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HOMOGENEOUS CONTACT RIEMANNIAN THREE-MANIFOLDS

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1. Introduction

A contact manifold (M, ω) is said to be *homogeneous* [8] if there is a connected Lie group G acting transitively as a group of diffeomorphisms on M which leave the contact form ω invariant. As is well known, this class extends the class of contact manifolds given by odd-dimensional spheres. If g is a metric associated to ω and G is a group acting transitively as a group of isometries which leave ω invariant, then (ω, g) is called a *homogeneous contact Riemannian structure* on M. When (M, ω) is a compact homogeneous contact manifold, by the Boothby-Wang fibration one can consider a homogeneous Sasakian structures (ω, g) on M. In this context Goldberg [10] showed that the sphere is the only simply connected homogeneous contact manifold which can be equipped with an invariant contact metric of positive sectional curvature (we note that a homogeneous Riemannian manifold is complete and hence compact when its sectional curvatures are positive). More recently, it has been proved in [13],[14] that the spheres S^3, S^5 and the Stiefel manifold $T^1(S^3)$ are the only compact simply connected n-dimensional manifolds, $n = 3, 5$, which admit a homogeneous contact structure.

The purpose of this paper is to study simply connected homogeneous contact Riemannian 3-manifolds without the condition of compactness. In Section 3, we prove that all these manifolds are Lie groups equipped with a left invariant contact Riemannian structure. In the unimodular case the torsion τ satisfies $\delta\psi = 0$, where δ is the Berger-Ebin operator [1] and $\psi = -\tau\phi$, the so-called ϕ-torsion. Moreover, the Webster scalar curvature W and the torsion invariant $\|\tau\|$, introduced by Chern-Hamilton [9], characterize such manifolds. In particular, the 3-sphere S^3 is the only simply connected 3-manifold which admits a homogeneous contact Riemannian structure, with scalar curvature $r > -2(1 - \frac{\|\tau\|}{2\sqrt{2}})^2$. Moreover, the Heisenberg group H and the Lie group $\tilde{SL}(2, R)$ are the only simply connected 3-manifolds which admit an unimodular homogeneous contact Riemannian structure with Webster scalar curvature $W = 0$. Finally, in Section 4 we show that unimodular homogeneous contact Riemannian 3-manifolds are locally ϕ-symmetric.

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Remark 1.1. Boothby-Wang ([8] p. 729) proved that if G is a semi-simple Lie group on which is defined a contact form ω, invariant under left translations, then G is locally isomorphic with either $SO(3)$ or $SL(2, R)$.

Remark 1.2. The unimodular Lie groups found in Theorem 3.1 admit a contact metric structure with $\xi \in (k, \mu)-$nullity distribution [5].

2. Preliminaries on contact Riemannian manifolds

A contact manifold is a differentiable $(2n+1)$-manifold M equipped with a global 1-form ω such that $\omega \wedge (d\omega)^n \neq 0$ everywhere on M. It has an underlying almost contact structure (ω, ϕ, ξ) where ξ is a global vector field (called the characteristic vector field) and ϕ a global tensor of type (1,1) such that

$$\omega(\xi) = 1, \quad \phi(\xi) = 0, \quad \textit{and} \quad \phi^2 = -I + \omega \otimes \xi.$$

A metric g can be found such that

$$\omega(X) = g(\xi, X), \quad (d\omega)(X, Y) = g(X, \phi Y), \quad g(X, \phi Y) = -g(\phi X, Y).$$

We refer to (ω, g) or (ω, g, ξ, ϕ) as a contact Riemannian structure. In the sequel the curvature tensor R of (M, g) is defined by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$$

where ∇ is the Levi-Civita connection with respect to g.

