, $u \circ L_m$ and $v \circ L_m$ are both holomorphic, or $\bar{u} \circ L_m$ and $\bar{v} \circ L_m$ are both holomorphic, or a nontrivial linear combination of $u \circ L_m$ and

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$v \circ L_m$ is constant. Stroethoff [S] also obtained characterizations in terms of boundary vanishing properties of certain integral and differential quantities.

In this paper, we naturally consider the same characterizing problem on the ball with pluriharmonic symbols. Recall that a function $u \in C^2(B)$ is said to be pluriharmonic if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. As is well known, every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function. Hence, harmonic and pluriharmonic functions coincide on the unit disk.

Stroethoff's paper [S] uses some corona techniques special to the disk. So, in considering the problem on the ball, we cannot use the straightforward modification of Stroethoff's approach. However, we realize that the corona techniques are not essential in the approach used by Stroethoff and we use some other way of approching the problem. Because the corona theorem is unsolved in the ball, the only Hoffman maps used are those lying in Φ (where Φ is described in the next paragraph). To be more precise, let us introduce some notations and basic facts on the maximal ideal space of the ball.

Let H^∞ be the space of all bounded holomorphic functions on B. The maximal ideal space $\mathcal{M}$ of the ball is the set of all multiplicative linear functionals on H^∞. If we think of $\mathcal{M}$ as a subspace of the dual space of H^∞ with weak-star topology, the space $\mathcal{M}$ becomes a compact Hausdorff space. Identifying a point of B with the functional of evaluation at that point, we can regard B as a subset of $\mathcal{M}$. For $z \in B$, let φ_z denote the canonical automorphism (see Section 2) of B. Since B is a subset of $\mathcal{M}$, we can think of φ_z as a map from B to $\mathcal{M}$. The compactness of $\mathcal{M}$ implies that for any net $\{\varphi_{z_\alpha}\}$ of automorphisms, there is a subnet $\{\varphi_{z_\beta}\}$ of $\{\varphi_{z_\alpha}\}$ such that φ_{z_β} converges (pointwise) to a map $\varphi\colon B \to \mathcal{M}$. We let

$$\Phi = \text{closure}\{\varphi_z\colon z \in B\} \setminus \{\varphi_z\colon z \in B\}.$$

By using the Gelfand transform, we can think of H^∞ as a subset of the space of all continuous functions on $\mathcal{M}$. Moreover, it turns out [Z2, Proposition 8] that each bounded pluriharmonic function on B extends to a continuous function on $\mathcal{M}$. We will use the same notation for a bounded pluriharmonic function and its continuous extension on $\mathcal{M}$. In addition, it is also known [Z2, Proposition 9] that if a net $\{\varphi_{z_\alpha}\}$ of automorphisms converges to some $\varphi \in \Phi$, then for any bounded pluriharmonic function u, the function $u \circ \varphi_{z_\alpha}$ converges to $u \circ \varphi$ uniformly on every compact subset of B and hence $u \circ \varphi$ is also a bounded pluriharmonic function on B. For some more information on $\mathcal{M}$, see [Z2].

For $u \in C^2(B)$, the invariant Laplacian $\widetilde{\Delta} u$ is defined by

$$(\widetilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0) \qquad (z \in B)$$

where Δ denotes the ordinary Laplacian. The operator $\widetilde{\Delta}$ commutes with automorphisms in the sense that $\widetilde{\Delta}(u \circ \varphi) = (\widetilde{\Delta} u) \circ \varphi$ for all automorphisms φ of B. For details, see [R, Chapter 4].

Our main result is the following theorem.

THEOREM 1. *Let u, v be two bounded pluriharmonic functions on B and assume $u = f + \bar{g}$, $v = h + \bar{k}$ for some functions f, g, h, k holomorphic on B. Then the following statements are equivalent*:

(a) *T_u and T_v are essentially commuting on A^2.*
(b) $\lim_{|a|\to 1} \widetilde{\Delta}[f\bar{k} - h\bar{g}](a) = 0.$
(c) *For each $\varphi \in \Phi$,* (i) *both $u \circ \varphi$ and $v \circ \varphi$ are holomorphic on B, or* (ii) *both $\bar{u} \circ \varphi$ and $\bar{v} \circ \varphi$ are holomorphic on B, or* (iii) *there exist constants α and β, not both* 0, *such that $\alpha(u \circ \varphi) + \beta(v \circ \varphi)$ is constant on B.*
(d) $\lim_{|a|\to 1} \int_B |(f\circ\varphi_a - f(a))(\bar{k}\circ\varphi_a - \bar{k}(a)) - (h\circ\varphi_a - h(a))(\bar{g}\circ\varphi_a - \bar{g}(a))|\, dV = 0.$

In Section 2, we collect some preliminary results on Toeplitz operators which we need in the proof of Theorem 1. In Section 3, we prove Theorem 1 and, as an immediate consequence, give a characterization of essentially normal Toeplitz operators.

