LOCAL PROPERTIES OF SELF-SIMILAR MEASURES

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1. Introduction

Let (X, d) be a complete, separable metric space and let μ be a Borel probability measure on X. Let $B_\epsilon(x)$ be the closed ball of radius ϵ centered at x. For $x \in X$, $\alpha \geq 0$ we are interested in the quantity

$$\alpha = \lim_{\epsilon \to 0} \frac{\log \mu B_\epsilon(x)}{\log \operatorname{diam} B_\epsilon(x)},$$

if the limit exists. α is often referred to as the local dimension of μ at x. Typical questions involve the conditions on the measure that would ensure the existence of the limit, characterization of points for which this limit exists, the range of possible values for the local dimension and so forth. Several authors investigated this phenomenon using thermodynamic formalism. Cawley and Mauldin [3] were the first to provide a geometric measure-theoretical framework for such a decomposition for Moran fractals. Attention has primarily focussed on the situation where X is taken to be d-dimensional euclidean space, and μ is a self-similar measure with totally disconnected support. It is therefore natural to ask if such a decomposition may be carried out for these measures by weakening the condition that their support be totally disconnected, but requiring that they satisfy the open set condition (OSC). This problem was posed in [7, Question (d)] and [9, Question 7.8], among others, and was finally settled by Arbeiter and Patzschke [1], for a random self-similar measure.

We present an alternate treatment of the above questions. We restrict our attention to the case where $X = \mathbb{R}^d$ and μ is a graph directed self-similar measure satisfying the open set condition. Our setting is therefore more restrictive than the random setting in [1], but more general than the class of self-similar measures. Moreover our approach yields stronger results, and may be readily generalized to a broader class of measures that includes the ones stated above. We are able to establish some explicit local properties, which enable us to see the geometric measure-theoretic interplay between the sets that are used to construct the measure (called cylinders), and the sets used to study the local geometry (the balls). This approach allows us to transfer results about Hausdorff [4] and Packing dimensions [5] from the string space (which is easy to analyze) to the metric space in question. The paper is arranged as follows: Section 2 describes the setting. Section 3 begins with the notion of *stoppings*. We

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then investigate the local properties. We apply these results in Section 4 to study the local dimensions.

2. Preliminaries

Let (V, E) be a directed multigraph, where V is the set of vertices and E is the set of edges. For $u, v \in V$, there is a subset E_{uv} of E known as the edges from u to v. Let $E_u = \bigcup_{v \in V} E_{uv}$. For $e \in E$, let $i(e)$ denote the initial vertex of e, and $t(e)$ be the terminal vertex of e. A path in the graph is a finite string $\gamma = \gamma_1 \gamma_2 \ldots \gamma_k$ of edges, such that $t(\gamma_i) = i(\gamma_{i+1})$. Let $E_{uv}^{(k)}$ be the set of all paths of length k that begin at u and end at v. Let $E_u^{(k)}$ be the set of all paths that are of length k and begin at u. Let $E_u^\star$ be the set of all finite paths of any length that begin at u and $E^\star = \bigcup_{u \in V} E_u^\star$. An infinite string over the alphabet E corresponds to an infinite path if the terminal vertex for each edge matches the initial vertex of the next edge. Let $E_u^{(\omega)}$ be the set of all infinite paths starting at $u \in V$; let $E^{(\omega)} = \bigcup_{u \in V} E_u^{(\omega)}$.

For $\alpha \in E^\star$, $\sigma \in E^{(\omega)}$, we say that $\alpha \prec \sigma$ if there exists $\tau \in E_{t(\alpha)}^{(\omega)}$ such that $\sigma = \alpha\tau$. If $\alpha \in E^\star$, we let $[\alpha] = \{\sigma \in E^{(\omega)} : \alpha \prec \sigma\}$ and call it the cylinder set generated by α. Let $\sigma|k$ denote the finite string comprising of the first k symbols of σ.

Let $((V, E), (X_u)_{u \in V}, (S_e)_{e \in E}, (r_e)_{e \in E}, (p_e)_{e \in E})$, be a Mauldin-Williams (MW) Graph with probabilities. Due to rescaling, we may assume that $r_e < 1$, $\forall e \in E$. Also, as a result of strong connectedness, $p_e < 1$, $\forall e \in E$. Let $X_u, u \in V$, be compact subsets of $\mathbb{R}^d$. There exists a unique invariant list $(K_u)_{u \in V}$, where each K_u is a nonempty compact subset of X_u, satisfying

$$K_u = \bigcup_{e \in E_u} S_e(K_{t(e)}), \ \forall u \in V.$$

See [6] for details.

G is said to satisfy the open set condition (OSC) iff there exists a list $(U_u)_{u \in V}$ of sets, where U_u is a non-empty, open and bounded subset of X_u satisfying

(i) $\bigcup_{e \in E_u} S_e(U_{t(e)}) \subset U_u, \forall u \in V$.
(ii) $S_e(U_{t(e)}) \cap S_{e'}(U_{t(e')}) = \emptyset, \ \forall u \in V, e, e' \in E_u, e \neq e'$.

Further, if $U_u \cap K_u \neq \emptyset$, $\forall u \in V$, then G is said to satisfy the strong OSC (SOSC). These notions are extensively discussed in [10] and [12] among others.

By the boundary of a set $A \subseteq \mathbb{R}^d$, denoted ∂A, we will mean the collection of all points belonging to the closure of the set A and to the closure of the complement of A in $\mathbb{R}^d$.

Let J_u be a non-empty compact subset of X_u. Its existence and properties have been guaranteed by Schief [10] and Wang [12]. Since $\mathrm{int}(J_u) \neq \emptyset$, $\mathcal{L}^d(J_u) > 0$, where $\mathcal{L}^d$ denotes the d - dimensional Lebesgue measure. For $e = e_1 e_2 \ldots e_n \in E^\star$,

we let $S(e) = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_n}$, and let

$$J_u^{(n)} = \bigcup_{e \in E_u^{(n)}} S_e(J_{t(e)}).$$

Then

$$K_u = \bigcap_{n=1}^{\infty} J_u^{(n)}.$$

For each $u \in V$, there is a model map $h_u \colon E_u^{(\omega)} \longrightarrow X_u$, defined so that $h_u(\sigma)$ is the unique element of the set $\bigcap_{n=1}^{\infty} S_{\sigma|n}(J_{t(\sigma|n)})$. So $K_u = h_u(E_u^{(\omega)})$. For $\tau = \tau_1 \tau_2 \ldots \tau_n \in E^*$, we let $K(\tau) = S_\tau(K_{t(\tau)})$, $r(\tau) = r(\tau_1)r(\tau_2)\ldots r(\tau_n)$, $p(\tau) = p(\tau_1)p(\tau_2)\ldots p(\tau_n)$, $r_{\max} = \max\{r(e) \colon e \in E\}$, $r_{\min} = \min\{r(e) \colon e \in E\}$, $p_{\max} = \max\{p(e) \colon e \in E\}$, $p_{\min} = \min\{p(e) \colon e \in E\}$.