Given a contact Riemannian structure on M, we can define the tensors

$$h = \frac{1}{2} L_\xi \phi \quad \tau = L_\xi g, \quad l = R(\xi, \cdot)\xi$$

where L_ξ denotes the Lie derivation by ξ. The tensors h, τ and l are symmetric and satisfy (see [2], [11]):

$$\tau(X, Y) = 2g(\phi X, hY), \quad h\phi = -\phi h, \quad h(\xi) = 0, \quad \tau(\xi, X) = 0,$$

$$\tau(\phi X, \phi Y) = -\tau(X, Y), \qquad \tau(\phi X, Y) = \tau(X, \phi Y),$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.1}$$

$$\phi l \phi - l = 2(h^2 + \phi^2). \tag{2.2}$$

The tensors $\nabla_\xi h$ and $\nabla_\xi \tau$ satisfy the same properties as h and τ, respectively. Since h anticommutes with ϕ, if e is an eigenvector of h corresponding to the eigenvalue λ, then ϕe is an eigenvector of h corresponding to the eigenvalues $-\lambda$.

If ξ is a Killing vector field with respect to g, then M is said to be *K-contact manifold*. Clearly, M is K-contact if and only if $\tau = 0$ (or equivalently $h = 0$). If the almost complex structure J on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \omega(X) d/dt)$$

is integrable, M is said to be *Sasakian*. This condition is satisfied if and only if

$$R(X, Y)\xi = \omega(X)Y - \omega(Y)X.$$

Note that, by (2.2), a Sasakian manifold is K-contact, but the converse holds only in dimension 3. Moreover, for a 3-dimensional contact Riemannian manifold, the *Webster scalar curvature* is given by (see [9], p. 284)

$$W = \frac{1}{8}(r - \varrho(\xi, \xi) + 4) = \frac{1}{8}\left(r + 2 + \frac{\|\tau\|^2}{4}\right) \tag{2.3}$$

where ϱ is the Ricci tensor, r the scalar curvature and the length $\|\tau\|$ is the *torsion invariant* introduced by Chern and Hamilton [9] in their study of contact Riemannian 3-manifolds.

Now we consider the so-called ϕ-torsion ψ introduced in [11], namely the (0,2) symmetric tensor

$$\psi(X, Y) = 2g(hX, Y) = -\tau(X, \phi Y),$$

and denote by $T_g\Lambda$ the tangent space at g to the space Λ of all Riemannian metrics on M. Then we have the following result:

PROPOSITION 2.1. *In a contact Riemannian* 3*-manifolds* (M, ω, g), *the following properties are equivalent*:

(a) $\varrho(\xi, X) = 0$ *for every* $X \in \operatorname{Ker}\omega$;
(b) $(\delta\tau)(X) = 0$ *for every* $X \in \operatorname{Ker}\omega$;
(c) $\delta\psi = 0$, *i.e., the tangent vector* $\psi \in T_g\Lambda$ *is perpendicular to the orbit* O_g *of* g *under the group of diffeomorphims of* M (*see* [1]).

Proof. Since $\delta(S)(X) = -\operatorname{tr}(\nabla S)(X, \cdot, \cdot)$, where S is a (0,2) symmetric tensor, by a direct calculation one obtains

$$\delta\tau = -2\varrho(\xi, e)\theta^1 - 2\varrho(\xi, \phi e)\theta^2 + \frac{1}{2}\|\tau\|^2\omega, \tag{2.4}$$

$$\delta\psi = 2\varrho(\xi, \phi e)\theta^1 - 2\varrho(\xi, e)\theta^2 \tag{2.5}$$

where $(\theta^1, \theta^2, \omega)$ is the dual basis of the ϕ-basis $(e, \phi e, \xi)$. So Proposition 2.1 follows from (2.4) and (2.5). $\square$

Remark 2.1. From (2.4) and (2.5) we obtain the formula

$$\delta\tau = (\delta\psi)\circ\phi + \frac{1}{2}\|\tau\|^2\omega.$$

3. Homogeneous contact Riemannian 3-manifolds

We start with the main result of this note. All manifolds are supposed to be connected.