2. Toeplitz operators

For $z \in B$, $z \neq 0$, the explicit formula for the canonical automorphism (= biholomorphic self map) φ_z is given by

$$\varphi_z(w) = \frac{z - |z|^{-2}(w \cdot \bar{z})z - \sqrt{1 - |z|^2}[w - |z|^{-2}(w \cdot \bar{z})z]}{1 - w \cdot \bar{z}}$$

and $\varphi_0(w) = -w$ for $w \in B$. It is well known that $\varphi_z \circ \varphi_z$ is the identity on B and the real Jacobian $J_R\varphi_z$ of φ_z is given by

$$J_R\varphi_z(w) = \left(\frac{1 - |z|^2}{|1 - w \cdot \bar{z}|^2}\right)^{n+1} \qquad (w \in B). \tag{2}$$

In addition, the identity

$$1 - \varphi_z(a) \cdot \overline{\varphi_z(b)} = \frac{(1 - |z|^2)(1 - a \cdot \bar{b})}{(1 - a \cdot \bar{z})(1 - z \cdot \bar{b})} \tag{3}$$

holds for every $a, b \in B$. See [R, Chapter 2] for details.

For $a \in B$, we put

$$k_a(z) = \left(\frac{\sqrt{1 - |a|^2}}{1 - z \cdot \bar{a}}\right)^{n+1} \qquad (z \in B)$$

for notational simplicity. By (2) and (3), we have a useful change-of-variable formula:

$$\int_B h\,dV = \int h \circ \varphi_a |k_a|^2\,dV \qquad (a \in B) \tag{4}$$

for all measurable h on B whenever the integrals make sense.

We start with an observation on how the product of two Toeplitz operators acts on the kernels k_a.

LEMMA 2. *Let $u, v \in L^\infty$ and $h \in A^2$. Then we have*

$$\langle T_{u\circ\varphi_b} T_{v\circ\varphi_b} k_a, h\rangle = \langle T_u T_v[(k_a \circ \varphi_b)k_b], (h \circ \varphi_b)k_b\rangle$$

for every $a, b \in B$. Here and elsewhere, the notation $\langle\ ,\ \rangle$ denotes the usual inner product in L^2.

Given $u \in L^\infty$, it is easy to see that the adjoint operator T_u^* of T_u is $T_{\bar u}$.

Proof. A routine manipulation using (3) yields

$$\frac{(1 - \varphi_z(a) \cdot \bar w)^{n+1}}{(1 - a \cdot \overline{\varphi}_z(w))^{n+1}} = \frac{k_z(a)}{\overline{k_z}(w)}$$

for $a, z, w \in B$. Thus, by the explicit formula (1) for the Bergman projection P, and change-of-variable formula (4), one can see that

$$T_{\psi\circ\varphi_z} g = k_z P[(g \circ \varphi_z)\psi k_z] \circ \varphi_z \qquad (g \in A^2)$$

for every $\psi \in L^\infty$ and $z \in B$. It follows from change-of-variable formula (4) again that

$$\begin{aligned}
\langle T_{u\circ\varphi_b} T_{v\circ\varphi_b} k_a, h\rangle &= \langle T_{v\circ\varphi_b} k_a, T^*_{u\circ\varphi_b} h\rangle \\
&= \langle T_{v\circ\varphi_b} k_a, T_{\bar u\circ\varphi_b} h\rangle \\
&= \langle k_b P[(k_a \circ \varphi_b) v k_b](\varphi_b), k_b P[(h \circ \varphi_b)\bar u k_b](\varphi_b)\rangle \\
&= \int_B P[(k_a \circ \varphi_b) v k_b](\varphi_b) \overline{P[(h \circ \varphi_b)\bar u k_b](\varphi_b)} |k_b|^2\,dV \\
&= \int_B P[(k_a \circ \varphi_b) v k_b] \overline{P[(h \circ \varphi_b)\bar u k_b]}\,dV \\
&= \langle P[(k_a \circ \varphi_b) v k_b], P[(h \circ \varphi_b)\bar u k_b]\rangle \\
&= \langle T_v[(k_a \circ \varphi_b)k_b], T_u^*[(h \circ \varphi_b)k_b]\rangle \\
&= \langle T_u T_v[(k_a \circ \varphi_b)k_b], (h \circ \varphi_b)k_b\rangle
\end{aligned}$$

for every $a, b \in B$. The proof is complete. □

The notation $\| \ \|_p$ denotes the usual L^p-norm with respect to the measure V. The following lemma shows that the essentially commuting problem for Toeplitz operators can be reduced to the commuting one in certain cases.