Let $(\mu_u)_{u \in V}$ be the unique, invariant self-similar measure list corresponding to $(K_u)_{u \in V}$, and for $q \in \mathbb{R}$, let $\hat{\mu}_u^{(q)}$ denote the multifractal measures. For details concerning these measures, please see [7]. For our purposes, we need a measure defined for the entire graph. This may be done using the stationary distribution as follows.

Let $A(q, \beta)$ be the square matrix with rows and columns indexed by V:

$$A_{uv}(q, \beta) = \sum_{e \in E_{uv}} p(e)^q r(e)^\beta.$$

For a given q, there exists unique β so that the spectral radius of $A(q, \beta)$ is 1. Since $A(q, \beta)$ has spectral radius 1, there exist positive right and left eigenvectors ρ_v, λ_v with

$$\sum_{v \in V} \sum_{e \in E_{uv}} p(e)^q r(e)^\beta \rho_v = \rho_u, \quad \forall u \in V.$$

$$\sum_{u \in V} \sum_{e \in E_{uv}} \lambda_u p(e)^q r(e)^\beta = \lambda_v, \quad \forall v \in V.$$

By the Perron-Frobenius Theorem [11], λ_v, $\rho_v > 0$. Let $P(e) = \rho_u^{-1} p(e)^q r(e)^\beta \rho_v$ and let

$$P'(e) = \pi_u P(e)$$

where $e \in E_{uv}$, $\pi_u = \lambda_u \rho_u$, $u \in V$. Then

$$\sum_{u,v \in V} \sum_{e \in E_{uv}} P'(e) = \sum_{u \in V} \lambda_u \left(\sum_{v \in V} \sum_{e \in E_{uv}} p(e)^q r(e)^\beta \rho_v \right)$$

$$= \sum_{u \in V} \lambda_u \rho_u = 1.$$

For $\gamma \in E^*$, we let

$$\hat{\mu}^{(q)}([\gamma]) = \lambda_{i(\gamma)} p(\gamma)^q r(\gamma)^\beta \rho_{t(\gamma)}.$$

This gives us a unique probability measure $\hat{\mu}^{(q)}$ on $E^{(\omega)}$. It is therefore clear from the uniqueness of the extended measures that

$$\hat{\mu}_u^{(q)} = \frac{\hat{\mu}^{(q)}|_{E_u^{(\omega)}}}{\hat{\mu}^{(q)}(E_u^{(\omega)})}.$$

For a given q, let $\Phi(q, \beta)$ be the spectral radius of $A(q, \beta)$, and let β be the unique value such that $\Phi(q, \beta) = 1$. This defines β as an analytic function of q. Define $\alpha = -\frac{d\beta}{dq}$ and $f(\alpha) = q\alpha + \beta$. We may sometimes write $\alpha = \alpha_\mu$ to emphasize its dependence on the measure μ. These functions have been studied in [7].

For $x \in K_u$, $\alpha \in \mathbb{R}$, $\alpha > 0$, we will call α the local dimension of μ at x if

$$\lim_{\epsilon \to 0} \frac{\log \mu(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} = \alpha.$$

Note that the existence of the limit is part of the definition. Most of our attention will be directed toward the following sets:

$$K_u^{(\alpha)} = \left\{ x \in K_u : \lim_{\epsilon \to 0} \frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} = \alpha \right\},$$

$$\widehat{K}_u^{(\alpha)} = \left\{ \sigma \in E_u^{(\omega)} : \lim_{k \to \infty} \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha \right\}.$$

Since we will be dealing with ratios of logarithms, we will adopt the following conventions. For $0 < \varrho, \varsigma < 1$,

$$\frac{\log \varrho}{\log 0} = \frac{\log 1}{\log \varsigma} = \frac{\log 1}{\log 0} = 0,$$

$$\frac{\log 0}{\log \varsigma} = \frac{\log \varrho}{\log 1} = \frac{\log 0}{\log 1} = \infty,$$

$$\frac{\log 0}{\log 0} = \frac{\log 1}{\log 1} = 1.$$

3. Local structure

Fractal measures are often constructed using an iterative process. One of the consequences of the iterative process is the construction of a collection of sets C, called cylinders. In our case, the cylinders are the sets $J_u(\tau), u \in V, \tau \in E_u^*$. Computations involving the measure are carried out more naturally by using these cylinder sets. But for a given metric structure, we are really interested in deducing

the behaviour of the measure with respect to the closed balls, B. This immediately leads us to the following questions. For $x \in K_u$, $\sigma = h_u^{-1}(x)$:

1. When does $\lim\limits_{\epsilon \to 0} \dfrac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} = \lim\limits_{k \to \infty} \dfrac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha$?

2. Does the Hausdorff dimension of a subset of K_u when computed using B coincide with that using C?

3. Does the Packing dimension of a subset of K_u when computed using B coincide with that using C?

Questions 2 and 3 are answered in [4] and [5] respectively. In this paper, we prove the following result:

MAIN RESULT.   *Let $u \in V$. There exists a set D_u such that for all $x \in K_u \setminus D_u$,*

$$\lim_{\epsilon \to 0} \frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} = \lim_{k \to \infty} \frac{\log p(\sigma|k)}{\log r(\sigma|k)},$$

whenever one of these limits exist. Moreover, $\mu_u^{(q)}(D_u) = 0$, $\forall q \in \mathbb{R}$.

3.1. *Stoppings.*   The notion of *stoppings* arose from the idea of mapping the set J_u into itself, and identifying strings τ of shortest length, so that $J_u(\tau) \subseteq \operatorname{int} J_u(\tau|1)$. Although our definition is more restrictive than the one in [1], it turns out to be more useful in obtaining exact results.

Let $E_u^{(1)} = \{e_1, e_2, \ldots, e_j\}$; $J_u(e_k) = S_{e_k}(J_{t(e_k)})$, $k = 1, 2, \ldots, j$. Note that j depends on the particular choice of $u \in V$. Let $\tau \in E_u^\star$. Then τ is called a *u-stopping* iff $K(\tau) \cap J_u(e_k) = \emptyset$, $\forall k = 1, \ldots, j$; $k \neq \tau|1$. Let $N(u) = \{v \in V: E_{uv} \neq \emptyset\}$.

LEMMA 3.1.   *Let SOSC hold. Then for each $v \in N(u)$, there exists $\tau_v \in E_v^\star$ such that $e\tau_v$ is a stopping for every $e \in E_{uv}$.*

Proof.   Let $u \in V$, $v \in N(u)$. By the SOSC, $\operatorname{int} J_v \cap K_v \neq \emptyset$. Thus we may choose $x_v \in \operatorname{int} J_v \cap K_v$. By the definition of h_v and our choice of x_v, $\sigma_v = h_v^{-1}(x_v)$ must be unique. For each $e \in E_{uv}$, $S_e(x_v) \equiv x_e(v) \in \operatorname{int} J_u \cap K_u$. Therefore, by the same reasoning as before and by self-similarity, $\sigma_e(v) = h_u^{-1}(S_e(x_v))$ must also be unique. Now

$$x_e(v) \notin \partial J_u(e_k) \cap \partial J_u(e_{k'}) \text{ for any } e_k, e_{k'} \in E_u^{(1)}.$$

Hence there exists $k_e(v) \in \mathbb{N}$, smallest, such that

$$J_u(\sigma_e(v)|k_e(v) + 1) \bigcap J_u(e_k) = \emptyset \text{ for every } e_k \in E_u^{(1)}, \ e_k \neq e.$$

Let $k(v) = \max\{k_e(v): e \in E_{uv}\}$; $\tau_v = (h_v^{-1}(x_v)|k(v))$. Then, by construction, $\tau_v \in E_v^\star$, and $e\tau_v$ is a stopping for every $e \in E_{uv}$. $\square$

Remark 3.2.   It is clear that if σ is a u-stopping, and if $\sigma \prec \tau$ for some $\tau \in E_u^\star$, then τ is also a u-stopping. Thus we can take

$$k_u = \max\{k(v)\colon v \in N(u)\}$$

and finally

$$m = 1 + \max\{k_u\colon u \in V\}.$$

We then obtain that for each $u \in V$, $v \in N(u)$, there exists $e\tau_v \in E_v^\star$, such that $e\tau_v$ is a stopping for every $e \in E_{uv}$, and that moreover, $|e\tau_v| = m$, $\forall u \in V$, $v \in N(u)$, $e \in E_{uv}$.