THEOREM 3.1. *Let (M, ω, g) be a simply connected homogeneous contact Riemannian 3-manifold. Then M is a Lie group G and (ω, g) is a left invariant contact Riemannian structure. More precisely, we have the following classification.*

(1) *If G is unimodular, then it is one of the following Lie groups:*
the Heisenberg group H when $W = \|\tau\| = 0$;
the 3-sphere group $SU(2)$ when $4\sqrt{2}W > \|\tau\|$;
the group $\tilde{E}(2)$, universal covering of the group of rigid motions of Euclidean 2-space, when $4\sqrt{2}W = \|\tau\| > 0$;
the group $\tilde{SL}(2, R)$ when $-\|\tau\| \neq 4\sqrt{2}W < \|\tau\|$;
the group $E(1, 1)$ of rigid motions of Minkowski 2-space when $4\sqrt{2}W = -\|\tau\| < 0$.

Moreover, in all these cases the contact Riemannian structure satisfies

$$\delta\psi = 0.$$

(2) *If G is non-unimodular, its Lie algebra is given by*

$$[e_1, e_2] = \alpha e_2 + 2\xi, \qquad [e_1, \xi] = \gamma e_2, \qquad [e_2, \xi] = 0,$$

where $\alpha \neq 0$, with $e_1, e_2 = \phi e_1 \in \operatorname{Ker}\omega$, and $4\sqrt{2}W < \|\tau\|$. Moreover, if $\gamma = 0$ the torsion $\tau = 0$ (i.e., the structure is Sasakian) and the Webster scalar curvature $W = -\frac{\alpha^2}{4}$.

Since

$$4W - \frac{\|\tau\|}{\sqrt{2}} = \frac{r}{2} + \left(1 - \frac{\|\tau\|}{2\sqrt{2}}\right)^2,$$

from Theorem 3.1 we get the following interesting consequence.

COROLLARY 3.2. *The 3-sphere group $SU(2)$ is the only simply connected 3-manifold which admits a homogeneous contact Riemannian (resp. Sasakian) structure with scalar curvature $r > -2(1 - \frac{\|\tau\|}{2\sqrt{2}})^2$ (resp. $r > -2$).*

Remark 3.1. Corollary 3.2 extends the Wallach's theorem ([12] p. 297): $SU(2)$ is the only simply connected Lie group which admits a left invariant metric of positive sectional curvature.

Another consequence of Theorem 3.1 is the following:

COROLLARY 3.3. *The Heisenberg group and $\tilde{S}L(2, R)$ are the only simply connected 3-manifolds which admit an unimodular homogeous contact Riemannian structure with Webster scalar curvature $W = 0$.*

Remark 3.2. Let (M, ω) be a compact contact (2n+1)-manifold. Then the condition $\nabla_\xi \tau = -2\psi$ is the critical point condition of the functional $E(g) = \frac{1}{2}\int_M \|\tau\|^2$ defined on the set of all metrics associated to the contact form ω (see [3],[19]). $\nabla_\xi \tau = 0$ and $\psi = 0$ are, respectively, the critical point conditions, when $\dim M = 3$, of the functionals $I(g) = \int_M r$ and $F(g) = \int_M W$ (see [15], [9]). On the other hand, each of these unimodular Lie group G possesses a discrete subgroup Γ so that the quotient $\Gamma\backslash G$ is compact (see [12]).

Proof of Theorem 3.1. Let (M, ω, g) be a simply connected homogeneous contact Riemannian 3-manifold. Denote by G the Lie group acting transitively as a group of isometries which leaves ω invariant. Since (M, g) is a simply connected homogeneous Riemannian 3-manifold, a result of Sekigawa [17] shows that (M, g) is isometric to either

(I) *a Lie group manifold endowed with a left invariant metric,*

or

(II) *a symmetric space.*

We examine separately the two cases.