LEMMA 3. *Let u, v be bounded pluriharmonic symbols. If T_u and T_v are essentially commuting on A^2, then $T_{u\circ\varphi}$ and $T_{v\circ\varphi}$ commute on A^2 for every $\varphi \in \Phi$.*

Proof. Let $\varphi \in \Phi$ and choose a net $\{w_\alpha\}$ in B such that $\varphi_{w_\alpha} \to \varphi$. We note, as mentioned in Section 1, $u \circ \varphi_{w_\alpha} \to u \circ \varphi$ and $v \circ \varphi_{w_\alpha} \to v \circ \varphi$ uniformly on every compact subset of B as $\varphi_{w_\alpha} \to \varphi$. Let $h \in A^2$ and $a \in B$. Since u and v are bounded by assumption, by an application of the dominated convergence theorem and Lemma 2 we can see that

$$\begin{aligned}
&|\langle (T_{u\circ\varphi}T_{v\circ\varphi} - T_{v\circ\varphi}T_{u\circ\varphi})k_a, h\rangle| \\
&\quad= \lim_\alpha |\langle (T_{u\circ\varphi_{w_\alpha}}T_{v\circ\varphi_{w_\alpha}} - T_{v\circ\varphi_{w_\alpha}}T_{u\circ\varphi_{w_\alpha}})k_a, h\rangle| \\
&\quad= \lim_\alpha |\langle (T_uT_v - T_vT_u)[(k_a \circ \varphi_{w_\alpha})k_{w_\alpha}], (h \circ \varphi_{w_\alpha})k_{w_\alpha}\rangle| \\
&\quad\le \lim_\alpha \|(T_uT_v - T_vT_u)[(k_a \circ \varphi_{w_\alpha})k_{w_\alpha}]\|_2 \|(h \circ \varphi_{w_\alpha})k_{w_\alpha}\|_2.
\end{aligned}$$

On the other hand, by the change-of-variable formula (4), we see that

$$\|(h \circ \varphi_{w_\alpha})k_{w_\alpha}\|_2^2 = \int_B |h \circ \varphi_{w_\alpha}|^2 |k_{w_\alpha}|^2 \, dV = \int_B |h|^2 \, dV. \tag{5}$$

Moreover, note that $(k_a \circ \varphi_{w_\alpha})k_{w_\alpha}$ converges to 0 weakly in A^2 for every $a \in B$. Hence, the compactness of $T_uT_v - T_vT_u$, together with (5), implies that

$$\langle (T_{u\circ\varphi}T_{v\circ\varphi} - T_{v\circ\varphi}T_{u\circ\varphi})k_a, h\rangle = 0 \qquad (a \in B).$$

Since $h \in A^2$ is arbitrary, we have $(T_{u\circ\varphi}T_{v\circ\varphi} - T_{v\circ\varphi}T_{u\circ\varphi})k_a = 0$ for every $a \in B$. Now, the result follows from the fact (see, for example, [L, Theorem 4.1]) that the set $\{k_a\colon a \in B\}$ spans a dense subset of A^2. This completes the proof. □

Before turning to the proof, we also need a recent result of Zheng [Z1] on commuting Toeplitz operators. The original statement in [Z1] is in a slightly different form.

LEMMA 4. *Let $u = f + \bar{g}$ and $v = h + \bar{k}$ be bounded pluriharmonic symbols satisfying the hypothesis of Theorem 1. Then the following statements are equivalent:*

(a) *T_u and T_v are commuting on A^2.*
(b) $\widetilde{\Delta}[f\bar{k} - h\bar{g}] = 0$.

(c) *u, v are both holomorphic or both antiholomorphic or there exist constants α and β, not both* 0, *such that $\alpha u + \beta v$ is constant on B.*

Proof. See [Z1, Main Theorem] for the equivalence (a) $\Longleftrightarrow$ (c). In [Z1] the equivalence of condition (b) is implicit in the proof of [Z1, Main Theorem]. See also [CL] for a proof of (a) $\Rightarrow$ (b). □

3. Proof

In this section, we will give a proof of Theorem 1. Before proceeding to the proof, we recall the well-known Bloch space and Hankel operators. The Bloch space $\mathcal{B}$ is the space of all holomorphic functions f on B for which

$$\|f\| = \sup_{w\in B}(1 - |w|^2)|\nabla f(w)| < \infty$$

where ∇f is the complex gradient of f. Note that $\mathcal{B} \subset A^p$ for all $p < \infty$. Moreover, it turns out [HY, Theorem 3.8] that the Bloch norm can be estimated by a certain integral quantity: For $1 \le p < \infty$, there is a positive constant C_p such that

$$C_p^{-1}\|f\| \le \sup_{a\in B}\|f\circ\varphi_a - f(a)\|_p \le C_p\|f\| \tag{6}$$

for all functions f holomorphic on B.