For each $u \in V$, $v \in N(u)$, let τ_v be the smallest (in terms of length) stopping guaranteed by Lemma 3.1. Let

$$\bar{S}_v = \{s \in E_v^\star \colon |s| = |\tau_v|, \, es \text{ is a stopping for every } e \in E_{uv}\}.$$

So $\bar{S}_v \neq \emptyset$, by Lemma 3.1, $|\bar{S}_v| < \infty$. Finally, let

$$S_u = \{es\colon s \in \bar{S}_v, \, v \in N(u), \, e \in E_{uv}\}.$$

Let $S = \bigcup_{u \in V} S_u$. So for every $\tau \in S$, $|\tau| = m$.

Then S is the set of stoppings of length m.

The idea is that we take an infinite string and look for the occurrence of a finite string from S. Thus we get a finite cylinder, and using self-similarity, we would know the geometry in the neighbourhood of this cylinder. We make these ideas precise in the next section.

3.2. *Geometric results.*   Let $\tau \in S_u$. Suppose $\tau = e\tau_1\tau_2 \ldots \tau_{m-1}$, where $e \in E_{uv}$ for some $v \in N(u)$. Then $\tau^- = e\tau_1, \ldots, \tau_{m-2}$. Let

$$d(\tau) = \min\{d(x, y)\colon x \in \partial J_u(\tau), \, y \in \partial J_u(e_k), \, e_k \in E_u^{(1)}, \, e_k \neq e\}.$$

If $r(\tau^-) < d(\tau)$ let $N_\tau = 1$. If $r(\tau_-) \geq d(\tau)$, then choose $N_\tau \in \mathbb{N}$, smallest, such that $r(\tau^-)\, r_{\max}^{N_\tau} < d(\tau)$. Clearly, such N_τ exists.

LEMMA 3.3.   *Let* $u \in V$, $\tau_1 \in E^\star$ *such that* $t(\tau_1) = u$. *Let* $\tau \in S_u$, $\tau = e\tau_1\tau_2 \ldots \tau_{m-1}$ *where*, $e \in E_{uv}$, *for some* $v \in N(u)$, *and*

$$d(\tau_1\tau) = \min\{d(x, y)\colon x \in \partial J_{i(\tau_1)}(\tau_1\tau), \, y \in \partial J_{i(\tau_1)}(\tau_1 e_k), \, e_k \in E_u^{(1)}, \, e_k \neq e\}.$$

Then

(i) $d(\tau_1\tau) = r(\tau_1)d(\tau)$,
(ii) $r(\tau_1\tau^-)r_{max}^{N_\tau} < d(\tau_1\tau)$.

Proof. (i) Since $J_u(\tau)$ and $J_u(e_k)$ are compact sets, there exists $x \in \partial J_u(\tau)$, $y \in \partial J_u(e_k)$, $e_k \neq e$, such that $d(\tau) = d(x, y)$. By self-similarity,

$$d(\tau_1 \tau) = d(S_{\tau_1}(x), S_{\tau_1}(y))$$
$$= r(\tau_1)d(x, y)$$

so

$$d(\tau_1 \tau) = r(\tau_1)d(\tau).$$

(ii)

$$r(\tau_1 \tau^-) \, r_{\max}^{N_\tau} = r(\tau_1) \, r(\tau^-) \, r_{\max}^{N_\tau}$$
$$< r(\tau_1) \, d(\tau)$$
$$= d(\tau_1 \tau) \text{ by (i).} \qquad \square$$

The following definitions are motivated by [10] and [2].
Given $k \in E_u^\star$, $x \in K_u \cap J_u(k)$, $\alpha > 0$, let

$$I_1(k, x, \alpha) = \{\tau \in E_u^\star: r(\tau) \leq r(k) < r(\tau^-); \ J_u(\tau) \cap B_\alpha(x) \neq \emptyset\},$$

$$I_2(k, x, \alpha) = \{\tau \in E_u^\star: p(\tau) \leq p(k) < p(\tau^-); \ J_u(\tau) \cap B_\alpha(x) \neq \emptyset\},$$

$$I_3(k, x, \alpha) = \{\tau \in E_u^\star: |\tau| = |k|; \ J_u(\tau) \cap B_\alpha(x) \neq \emptyset\},$$

$$I_1(k) = \{\tau \in E_u^\star: r(\tau) \leq r(k) < r(\tau^-); \ J_u(\tau) \cap J_u(k) \neq \emptyset\},$$

$$I_2(k) = \{\tau \in E_u^\star: p(\tau) \leq p(k) < p(\tau^-); \ J_u(\tau) \cap J_u(k) \neq \emptyset\},$$

$$I_3(k) = \{\tau \in E_u^\star: |\tau| = |k|; \ J_u(\tau) \cap J_u(k) \neq \emptyset\}.$$

LEMMA 3.4. *Let $u \in V$, $\tau_1 \in E^*$ such that $t(\tau_1) = u$. Let $\tau \in S_u$. Then:*

(i) $I_j(\tau_1 \tau) = \{\tau_1 s: \ s \in I_j(\tau)\}$, $j = 1, 2, 3$.
(ii) $\exists \gamma: \#I_j(\tau_1 \tau) \leq \gamma, \forall \, \tau_1 \in E^*, t(\tau) = u, \forall \tau \in S_u, \forall u \in V, j = 1, 2, 3$.