For a function $u \in L^2$, the Hankel operator H_u with symbol u is defined by

$$H_u f = uf - P(uf)$$

for functions $f \in A^2$. As in the case of Toeplitz operators, the operator H_u: $A^2 \to (A^2)^\perp$ is densely defined and not necessary bounded in general. However, it is known that the antiholomorphic symbols of bounded Hankel operators are precisely the conjugates of Bloch functions [BZ, Theorem C]: For $u \in A^2$, $H_{\bar u}$ is bounded if and only if $u \in \mathcal{B}$.

There is a connection between Toeplitz operators and Hankel operators, which is useful for our purpose. More explicitly, the formula

$$T_uT_v - T_vT_u = H_{\bar v}^*H_u - H_{\bar u}^*H_v$$

can be verified by a straightforward calculation. Note that Hankel operators with holomorphic symbols are the zero operator. Thus, for two bounded pluriharmonic symbols $u = f + \bar g$ and $v = h + \bar k$ which we are considering in Theorem 1, the above formula yields

$$T_uT_v - T_vT_u = H_{\bar h}^*H_{\bar g} - H_{\bar f}^*H_{\bar k}. \tag{7}$$

Moreover, since $u = f + \bar{g}$, $v = h + \bar{k}$ are bounded, it is not hard to see (see, for example, [Z2, Proposition 10]) that functions f, g, h, and k all belong to $\mathcal{B}$. Thus, the operator in the right side of (7) is bounded by the result mentioned above.

Recall that if u is a bounded pluriharmonic function and if $\{\varphi_{w_\alpha}\}$ is a net such that $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$, then $u \circ \varphi_{w_\alpha} \to u \circ \varphi$ uniformly on every compact subset of B and thus $u \circ \varphi$ is also a bounded pluriharmonic function. We first prove a lemma which shows that the invariant Laplacian behaves well with such a limiting process.

LEMMA 5. *Suppose $u = f + \bar{g}$, $v = h + \bar{k}$ are bounded pluriharmonic functions as in the hypothesis of Theorem* 1 *and let $\{\varphi_{w_\alpha}\}$ be a net such that $\varphi_{w_\alpha} \to \varphi \in \Phi$. If $u \circ \varphi = F + \bar{G}$, $v \circ \varphi = H + \bar{K}$ where F, G, H, K are functions holomorphic on B, then*

$$\widetilde{\Delta}[f\bar{k}] \circ \varphi_{w_\alpha} \to \widetilde{\Delta}[F\bar{K}], \quad \widetilde{\Delta}[h\bar{g}] \circ \varphi_{w_\alpha} \to \widetilde{\Delta}[H\bar{G}].$$

Proof. Put $f_\alpha = f \circ \varphi_{w_\alpha} - f(w_\alpha)$ and $k_\alpha = k \circ \varphi_{w_\alpha} - k(w_\alpha)$ for simplicity. Since f_α, k_α are holomorphic, we have

$$\widetilde{\Delta}(f_\alpha \bar{k}_\alpha)(z) = 4(1 - |z|^2)[\nabla f_\alpha \cdot \overline{\nabla k_\alpha} - \mathcal{R} f_\alpha \overline{\mathcal{R} k_\alpha}] \qquad (z \in B) \tag{8}$$

by [R, Proposition 4.1.3] where $\mathcal{R}$ denotes the radial differentiation. Note that

$$u \circ \varphi_{w_\alpha} - u(w_\alpha) \to u \circ \varphi - u \circ \varphi(0)$$

uniformly on every compact subset of B. In particular, since u and v are bounded,

$$u \circ \varphi_{w_\alpha} - u(w_\alpha) \to u \circ \varphi - u \circ \varphi(0)$$

in L^2. Now, using the L^2-boundedness of the Bergman projection P, we have

$$P[u \circ \varphi_{w_\alpha} - u(w_\alpha)] \to P[u \circ \varphi - u \circ \varphi(0)]$$

in L^2. Note $P(\bar{\varphi}) = \bar{\varphi}(0)$ for every $\varphi \in A^2$. It follows that

$$P[u \circ \varphi_{w_\alpha} - u(w_\alpha)] = f_\alpha$$

and

$$P[u \circ \varphi - u \circ \varphi(0)] = F - F(0).$$

Hence, $f_\alpha \to F - F(0)$ in L^2. Since $f_\alpha \to F - F(0)$ in L^2, we have $f_\alpha \to F - F(0)$ uniformly on every compact subset of B and therefore $\nabla f_\alpha \to \nabla F$ and $\mathcal{R} f \to \mathcal{R} F$ uniformly on every compact subset of B. Now, applying the same reasoning to $\bar{v}$, we see that the same is true for k_α. Thus, taking the limit in (8), we have

$$\widetilde{\Delta}(f_\alpha \bar{k}_\alpha) \to \widetilde{\Delta}(F\bar{K}). \tag{9}$$

On the other hand, since $\widetilde{\Delta}$ annihilates (anti)holomorphic functions and commutes with automorphisms, we have

$$\begin{aligned}\widetilde{\Delta}(f_\alpha \bar{k}_\alpha) &= \widetilde{\Delta}[(f \circ \varphi_{w_\alpha} - f(w_\alpha))\overline{(k \circ \varphi_{w_\alpha} - k(w_\alpha))}] \\ &= \widetilde{\Delta}[f \circ \varphi_{w_\alpha} \overline{k \circ \varphi_{w_\alpha}}] \\ &= \widetilde{\Delta}[f\bar{k}] \circ \varphi_{w_\alpha}.\end{aligned}$$

Thus, by (9), $\widetilde{\Delta}[f\bar{k}] \circ \varphi_{w_\alpha} \to \widetilde{\Delta}(F\bar{K})$. Similarly, $\widetilde{\Delta}[h\bar{g}] \circ \varphi_{w_\alpha} \to \widetilde{\Delta}(H\bar{G})$. The proof is complete. □

Now, we are ready to prove Theorem 1.

Proof of (a) $\Rightarrow$ (b). It is sufficient to show that, for a given net $\{w_\alpha\}$ such that $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$,

$$\widetilde{\Delta}(f\bar{k} - h\bar{g})(w_\alpha) \to 0 \tag{10}$$

holds. So, fix a net $\{w_\alpha\}$ such that $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$ and let F, G, H, K be as in Lemma 5. By Lemma 3, $T_{u\circ\varphi}$ and $T_{v\circ\varphi}$ commute. Thus, by Lemma 4, $\widetilde{\Delta}(F\bar{K} - H\bar{G}) = 0$. Consequently, by Lemma 5 with evaluation at the origin, we have (10) as desired. The proof is complete. □

Proof of (b) $\Rightarrow$ (c). Let $\varphi \in \Phi$ and assume $\varphi_{w_\alpha} \to \varphi$. Let $u \circ \varphi = F + \bar{G}$, $v \circ \varphi = H + \bar{K}$ as before. Fix an arbitrary point $a \in B$ and put $z_\alpha = \varphi_{w_\alpha}(a)$. Since the automorphism $\varphi_a \circ \varphi_{w_\alpha} \circ \varphi_{z_\alpha}$ fixes the origin, it is a unitary transformation, say $U_{a,\alpha}$, by the Schwarz lemma. Thus we have

$$\varphi_{z_\alpha} = \varphi_{w_\alpha} \circ \varphi_a \circ U_{a,\alpha}. \tag{11}$$

Since the set of all unitary transformations is compact, we may assume $U_{a,\alpha}$ converges to some unitary transformation U_a. Now, for a given function $\psi \in H^\infty$, since $\psi \circ \varphi_{w_\alpha} \to \psi \circ \varphi$ uniformly on every compact subset of B and $\varphi_a \circ U_{a,\alpha} \to \varphi_a \circ U_a$, we see that $\psi \circ \varphi_{w_\alpha} \circ \varphi_a \circ U_{a,\alpha} \to \psi \circ \varphi \circ \varphi_a \circ U_a$. This, together with (11), shows $\varphi_{z_\alpha} \to \tilde{\varphi}$ where $\tilde{\varphi} = \varphi \circ \varphi_a \circ U_a$. By the same argument, we have $u \circ \varphi_{z_\alpha} \to u \circ \tilde{\varphi}$ and $v \circ \varphi_{z_\alpha} \to v \circ \tilde{\varphi}$ uniformly on every compact subset of B. Note that $\varphi \in \Phi$ implies $|w_\alpha| \to 1$ and thus $|z_\alpha| \to 1$. So, $\tilde{\varphi} \in \Phi$.

Now, since $u \circ \tilde{\varphi} = F \circ \varphi_a \circ U_a + \overline{G \circ \varphi_a \circ U_a}$ and $v \circ \tilde{\varphi} = H \circ \varphi_a \circ U_a + \overline{K \circ \varphi_a \circ U_a}$, it follows from Lemma 5 that

$$\begin{aligned}0 &= \lim_\alpha \widetilde{\Delta}[f\bar{k} - h\bar{g}](z_\alpha) \\ &= \widetilde{\Delta}[(F\bar{K} - H\bar{G}) \circ \varphi_a \circ U_a](0) \\ &= \widetilde{\Delta}[F\bar{K} - H\bar{G}](\varphi_a \circ U_a(0)) \\ &= \widetilde{\Delta}[F\bar{K} - H\bar{G}](a).\end{aligned}$$