Proof. (i) Follows from the definitions by using self-similarity.
(ii) Follows from (i) by choosing γ as follows:

$$\gamma = \max_{u \in V}(\max_{j=1,2,3}(\max\{\#I_j(\tau): \ \tau \in S_u\})) \qquad \square$$

LEMMA 3.5. *Let $u \in V$. Let SOSC hold. Let $\sigma = \tau_1 \tau \tau_2$ where $\tau_1 \in E^*$ with $t(\tau_1) = u$, $\tau \in S_u$, $\tau_2 \in E_{t(\tau)}^{N_r}$ where $N_r = \max_{\tau \in S} N_\tau$. Then there is a constant γ*

such that for any $\sigma \in E^\star$ as above, any $x \in J_{i(\tau_1)}(\sigma)$, any $u \in V$, and any $\alpha > 0$ with $\alpha \leq r(\sigma^-)$, and $x \in J_{i(\tau_1)}(\sigma) \subseteq B_\alpha(x)$, we must have

$$\#I_j(\sigma, x, \alpha) \leq \gamma \quad \text{for } j = 1, 2, 3.$$

Proof. Fix $j \in \{1, 2, 3\}$, $u \in V$. If $\sigma = \tau_1 \tau \tau_2$, then by Lemma 3.3 and self-similarity,

$$I_j(\sigma, x, \alpha) = \{\tau_1 s : s \in I_j(\tau\tau_2, S_{\tau_1}^{-1}(x), \bar{\alpha}), \ \bar{\alpha} \leq r(\tau\tau_2^-)\}.$$

It suffices to consider $I_j(\tau\tau_2, x, \alpha)$, where $\alpha \leq r(\tau\tau_2^-)$ and $x \in J_u(\tau\tau_2) \subseteq B_\alpha(x)$. By our choice of N_r, since $\tau_2 \in E_{t(\tau)}^{(N_r)}$, we have $r(\tau\tau_2^-) < d(\tau)$. Thus $B_\alpha(x) \subseteq J_u(\tau|1)$. And finally

$$\#I_j(\tau\tau_2, x, \alpha) \ \leq \ |E^{(|\tau|+N_r)}| \leq |E|^{(m+N_r)} < \infty. \qquad \square$$

3.3. *Measuring the overlap.* In this section, we show that under the assumptions of the OSC, the measure of the intersection of any two adjacent cylinders is zero. We begin with some definitions.

Let $\sigma \in E_u^{(\omega)}$. Let A_i be the event that $\sigma_i\sigma_{i+1}\ldots\sigma_{i+m-1} \in S$. Let $X_i = 1_{A_i}$, and suppose $j_k(\sigma)$ is chosen so that

$$\sum_{k=0}^{j_k(\sigma)} X_i = k + 1.$$

If $x \in K_u$ such that $h_u^{-1}(x) = \sigma$ is unique, then let

$$s_k(\sigma) = s_k(x) = j_k(\sigma) + N_r + m - 1.$$

When σ is fixed or its meaning is clear from the context, then we will write $j_k(\sigma) = j_k$, $s_k(\sigma) = s_k$. Let

$$\widehat{K}^{(\alpha)} \ = \ \bigcup_{u \in V} \widehat{K}_u^{(\alpha)},$$

$$D_u \ = \ \left\{\sigma \in E_u^{(\omega)} : \frac{k}{j_k(\sigma)} \to 0\right\},$$

$$D \ = \ \bigcup_{u \in V} D_u,$$

$$K \ = \ \bigcup_{u \in V} K_u.$$

We begin by finding the support of $\hat{\mu}^{(q)}$.

LEMMA 3.6.   *For q, α as defined earlier we have*

$$\hat{\mu}^{(q)}(\widehat{K}^{(\alpha)}) = 1.$$

Proof.   For each $u \in V$, $\hat{\mu}_u^{(q)}(\widehat{K}_u^{(\alpha)}) = 1$ (see [7]). So

$$\begin{aligned}
\hat{\mu}^{(q)}|_{E_u^{(\omega)}}(\widehat{K}_u^{(\alpha)}) &= \hat{\mu}^{(q)}(E_u^{(\omega)}) \\
&= \sum_{v \in V} \sum_{e \in E_{uv}} P'(e) \\
&= \pi_u \sum_{v \in V} \sum_{e \in E_{uv}} P(e) \\
&= \pi_u.
\end{aligned}$$

And thus

$$\begin{aligned}
\hat{\mu}^{(q)}(\widehat{K}^{(\alpha)}) &= \hat{\mu}^{(q)}\left(\bigcup_{u \in V} \widehat{K}_u^{(\alpha)}\right) \\
&= \sum_{u \in V} \hat{\mu}^{(q)}(\widehat{K}_u^{(\alpha)}) \\
&= \sum_{u \in V} \hat{\mu}^{(q)}|_{E_u^{(\omega)}}(\widehat{K}_u^{(\alpha)}) \\
&= \sum_{u \in V} \pi_u \\
&= 1. \qquad \qquad \square
\end{aligned}$$

LEMMA 3.7.   *Let G satisfy OSC. Then $\forall\, \sigma, \tau \in E^\star$, $q \in \mathbb{R}$, we have*:

  (i) $\mu^{(q)}(K(\sigma) \cap K(\tau)) = 0$ *if* $\sigma \not\prec \tau$ *and* $\tau \not\prec \sigma$.
  (ii) $\mu^{(q)}(K(\sigma)) = P'(\sigma)$.

Proof.   This is a generalization of [8, Lemma 3.3], where it was proven for the self-similar case. The same proof works with the appropriate modifications for the graph-directed case. $\square$

COROLLARY 3.8. $\mu^{(q)}\{x \in K_u\colon h_u^{-1}(x) \text{ is not unique}\} = 0.$

Proof. Clearly, h_u^{-1} is not unique if and only if

$$\exists s, t \in E_u^\star, s \not\prec t, t \not\prec s, \text{ such that } x \in K(s) \cap K(t).$$

So,

$$\{x \in K_u \colon h_u^{-1}(x) \text{ is not unique}\} \subseteq \bigcup_n \bigcap_{s, t \in E_u^{(n)}, s \neq t} (K(s) \cap K(t)).$$

By Lemma 3.7, we obtain the desired result. $\square$

3.4. *Nestedness of balls and cylinders.* Given $k \in \mathbb{N}$, let

$$\epsilon_k(x) = \epsilon_k(\sigma) = \max\{d(x, y) \colon y \in \partial J_u(\sigma|k)\}$$
$$m_1(\sigma|s_k) = \max\{r(t) \colon t \in I_1(\sigma|s_k, x, \epsilon_{s_k})\}.$$
$$m_2(\sigma|s_k) = \max\{p(t) \colon t \in I_2(\sigma|s_k, x, \epsilon_{s_k})\}.$$

LEMMA 3.9. *Let* $x \in K_u$ *such that* $h_u^{-1}(x) = \sigma$ *is unique. Then*:

(i) $x \in \bigcap_k J_u(\sigma|s_k)$.

(ii) $r(\sigma|s_k) < d(\sigma|s_k - N_r)$.

(iii) $x \in J_u(\sigma|s_k) \subseteq B_{\epsilon_{s_k}}(x)$ *and*

$$K_u \cap B_{\epsilon_{s_k}}(x) \subseteq \bigcup_{t \in I_1(\sigma|s_k, x, \epsilon_{s_k})} J_u(t) \subseteq J_u(\sigma|j_k - 1) \subseteq J_u(\sigma|j_{k-1}).$$

(iv) $x \in J_u(\sigma|s_k) \subseteq B_{\epsilon_{s_k}}(x)$ *and*

$$K_u \cap B_{\epsilon_{s_k}}(x) \subseteq \bigcup_{t \in I_2(\sigma|s_k, x, \epsilon_{s_k})} J_u(t) \subseteq J_u(\sigma|j_k - 1) \subseteq J_u(\sigma|j_{k-1}).$$

(v) $\lim_{k \to \infty} \dfrac{\log m_1(\sigma|s_k)}{\log r(\sigma|s_k)} = 1 = \lim_{k \to \infty} \dfrac{\log m_2(\sigma|s_k)}{\log p(\sigma|s_k)}$.

Proof. (i) Since $h_u^{-1}(x) = \sigma$ is unique, we must have $x \notin \partial J_u(e) \cap \partial J_u(e')$ for every $e, e' \in E_u^{(1)}$. So by Lemma 3.1, there exists $\tau_1 \in E_u^\star, \tau \in S$, such that $x \in J_u(\tau_1 \tau) \subset J_u(\sigma|1)$. Let $j_1 = |\tau_1| + 1$. Suppose j_k has been chosen. Since $h_u^{-1}(x)$ is unique, we must have $x \notin \partial J_{t(\sigma|j_k)}(e) \cap \partial J_{t(\sigma|j_k)}(e')$ for every $e, e' \in E_{t(\sigma|j_k)}^{(1)}$. Again, by Lemma 3.1 , there exists $t \in E_{t(\sigma|j_k)}^\star, \tau \in S$, such that $x \in J_u(\sigma_1 \sigma_2 \ldots \sigma_{j_k} t \tau) \subset J_u(\sigma|j_k)$. Let $j_{k+1} = j_k + |\tau| + 1$. Thus, inductively, we see that if $h_u^{-1}(x)$ is unique, then $X_k = 1$ for infinitely many values of k. Since $s_k = j_k + N_r + m - 1$, we see that $x \in \bigcap_k J_u(\sigma|s_k)$.

(ii)

$$r(\sigma|s_k) < r(\sigma|s_k - 1)$$
$$\leq r(\sigma|s_k - N_r - 1) \, r_{max}^{N_r}$$
$$< d(\sigma|s_k - N_r) \text{ by Lemma 3.3(ii).}$$

(iii) Since $x \in J_u(\sigma|s_k)$ (by part (i) above), for any $y \in \partial J_u(\sigma|s_k)$,

$$d(x, y) < r(\sigma|s_k) < d(\sigma|s_k - N_r).$$

By (ii) above,

$$\epsilon_{s_k} \leq d(\sigma|s_k - N_r)$$

$$= \min\{d(y_1, y_2): y_1 \in \partial J_u(\sigma|s_k - N_r), \ y_2 \in \partial J_u(\sigma_1\sigma_2\ldots\sigma_{j_k}e)\}$$

where $e \in E^{(1)}_{t(\sigma|j_k)}, e \neq \sigma_{j_k+1}$. And so,

$$K_u \cap B_{\epsilon_{s_k}}(x) \subseteq \bigcup_{t \in I_1(\sigma|s_k, x, \epsilon_{s_k})} J_u(t)$$

$$\subseteq J_u(\sigma|j_k - 1) \subseteq J_u(\sigma|j_{k-1}).$$

(iv) Same as (iii).

(v) Recall that $m_1(\sigma|s_k) = \max\{r(t): t \in I_1(\sigma|s_k, x, \epsilon_{s_k})\}$. Then $\log m_1(\sigma|s_k) \leq \log r(\sigma|s_k) \leq \log m_1(\sigma|s_k) - \log r_{\min}$ so

$$\frac{\log m_1(\sigma|s_k) - \log r_{\min}}{\log m_1(\sigma|s_k)} \leq \frac{\log r(\sigma|s_k)}{\log m_1(\sigma|s_k)} \leq 1$$

and therefore

$$\lim_k \frac{\log r(\sigma|s_k)}{\log m_1(\sigma|s_k)} = 1.$$

The limit involving the measures follows in the same way.   □

3.5. Local dimension using stoppings.

PROPOSITION 3.10. *For any $x \in K_u$ such that $\sigma = h_u^{-1}(x)$ is unique, we have*

(i) $\displaystyle \liminf_k \frac{\log \mu_u(B_{\epsilon_{s_k}}(x))}{\log \operatorname{diam}(B_{\epsilon_{s_k}}(x))} = \liminf_k \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)},$

(ii) $\displaystyle \limsup_k \frac{\log M_u(B_{\epsilon_{s_u}}(x))}{\log \operatorname{diam}(B_{\epsilon_{s_u}}(x))} = \limsup_k \frac{\log p(\sigma|s_u)}{\log r(\sigma|s_k)},$

Proof. By Lemma 3.9(iii),

$$K_u \cap B_{\epsilon_{s_k}}(x) \subseteq \bigcup_{t \in I_1(\sigma|s_k, x, \epsilon_{s_k})} J_u(t).$$

By Lemma 3.5, $\operatorname{diam}(B_{\epsilon_{s_k}}(x)) \leq \gamma m_1(\sigma|s_k)$. So

$$\frac{\log \mu_u(B_{\epsilon_{s_k}}(x))}{\log \operatorname{diam}(B_{\epsilon_{s_k}}(x))} \leq \frac{\log p(\sigma|s_k)}{\log \gamma m_1(\sigma|s_k)}$$

$$= \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)} \frac{\log r(\sigma|s_k)}{\log \gamma m_1(\sigma|s_k)}.$$

By Lemma 3.9(v),

$$\liminf_{\epsilon_{s_k}} \frac{\log \mu_u(B_{\epsilon_{s_k}}(x))}{\log \operatorname{diam}(B_{\epsilon_{s_k}}(x))} \leq \liminf_{s_k} \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)}.$$

By Lemma 3.9(iv),

$$K_u \cap B_{\epsilon_{s_k}}(x) \subseteq \bigcup_{t \in I_2(\sigma|s_k, x, \epsilon_{s_k})} J_u(t).$$

By Lemma 3.5, $\mu_u(B_{\epsilon_{s_k}}(x)) \leq \gamma m_2(\sigma|s_k)$. So

$$\frac{\log \mu_u(B_{\epsilon_{s_k}}(x))}{\log \operatorname{diam}(B_{\epsilon_{s_k}}(x))} \geq \frac{\log \gamma m_2(\sigma|s_k)}{\log r(\sigma|s_k)}.$$

By Lemma 3.9(v),

$$\liminf_{\epsilon_{s_k}} \frac{\log \mu_u(B_{\epsilon_{s_k}}(x))}{\log \operatorname{diam}(B_{\epsilon_{s_k}}(x))} \geq \liminf_{s_k} \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)}. \qquad \square$$

3.6. *Relative measures of successive cylinders.* Let $x \in K_u$. Suppose $\sigma = h_u^{-1}(x)$ is unique. For each $e \in E$, let $N_k(x, e)$ be the number of times edge e is traversed in the first k steps of σ. Let $p(x, e) = \lim_k \frac{N_k(x,e)}{k}$, if this limit exists.
For $0 < p(x, e) < 1$,

$$\lim_k \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \frac{\sum_{e \in E} p(x, e) \log p(e)}{\sum_{e \in E} p(x, e) \log r(e)}.$$

Let $S(x) \subseteq E$ be such that:

(i) $N_k(x, e) \to \infty$ as $k \to \infty$ for every $e \in S(x)$.
(ii) $\sup_k N_k(x, e) < \infty$ for every $e \in E \setminus S(x)$.