Hence, by Lemma 4, (c) holds. This completes the proof. □

Proof of (c) $\Rightarrow$ (d). Let $\{w_\alpha\}$ be a given net such that $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$. We shall continue using notations introduced in the proof of Lemma 5. To prove (d), it is sufficient to show

$$\int_B |f_\alpha \bar{k}_\alpha - h_\alpha \bar{g}_\alpha| \, dV \to 0 \tag{12}$$

where g_α, h_α are functions defined similarly. In the proof of Lemma 5, we have seen that $f_\alpha \to F - F(0)$ in L^2. Of course, the same is true for g_α, h_α, and k_α. Thus, we have $f_\alpha \bar{k}_\alpha - h_\alpha \bar{g}_\alpha \to (F - F(0))(\bar{K} - \bar{K}(0)) - (H - H(0))(\bar{G} - G(0))$ in L^1. In particular, we have

$$\begin{aligned}\int_B |f_\alpha \bar{k}_\alpha - h_\alpha \bar{g}_\alpha| \, dV \qquad\qquad (13)\\ \to \int_B |(F - F(0))(\bar{K} - \bar{K}(0)) - (H - H(0))(\bar{G} - \bar{G}(0))| \, dV.\end{aligned}$$

If (i) or (ii) in assumption (c) holds, then G, K or F, H are constants, respectively and hence $(F - F(0))(\bar{K} - \bar{K}(0)) - (H - H(0))(\bar{G} - \bar{G}(0)) = 0$. Also, if we assume (iii) and, in addition, $\alpha \neq 0$, then $(F + c_1 H) + \overline{G + \bar{c}_1 K} = c_2$ for some constants c_1, c_2. It follows that $F + c_1 H = d_1$ and $G + \bar{c}_1 K = d_2$ for some constants d_1, d_2. Hence, $F = d_1 - c_1 H$ and $G = d_2 - \bar{c}_1 K$. Inserting these into $(F - F(0))(\bar{K} - \bar{K}(0)) - (H - H(0))(\bar{G} - \bar{G}(0))$, we also have $(F - F(0))(\bar{K} - \bar{K}(0)) - (H - H(0))(\bar{G} - \bar{G}(0)) = 0$. Now, (12) follows from (13). The proof is complete. □

For $a \in B$, we let K_a be the Bergman kernel given by

$$K_a(z) = \frac{1}{(1 - z \cdot \bar{a})^{n+1}} \qquad (z \in B).$$

Then, by (1), we have the following reproducing property:

$$F(a) = \langle F, K_a \rangle \tag{14}$$

for every $F \in A^2$. We note that the above reproducing property (14) still remains valid for functions $F \in A^1$. See Theorem 7.1.4 of [R] for details.

In the proof below, the same letter C will denote the various positive constants which may change from one occurrence to the next.

Proof of (d) $\Rightarrow$ (a). Assume (d) and show (a). To show (a), it is sufficient to show the compactness of $H_{\bar{f}}^* H_{\bar{k}} - H_{\bar{h}}^* H_{\bar{g}}$ by (7). For notational simplicity, we put

$$R(z, a) = \big(f(z) - f(a)\big)\big(\bar{k}(z) - \bar{k}(a)\big) - \big(h(z) - h(a)\big)\big(\bar{g}(z) - \bar{g}(a)\big)$$

for $z, a \in B$. Then, by the Cauchy-Schwarz inequality and (6), one can see that for each $1 \leq p < \infty$,

$$\sup_{a\in B} \int_B |R(\varphi_a(z), a)|^p \, dV(z) \leq C(\|f\|^p \|k\|^p + \|h\|^p \|g\|^p) < \infty \tag{15}$$

for some constant $C = C(n, p)$. The last inequality follows from the fact that f, g, h and k are all in $\mathcal{B}$. Also, by assumption, we have

$$\lim_{|a|\to 1} \int_B |R(\varphi_a(z), a)| \, dV(z) = 0. \tag{16}$$

By the reproducing property (14), it is not hard to see that $P(\bar{F} K_a) = \bar{F}(a) K_a$ for every $F \in A^2$. Let $\psi \in A^2$ and pick a point $a \in B$. It follows from (14) that

$$\begin{aligned} H_{\bar{f}}^* H_{\bar{k}} \psi(a) &= \langle H_{\bar{f}}^* H_{\bar{k}} \psi, K_a \rangle \\ &= \langle H_{\bar{k}} \psi, H_{\bar{f}} K_a \rangle \\ &= \langle \bar{k}\psi - P(\bar{k}\psi), \bar{f} K_a - P(\bar{f} K_a) \rangle \\ &= \langle \bar{k}\psi, (\bar{f} - \bar{f}(a)) K_a \rangle. \end{aligned}$$

We also see that

$$\langle \psi, (\bar{f} - \bar{f}(a)) K_a \rangle = \langle \psi(f - f(a)), K_a \rangle = 0.$$

This follows from the reproducing property (14). It follows that

$$\begin{aligned} H_{\bar{f}}^* H_{\bar{k}} \psi(a) &= \langle (\bar{k} - \bar{k}(a))\psi, (\bar{f} - \bar{f}(a)) K_a \rangle \\ &= \int_B \frac{(f(z) - f(a))(\bar{k}(z) - \bar{k}(a))}{(1 - a \cdot \bar{z})^{n+1}} \psi(z) \, dV(z). \end{aligned}$$

Similarly, we also have

$$H_{\bar{h}}^* H_{\bar{g}} \psi(a) = \int_B \frac{(h(z) - h(a))(\bar{g}(z) - \bar{g}(a))}{(1 - a \cdot \bar{z})^{n+1}} \psi(z) \, dV(z).$$

Hence, we can represent $H_{\bar{f}}^* H_{\bar{k}} - H_{\bar{h}}^* H_{\bar{g}}$ as an integral operator as follows:

$$(H_{\bar{f}}^* H_{\bar{k}} - H_{\bar{h}}^* H_{\bar{g}}) \psi(a) = \int_B \frac{R(z, a)}{(1 - a \cdot \bar{z})^{n+1}} \psi(z) \, dV(z).$$

For each $\rho \in (0, 1)$, define $S_\rho \colon A^2 \to L^2$ by

$$S_\rho \psi(a) = \chi_{\rho B}(a) \int_B \frac{R(z, a)}{(1 - a \cdot \bar{z})^{n+1}} \psi(z) \, dV(z)$$

where the notation χ_E denotes the usual characteristic function for $E \subset B$. We first show that each S_ρ is compact. To see this, it is sufficient to see that its kernel function is in $L^2(B \times B)$. That is,

$$\int_B \int_B \left| \frac{\chi_{\rho B}(a) R(z,a)}{(1 - a \cdot \bar{z})^{n+1}} \right|^2 dV(z) dV(a) < \infty. \tag{17}$$

By change-of-variable formula (4), one obtains

$$\begin{aligned}
\int_B \int_B \left| \frac{\chi_{\rho B}(a) R(z,a)}{(1 - a \cdot \bar{z})^{n+1}} \right|^2 dV(z) dV(a) &= \int_{\rho B} \int_B \frac{|R(z,a)|^2 |k_a(z)|^2}{(1 - |a|^2)^{n+1}} dV(z) dV(a) \\
&= \int_{\rho B} \int_B \frac{|R(\varphi_a(z), a)|^2}{(1 - |a|^2)^{n+1}} dV(z) dV(a) \\
&\le \frac{1}{(1 - \rho^2)^{n+1}} \\
&\quad \times \int_{\rho B} \int_B |R(\varphi_a(z), a)|^2 dV(z) dV(a).
\end{aligned}$$

Now, (17) follows from (15). Hence, each S_ρ is compact. Put

$$T_\rho = H_{\bar{f}}^* H_{\bar{k}} - H_{\bar{h}}^* H_{\bar{g}} - S_\rho$$

for notational simplicity. We note

$$T_\rho \psi(a) = \chi_\rho(a) \int_B \frac{R(z,a)}{(1 - a \cdot \bar{z})^{n+1}} \psi(z)\, dV(z)$$

where $\chi_\rho = \chi_{B \setminus \rho B}$. By change-of-variable formula (4) and simple manipulations using (3), one obtains

$$\begin{aligned}
\int_B \frac{|R(z,a)|^2}{|1 - a \cdot \bar{z}|^{n+1} \sqrt{1 - |z|^2}} dV(z) &= \int_B \frac{|R(\varphi_a(z), a)|^2 |k_a(z)|^2}{|1 - a \cdot \overline{\varphi_a(z)}|^{n+1} \sqrt{1 - |\varphi_a(z)|^2}} dV(z) \\
&= \frac{1}{\sqrt{1 - |a|^2}} \int_B \frac{|R(\varphi_a(z), a)|^2}{|1 - a \cdot \bar{z}|^n \sqrt{1 - |z|^2}} dV(z) \\
&\le \frac{1}{\sqrt{1 - |a|^2}} \left(\int_B |R(\varphi_a(z), a)|^{2t}\, dV(z) \right)^{1/t} \\
&\quad \times \left(\int_B \frac{dV(z)}{|1 - a \cdot \bar{z}|^{sn} (1 - |z|^2)^{s/2}} \right)^{1/s}
\end{aligned}$$

where we use Hölder's inequality with the conjugate exponents $s = (4n+3)/(4n+2)$ and $t = 4n+3$. On the other hand, by the Cauchy-Schwarz inequality and (15) one can see that

$$\begin{aligned}\int_B |R(\varphi_a(z), a)|^{2t}\, dV(z) &\leq \left(\int_B |R(\varphi_a(z), a)|\, dV(z)\right)^{1/2} \\ &\quad \times \left(\int_B |R(\varphi_a(z), a)|^{4t-1}\, dV(z)\right)^{1/2} \\ &\leq C\left(\int_B |R(\varphi_a(z), a)|\, dV(z)\right)^{1/2}\end{aligned}$$