LEMMA 3.11. *Let $x \in K_u$. Suppose there exists a sequence (l_k) satisfying:*

(i) $\frac{l_{k-1}}{l_k} \to 1$ *as* $k \to \infty$.
(ii) $\frac{N_{l_k}(x,e)}{l_k} \to p'(x, e)$, $\forall e \in S(x)$,

where $0 < p'(x, e) < 1$. *Then*

$$\lim_k \frac{\log r(\sigma|l_{k-1})}{\log r(\sigma|l_k)} = 1 = \lim_k \frac{\log p(\sigma|l_{k-1})}{\log p(\sigma|l_k)}.$$

Proof. Let $\delta_1 > 0$ such that $0 < p'(x, e) - \delta_1 < p'(x, e) + \delta_1 < 1$, $\forall e \in S(x)$. Let $0 < \delta < \delta_1$. Then, for each $e \in S(x)$, there exists $N_e \in \mathbb{N}$ such that $\forall k \geq N_e$,

$$p'(x, e) - \delta < \frac{N_{l_k}(x, e)}{l_k} < p'(x, e) + \delta.$$

Let $M = \max_{e \in S(x)} N_e$. Then for $k \geq M$,

$$p'(x, e) - \delta < \frac{N_{l_k}(x, e)}{l_k} < p'(x, e) + \delta.$$

Thus

$$l_{k-1}(p'(x, e) + \delta) \log r_e < N_{l_{k-1}}(x, e) \log r_e < l_{k-1}(p'(x, e) - \delta) \log r_e, \ \forall e \in S(x).$$

Then

$$\frac{\sum_{e \in S(x)} l_{k-1}(p'(x, e) - \delta) \log r_e}{\sum_{e \in S(x)} l_k(p'(x, e) + \delta) \log r_e} < \frac{\sum_{e \in S(x)} N_{l_{k-1}}(x, e) \log r_e}{\sum_{e \in S(x)} N_{l_k}(x, e) \log r_e}$$

$$< \frac{\sum_{e \in S(x)} l_{k-1}(p'(x, e) + \delta) \log r_e}{\sum_{e \in S(x)} l_k(p'(x, e) - \delta) \log r_e}$$

And so

$$\lim_k \frac{l_{k-1}}{l_k} \frac{\sum_{e \in S(x)} (p'(x, e) - \delta) \log r_e}{\sum_{e \in S(x)} (p'(x, e) + \delta) \log r_e} \leq \lim_k \frac{\sum_{e \in S(x)} N_{l_{k-1}}(x, e) \log r_e}{\sum_{e \in S(x)} N_{l_k}(x, e) \log r_e}$$

$$\leq \lim_k \frac{l_{k-1}}{l_k} \frac{\sum_{e \in S(x)} (p'(x, e) + \delta) \log r_e}{\sum_{e \in S(x)} (p'(x, e) - \delta) \log r_e}.$$

Since $\frac{l_{k-1}}{l_k} \to 1$, we obtain

$$\frac{\sum_{e \in S(x)} (p'(x, e) - \delta) \log r_e}{\sum_{e \in S(x)} (p'(x, e) + \delta) \log r_e} \leq \lim_k \frac{\log r(\sigma|l_{k-1})}{\log r(\sigma|l_k)}$$

$$\leq \frac{\sum_{e \in S(x)} (p'(x, e) + \delta) \log r_e}{\sum_{e \in S(x)} (p'(x, e) - \delta) \log r_e}.$$

Since $\delta > 0$ was arbitrary, we let $\delta \to 0$ and obtain the desired result. $\square$

3.7. *Measuring the discarded set.*

PROPOSITION 3.12.   *For each $q \in \mathbb{R}$, $\hat{\mu}^{(q)}$ is an ergodic T-invariant probability measure on $E^{(\omega)}$.*

Proof.   This proof is the same as that of the ergodicity of μ^* in [12, Proposition 2.2.1], with very few modifications.   □

We may think of D as the set of points that are too "close" to the boundaries of the cylinder sets. Corollary 3.8 showed that the overlap has measure 0. We now show that the set D also has measure 0.

PROPOSITION 3.13.   *Let D be as before. Then for any $q \in \mathbb{R}$, we must have*

$$\hat{\mu}^{(q)}(D) = 0.$$

Proof.   Let $\sigma \in E^{(\omega)}$. Then

$$D = \bigcap_{\epsilon} \bigcup_{N \geq m} \bigcup_{n \geq N} \left\{ \omega : \left| \frac{\sum_{k=m}^{n} X_k(\omega)}{n} \right| < \epsilon \right\}.$$

Thus D is a Borel subset of $E^{(\omega)}$. Moreover it satisfies $T^{-1}D = D$. By Proposition 3.12, $\hat{\mu}^{(q)}D = 0$ or 1. Let $l_{uv} = \min\{|\gamma| : \gamma \text{ is a path from } u \text{ to } v\}$. Then strong connectedness of the graph implies that l_{uv} is defined for each $u, v \in V$. Let $l = \max_{u,v} l_{uv}$. Recall that $m = |\tau|, \tau \in S$ and let $L = \max(l, m)$, and let $M = L(L+1)\dots(2L-1)$. Consider the matrix A' indexed as

$$A'_{uv}(q, \beta) = \sum_{\gamma \in E_{uv}^{(M)}} p(\gamma)^q r(\gamma)^\beta.$$

By our choice of M, we obtain a new graph G' whose edges are given by paths of length M in G. G' is strongly connected, and it contains all possible transitions in G. Thus A' is irreducible. Modify S, the set of stoppings as follows:

$\tau \in S' \iff |\tau| = M$ and there exists $\gamma \in S, \gamma \prec \tau$.

Let $g \colon E^{(M)} \to \mathbb{R}$ be given by $g(\gamma) = 1_{S'}$. For $\sigma \in E^{(\omega)}$, let $\sigma = \sigma_1\sigma_2\dots$, where $\sigma_i \in E^{(M)}$. Let $X_i = g(\sigma_i)$. Then (X_i) is again an ergodic Markov chain, with stationary distribution π_u as before. By the ergodic theorem (see [11]), for $\hat{\mu}^{(q)}$-a.e. $\gamma \in E^{(\omega)}$,

$$\frac{\sum_{i=0}^{k-1} g(\gamma_i)}{k} \to \sum_{u,v \in V} \sum_{\tau \in E_{uv}^{(M)}} \pi_u P(\tau) g(\tau).$$

And so each stopping $\tau \in S'$ is visited $\pi_u P(\tau) > 0$. But $\sigma \in D$ implies that $\frac{\sum_{i=0}^{n-1} g(\gamma_i)}{n} \to 0$ for every $\tau \in S'$. Thus, $\hat{\mu}^{(q)}(D) < 1$.   □

4. Local dimension

4.1. *Existence of the local dimension: A consequence.*

PROPOSITION 4.1.   *Let $x \in K_u$. Let $\sigma = h_u^{-1}(x)$ be unique. If $\sigma \notin D$ then:*

(i) $\sigma \in \widehat{K}^{(\alpha)}$ $\Rightarrow$ $\lim_{k \to \infty} \frac{s_{k-1}}{s_k} = 1.$

(ii) $x \in K^{(\alpha)}$ $\Rightarrow$ $\lim_{k \to \infty} \frac{s_{k-1}}{s_k} = 1.$

Proof.   (i) Suppose $\sigma \in \widehat{K}^{(\alpha)}$. Then $\lim_k \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha$, and so

$$\lim_k \frac{\sum_{e \in S(x)} \frac{N_k(x,e)}{k} \log p(e)}{\sum_{e \in S(x)} \frac{N_k(x,e)}{k} \log r(e)} = \alpha.$$

Let S' be as in the proof of Proposition 3.13. Since $\sigma \notin D$, there exists $\tau \in S'$, such that $\limsup_k \frac{N_k(x,\tau)}{k} > 0$. Look at the first time τ shows up in the string σ. Call it T. Then for some $l \in \{0, 1, \dots, M-1\}$, τ must occur infinitely often at times $T + l + kM$. So we get

$$\lim_k \frac{\sum_{e \in S(x)} \frac{N_k(x,e)}{k} \log p(e)}{\sum_{e \in S(x)} \frac{N_k(x,e)}{k} \log r(e)} = \alpha$$

and so

$$\lim_k \frac{\sum_{e \in S(x)} \frac{N_{kM}(x,e)}{k} \log p(e)}{\sum_{e \in S(x)} \frac{N_{kM}(x,e)}{k} \log r(e)} = \alpha$$

which implies

$$\lim_k \frac{\sum_{u,v \in V} \sum_{\tau \in E_{uv}^{(M)}} \frac{N_{kM}(x,\tau)}{k} \log p(\tau)}{\sum_{u,v \in V} \sum_{\tau \in E_{uv}^{(M)}} \frac{N_{kM}(x,\tau)}{k} \log r(\tau)} = \alpha.$$

From this it follows that there exists $\tau \in S'$ such that $\lim_k \frac{N_k(x,\tau)}{k} = p'(x, \tau)$ exists, and is non-zero.

Let s_k' be the $(k+1)$-st occurrence of τ in σ. Then

$$\frac{s_{k-1}'}{s_k'} = \frac{s_{k-1}'}{N_{s_{k-1}'}(x, \tau)} \frac{N_{s_k'}(x, \tau)}{s_k'} \frac{N_{s_{k-1}'}(x, \tau)}{N_{s_{k-1}'}(x, \tau) + 1} \to 1.$$

But $s_{k-1}' \leq s_{k-1} < s_k = s_k'$. The result follows from this.

(ii) Suppose $x \in K^{(\alpha)}$. Then

$$\lim_k \frac{\log \mu_u(B_{\epsilon_{s_k}}(x))}{\log \mathrm{diam}(B_{\epsilon_{s_k}}(x))} = \alpha.$$

By Proposition 3.10, we get

$$\lim_k \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)} = \alpha.$$

By the same reasoning as in part (i) above, there exists $\tau \in S'$ such that $\lim_k \frac{N_{s_{kM}}(\tau)}{s_{kM}} = p''(x, \tau) > 0$. Now

$$\frac{s_{kM}}{s_{k(M-1)}} = \frac{s_{kM}}{N_{s_{kM}}(x, \tau)} \frac{N_{s_{kM}}(x, \tau)}{N_{s_{k(M-1)}}} \frac{N_{s_{k(M-1)}}(x, \tau)}{s_{k(M-1)}}.$$

But $N_{s_{k(M-1)}(x,\tau)} \leq N_{s_{kM}(x,\tau)} \leq N_{s_{k(M-1)}(x,\tau)} + M$, and so

$$\frac{s_{kM}}{s_{k(M-1)}} \rightarrow 1. \qquad\qquad \square$$

4.2. *The main result.*   We have already seen in Proposition 3.10 that if we restrict our attention to the stoppings, then the lim infs and lim sups of the local dimensions computed using the balls $B_{\epsilon_{s_k}}(x)$ and the cylinders $J(\sigma|s_k)$ corresponding to these stoppings, are equal, as long as $\sigma = h_u^{-1}(x)$ is unique. We now consider local dimension using arbitrary balls and arbitrary cylinders. So we will consider the expressions

$$\frac{\log \mu_u(B_\epsilon(x))}{\log \mathrm{diam}(B_\epsilon(x))} \text{ and } \frac{\log p(\sigma|k)}{\log r(\sigma|k)}. \tag{1}$$

The limits of these expressions as $\epsilon \rightarrow 0$ and $k \rightarrow \infty$ respectively may not even exist. In this section we show that if we discard all the points that are too "close" to the overlaps of the cylinders, then the existence of the limit of any one of the expressions in (1) would imply the existence of the other, and the two values would coincide. We summarize this idea in the statement of our main result:

THEOREM 4.2.   *Let $u \in V$, $x \in K_u \setminus h_u(D_u)$ and $\sigma = h_u^{-1}(x)$. Then*

$$\lim_{\epsilon \rightarrow 0} \frac{\log \mu_u(B_\epsilon(x))}{\log \mathrm{diam}(B_\epsilon(x))} = \lim_{k \rightarrow \infty} \frac{\log p(\sigma|k)}{\log r(\sigma|k)},$$

whenever one of these limits exist. Moreover, $\mu_u^{(q)}(h_u D_u) = 0$, $\forall q \in \mathbb{R}$.

This is an immediate consequence of the following two propositions.

PROPOSITION 4.3.   *Let $\sigma \in \widehat{K}_u^{(\alpha)} \setminus D_u$. Then $h_u(\sigma) = x \in K_u^{(\alpha)}$.*

Proof.   Let $\epsilon > 0$ be given. Choose $k_\epsilon \in \mathbb{N}$, smallest, such that

$$x \in J_u(\sigma|k_\epsilon) \cap K_u \subseteq B_\epsilon(x).$$

Since $x \in J_u(\sigma|k_\epsilon - 1)$, we must have $\epsilon < r(\sigma|k_\epsilon - 1)$. But $r(\sigma|k_\epsilon - 1)r_{\min} \leq r(\sigma|k_\epsilon)$. So $\epsilon < \frac{r(\sigma|k_\epsilon)}{r_{\min}}$. Then,

$$\frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} \leq \frac{\log p(\sigma|k_\epsilon)}{\log(2r(\sigma|k_\epsilon)/r_{\min})}$$

$$= \frac{\log p(\sigma|k_\epsilon)}{\log r(\sigma|k_\epsilon) + \log(2/r_{\min})}$$

Thus,

$$\limsup_\epsilon \frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} \leq \limsup_k \frac{\log p(\sigma|k)}{\log r(\sigma|k)} = \alpha.$$

By Proposition 3.10, we have $\limsup_\epsilon \frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} = \alpha$. Since $\sigma \notin D_u$, we can choose a sequence (s_k) of stoppings. Choose k such that $s_{k-1} < k_\epsilon \leq s_k$. By Lemma 3.9(iv), we get

$$K_u \cap B_{\epsilon_{s_{k-1}}}(x) \subseteq \bigcup_{i \in I_2(\sigma|s_{k-1}, x, \epsilon_{s_{k-1}})} J_u(i) \subseteq J_u(\sigma|j_k - 1).$$