and, by Proposition 1.4.10 of [R],

$$\int_B \frac{dV(z)}{|1 - a\cdot\bar{z}|^{sn}(1-|z|^2)^{s/2}} \leq C$$

for some constants C independent of $a \in B$. It follows that

$$\int_B \frac{|R(z,a)|^2}{|1-a\cdot\bar{z}|^{n+1}\sqrt{1-|z|^2}}\, dV(z) \leq \frac{C}{\sqrt{1-|a|^2}}\left(\int_B |R(\varphi_a(z), a)|\, dV(z)\right)^{1/2t}$$

for some constant C independent $a \in B$. Now, the Cauchy-Schwarz inequality yields

$$\begin{aligned}|T_\rho\psi(a)|^2 &\leq \left(\chi_\rho(a)\int_B \frac{|R(z,a)\psi(z)|}{|1-a\cdot\bar{z}|^{n+1}}\, dV(z)\right)^2 \\ &\leq \left(\int_B \frac{\chi_\rho(a)|R(z,a)|^2}{|1-a\cdot\bar{z}|^{n+1}\sqrt{1-|z|^2}}\, dV(z)\right) \\ &\quad \times \left(\int_B \frac{\sqrt{1-|z|^2}}{|1-a\cdot\bar{z}|^{n+1}}|\psi(z)|^2\, dV(z)\right) \\ &\leq C\frac{\chi_\rho(a)}{\sqrt{1-|a|^2}}\left(\int_B |R(\varphi_a(z), a)|\, dV(z)\right)^{1/2t} \\ &\quad \times \left(\int_B \frac{\sqrt{1-|z|^2}}{|1-a\cdot\bar{z}|^{n+1}}|\psi(z)|^2\, dV(z)\right)\end{aligned}$$

for some constant C independent of $a \in B$. It follows from Fubini's theorem that

$$\begin{aligned}\int_B |T_\rho\psi|^2\, dV &\leq C \sup_{a\in B\setminus\rho B}\left(\int_B |R(\varphi_a(z), a)|\, dV(z)\right)^{1/2t} \\ &\quad \times \int_B \sqrt{1-|z|^2}|\psi(z)|^2 \int_B \frac{dV(a)}{|1-a\cdot\bar{z}|^{n+1}\sqrt{1-|a|^2}}\, dV(z)\end{aligned}$$

for some constant C independent of ρ. Moreover, by Proposition 1.4.10 of [R], we have

$$\int_B \frac{dV(a)}{|1-a\cdot\bar{z}|^{n+1}\sqrt{1-|a|^2}} \le \frac{C}{\sqrt{1-|z|^2}} \qquad (z\in B)$$

for some constant C independent of $z \in B$. Therefore, we finally have

$$\int_B |T_\rho\psi|^2 dV \le C \sup_{a\in B\setminus\rho B}\left(\int_B |R(\varphi_a(z),a)|\,dV(z)\right)^{1/2t}\int_B|\psi|^2\,dV$$

for some constant C independent of ρ. In other words,

$$\|T_\rho\|^2 \le C \sup_{a\in B\setminus\rho B}\left(\int_B |R(\varphi_a(z),a)|\,dV(z)\right)^{1/(8n+6)}$$

for some constant C independent of ρ. Now, letting $\rho\to 1$, we have $T_\rho \to 0$ in the operator norm by (16). Hence, $H_{\bar f}^* H_{\bar k} - H_{\bar h}^* H_{\bar g}$ can be approximated by compact operators, so it is compact, as desired. □

We say that a bounded linear operator L on a Hilbert space is *essentially* normal if L and its adjoint operator L^* are essentially commuting. We conclude the paper with a simple application on essentially normal Toeplitz operators.

COROLLARY 6. *Let u be a bounded pluriharmonic symbol on B and assume $u = f+\bar g$ for some functions f, g holomorphic on B. Then, the following statements are equivalent*:

(a) *T_u is essentially normal on A^2.*
(b) $\lim_{|a|\to 1}\widetilde{\Delta}(|f|^2-|g|^2)(a)=0$.
(c) *For each $\varphi\in\Phi$, $u\circ\varphi$ maps B into a straight line in $\mathbb{C}$.*
(d) $\lim_{|a|\to 1}\int_B\left(|f\circ\varphi_a - f(a)|^2 - |g\circ\varphi_a - g(a)|^2\right)dV = 0.$

Proof. The equivalence of (a), (b) and (d) is a consequence of Theorem 1. Now, assume (a) and let $\varphi\in\Phi$. Then, Theorem 1 implies that both $u\circ\varphi$ and $\bar u\circ\varphi$ are holomorphic on B or a nontrivial linear combination of $u\circ\varphi$ and $\bar u\circ\varphi$ is constant on B. The first case implies $u\circ\varphi$ is constant on B, so we have (c). Also, the latter case implies that $u\circ\varphi(B)$ lies on some straight line in $\mathbb{C}$, so we also have (c). Finally, assume (c); then we see that a nontrivial linear combination of $u\circ\varphi$ and $\bar u\circ\varphi$ is constant on B. Hence, (a) is a consequence of Theorem 1. This completes the proof. □

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