Now, let $t = N_r + m - 1$. Then $s_{k-t} = j_{k-t} + t \leq j_k$. So $s_{k-t-1} \leq s_{k-t} - 1 \leq j_k - 1$. Hence $J_u(\sigma|j_k - 1) \subseteq J_u(\sigma|s_{k-t-1})$. Also, by our choice of k_ϵ, $J_u(\sigma|k_\epsilon - 1) \not\subseteq B_\epsilon(x)$ so that $\epsilon < \epsilon_{k_\epsilon - 1} \leq \epsilon_{s_{k-1}}$. Thus $J_u(\sigma|s_k) \subseteq B_\epsilon(x)$ and

$$B_\epsilon(x) \cap K_u \subseteq B_{\epsilon_{s_{k-1}}}(x) \cap K_u$$

$$\subseteq \bigcup_{i \in I_2(\sigma|s_{k-1}, x, \epsilon_{s_{k-1}})} J_u(i)$$

$$\subseteq J_u(\sigma|j_k - 1)$$

$$\subseteq J_u(\sigma|s_{k-t-1}).$$

So

$$\frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} \geq \frac{\log \gamma m_2(\sigma|s_{k-t-1})}{\log r(\sigma|s_k)} \text{ , by Lemma 3.5.}$$

$$= \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)} \frac{\log \gamma m_2(\sigma|s_{k-t-1})}{\log p(\sigma|s_{k-t-1})} \frac{\log p(\sigma|s_{k-t-1})}{\log p(\sigma|s_k)}.$$

Using Propositions 4.1, 3.10, and Lemmas 3.9, 3.11, we get

$$\liminf_\epsilon \frac{\log \mu_u(B_\epsilon(x))}{\log \operatorname{diam}(B_\epsilon(x))} \geq \lim_{s_k} \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)} = \alpha.$$

PROPOSITION 4.4. *Let $x \in K_u^{(\alpha)} \setminus h_u(D_u)$. Then $h_u^{-1}(x) = \sigma \in \widehat{K}_u^{(\alpha)}$.*

Proof.   By Proposition 3.10, we have $\lim_{s_k} \frac{\log p(\sigma|s_k)}{\log r(\sigma|s_k)} = \alpha$. Let $n \in \mathbb{N}$; then there exists s_k such that $s_{k-1} < n \le s_k$. Now, $B_{\epsilon_n}(x) \subseteq B_{\epsilon_{s_{k-1}}}(x)$. By Lemma 3.9(iv),

$$B_{\epsilon_{s_{k-1}}}(x) \cap K_u \subseteq \bigcup_{i \in I_2(\sigma|s_{k-1}, x, \epsilon_{s_{k-1}})} J_u(i) \subseteq J_u(\sigma|j_k - 1).$$

As in the proof of Proposition 4.3, let $t = N_r + m - 1$. Then $s_{k-t-1} \le j_k - 1$. So

$$J_u(\sigma|n) \subseteq B_{\epsilon_n}(x)$$

$$\subseteq B_{\epsilon_{s_{k-1}}}(x) \cap K_u$$

$$\subseteq \bigcup_{i \in I_2(\sigma|s_{k-1}, x, \epsilon_{s_{k-1}})} J_u(i)$$

$$\subseteq J_u(\sigma|s_{k-t-1}).$$

Thus

$$\frac{\log \mu_u(B_{\epsilon_n}(x))}{\log \operatorname{diam}(B_{\epsilon_n}(x))} \ge \frac{\log \gamma \, m_2(\sigma|s_{k-t-1})}{\log r(\sigma|n)},$$

by Lemma 3.5.

Using Proposition 4.1(ii) and Lemmas 3.9(v), 3.11, we get

$$\limsup_n \frac{\log p(\sigma|n)}{\log r(\sigma|n)} \le \limsup_\epsilon \frac{\log \mu_u(B_{\epsilon_n}(x))}{\log \operatorname{diam}(B_{\epsilon_n}(x))} = \alpha.$$

Again by Lemma 3.9(iii),

$$B_{\epsilon_{s_{k-1}}}(x) \subseteq \bigcup_{i \in I_1(\sigma|s_{k-1}, x, \epsilon_{s_{k-1}})} J_u(i) \subseteq J_u(\sigma|j_k - 1).$$

So, by Lemma 3.5,

$$\frac{\log \mu_u(B_{\epsilon_n}(x))}{\log \operatorname{diam}(B_{\epsilon_n}(x))} \le \frac{\log p(\sigma|n)}{\log \gamma m_1(\sigma|s_{k-t-1})}.$$

Again by Proposition 4.1(ii) and Lemmas 3.9(v), 3.11, we get

$$\liminf_n \frac{\log p(\sigma|n)}{\log r(\sigma|n)} \ge \liminf_\epsilon \frac{\log \mu_u(B_{\epsilon_n}(x))}{\log \operatorname{diam}(B_{\epsilon_n}(x))} = \alpha. \qquad \square$$

4.3. *Other measures.*   In this section, we consider multifractal decompositions with respect to other measures. The diameter of a cylinder set is nothing but the lebesgue measure, suitably normalized. Note that all the above results hold for any other measure ν defined by some different initial probability distribution. Thus all our notions and definitions make sense when considered with respect to measure ν. Given

measure μ, we will write α_μ for the local dimension, to emphasize the dependence on μ.

If we assign another set of probabilities $p'(e)$ of traversing the edge e, then we may change the matrix $A(q, \beta)$ so that its entry in row u and column v is

$$A_{uv}(q, \beta) = \sum_{e \in E_{uv}} p(e)^q p'(e)^\beta.$$

Let μ, ν be the probability measures corresponding to the probabilities $p(e)$, $p'(e)$, $e \in E$, respectively. Then we may define the functions $\beta_{\mu,\nu}$ and $\alpha_{\mu,\nu}$ as before. Standard computations show that $\alpha_{\mu,\nu} = \frac{\alpha_\mu}{\alpha_\nu}$, and similarly we may obtain $f(\alpha_{\mu,\nu}) = \frac{f(\alpha_\mu)}{f(\alpha_\nu)}$. Also it is easy to deduce that $\Phi(0, 1) = 1 = \Phi(1, 0)$ where $\Phi(q, \beta)$ is the spectral radius of $A(q, \beta)$ defined in terms of $p(e)$, $p'(e)$, $e \in E$.

Now it is clear that for $\sigma \notin D_u$,

$$\sigma \in \widehat{K}^{(\alpha_\mu)} \iff \sigma \in \widehat{K}^{(\alpha_\nu)}.$$

From Theorem 4.2, we thus deduce that for $x \notin h_u D_u$,

$$x \in K^{(\alpha_\mu)} \iff x \in K^{(\alpha_\nu)}.$$

Then, by arguing as in the proof of Theorem 4.2, we may obtain the following:

THEOREM 4.5.   *Let $u \in V$, $x \in K_u \setminus h_u(D_u)$ and $\sigma = h_u^{-1}(x)$. Then*

$$\lim_{\epsilon \to 0} \frac{\log \mu_u(B_\epsilon(x))}{\log \nu_u(B_\epsilon(x))} = \lim_{k \to \infty} \frac{\log p(\sigma|k)}{\log p'(\sigma|k)}$$

whenever one of these limits exist. Moreover, $\mu_u^{(q)}(h_u D_u) = 0$, $\forall q \in \mathbb{R}$.

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