Case I In this case, G is diffeomorphic to M by the projection $\pi\colon G \longrightarrow M, a \longmapsto ax_o$, where x_o is a fixed point of M, and $g^\star = \pi^\star g$ is a left invariant metric on G (see [17]). Then to the homogeneous contact form ω corresponds the contact form $\omega^\star = \pi^\star\omega$ on G which is invariant under left translations. In fact, if we denote by $\tilde{a}$ an element $a \in G$ when it is regarded as an isometry of M, then $\pi \circ L_a = \tilde{a} \circ \pi$ implies $L_a^\star\omega^\star = (\pi \circ L_a)^\star\omega = (\tilde{a} \circ \pi)^\star\omega = \pi^\star\tilde{a}^\star\omega = \omega^\star$. Moreover, it is easy to see that $g^\star$ is a metric associated to $\omega^\star$ with $\phi_\star = \pi_\star^{-1} \circ \phi \circ \pi_\star$. Therefore we can consider M as a Lie group G and (ω, g) as a left invariant contact Riemannian structure on the Lie group G. Now, from (2.1), $h = -I + \phi\nabla\xi + \omega \otimes \xi$ and hence h commutes with left translations: $(L_a)_\star h_{x_o} = h_{ax_o}(L_a)_\star$. Consequently, the eigenvectors of h are left invariant. In fact, if $\mathbf{g} \equiv T_{x_o}(G)$ denotes the Lie algebra of G and $e_{x_o} \in \mathbf{g}$

with $h_{x_o}e_{x_o} = \lambda e_{x_o}$ then $e_x = (L_a)_\star e_{x_o}$, $x = ax_o$, is a left invariant vector field which satisfies

$$h_x e_x = h_x(L_a)_\star e_{x_o} = (L_a)_\star h_{x_o}e_{x_o} = \lambda e_x.$$

In particular, the eigenvalues, λ and $-\lambda$, of h are constant. Following [12], our Lie group G can be either unimodular or non-unimudular.

Unimodular case Recall that G is called unimodular if its left invariant Haar measure is also right invariant. In terms of the Lie algebra **g**, G is unimodular if and only if the linear transformation ad_X has trace zero for every $X \in$ **g**. Choose an orientation for the Lie algebra **g** so that the cross product $u \times v$ is defined and hence the formula $L(u \times v) := [u, v]$ defines a linear mapping from **g** to itself. Then, since G is unimodular, L is selfadjoint (see [12], p. 305).

Consider an orthonormal basis (e_1, e_2, e_3) of **g** given by a ϕ-basis; i.e., $\xi = e_1$, $e = e_2$, $\phi e = e_3$ such that $he_2 = \lambda e_2$ and $he_3 = -\lambda e_3$. Then the linear map L is given by

$$L(e_1) = [e_2, e_3], \quad L(e_2) = [e_3, e_1], \quad L(e_3) = [e_1, e_2].$$

Put $L(e_i) = \sum_{j=1}^{3} a_{ji}e_j$, where a_{ji} are constants. Since L is self-adjoint, the matrix $A = (a_{ji})$ is symmetric and hence we have

$$\begin{aligned} [e_2, e_3] &= \lambda_1 e_1 + \alpha e_2 + \beta e_3, \\ [e_3, e_1] &= \alpha e_1 + \lambda_2 e_2 + \gamma e_3, \\ [e_1, e_2] &= \beta e_1 + \gamma e_2 + \lambda_3 e_3. \end{aligned}$$

Since $\nabla_\xi e$ is parallel to ϕe, and $\nabla_e \xi = -(\lambda + 1)\phi e$ by (2.1), then $[e_1, e_2] = [\xi, e]$ is parallel to $e_3 = \phi e$ and hence $\beta = \gamma = 0$. Analogously, $[e_1, e_3] = [\xi, \phi e]$ is parallel to e_2 and hence we have $\alpha = 0$. So our ϕ-basis (e_1, e_2, e_3) is an orthonormal basis consisting of eigenvectors for L:

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3. \tag{3.1}$$

Let $\theta^1 = \omega$, θ^2, θ^3 be the dual 1-forms of the vector fields ξ, e_2, e_3. Using (3.1), we have

$$\begin{aligned} (d\theta^1)(e_2, e_3) &= -(d\theta^1)(e_3, e_2) \\ &= -\frac{\lambda_1}{2} \text{ and } (d\theta^1)(e_i, e_j) = 0 \text{ for } (i, j) \neq (2, 3), (3, 2). \end{aligned}$$

Since g is a metric associated to $\theta^1 = \omega$, we must have $d\theta^1(e_2, e_3) = g(e_2, \phi e_3)$ and hence $\lambda_1 = 2$. Using (3.1) and the first Cartan structural equations, we get

$$(\nabla_{ei} e_j) = \begin{pmatrix} 0 & \frac{\lambda_2+\lambda_3-2}{2}e_3, & \frac{-\lambda_2-\lambda_3+2}{2}e_2 \\ \frac{\lambda_2-\lambda_3-2}{2}e_3 & 0 & \frac{\lambda_3-\lambda_2+2}{2}e_1 \\ \frac{\lambda_2-\lambda_3+2}{2}e_2 & \frac{\lambda_3-\lambda_2-2}{2}e_1 & 0 \end{pmatrix}. \tag{3.2}$$

Then, using (3.2), by a direct calculation we find

$$\varrho = \varrho_{11}\omega \otimes \omega + \varrho_{22}\theta^2 \otimes \theta^2 + \varrho_{33}\theta^3 \otimes \theta^3,$$

where

$$\begin{aligned}(\varrho_{11}, \varrho_{22}, \varrho_{33}) &= (\varrho(\xi, \xi), \varrho(e, e), \varrho(\phi e, \phi e)) \\ &= \left(2 - \frac{(\lambda_2 - \lambda_3)^2}{2}, \frac{\lambda_2{}^2 - \lambda_3{}^2 + 4\lambda_3 - 4}{2}, \frac{\lambda_3{}^2 - \lambda_2{}^2 + 4\lambda_2 - 4}{2}\right). \qquad (3.3)\end{aligned}$$

In particular, $\varrho(\xi, X) = 0$ for every $X \in \operatorname{Ker}\omega$ and hence, by Proposition 2.1, we obtain

$$\delta\psi = 0.$$

Using (3.2) again we get

$$\nabla\xi = \frac{\lambda_2 - \lambda_3 - 2}{2}\theta^2 \otimes e_3 + \frac{\lambda_2 - \lambda_3 + 2}{2}\theta^3 \otimes e_2,$$

$$\begin{aligned}\tau(X, Y) &= (L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= (\lambda_2 - \lambda_3)(\theta^2 \otimes \theta^3 + \theta^3 \otimes \theta^2)(X, Y). \qquad (3.4)\end{aligned}$$

Moreover, from (3.3), (3.4) and (2.3), we have

$$r = 2(\lambda_2 + \lambda_3) - 2 - \frac{(\lambda_2 - \lambda_3)^2}{2}, \qquad \lambda = \frac{(\lambda_3 - \lambda_2)}{2}, \qquad \|\tau\|^2 = 2(\lambda_3 - \lambda_2)^2 = 8\lambda^2,$$

and hence

$$W = \frac{(\lambda_2 + \lambda_3)}{4}.$$

Therefore, we have the following cases.

(a) If $W = \|\tau\| = 0$, then $\lambda_2 = \lambda_3 = 0$.
(b) If $2W > \frac{\|\tau\|}{2\sqrt{2}}$, then $\lambda_2 = 2W - \lambda > \frac{\|\tau\|}{2\sqrt{2}} - \lambda \geq 0$ and $\lambda_3 = 2W + \lambda > \frac{\|\tau\|}{2\sqrt{2}} + \lambda \geq 0$.
(c) If $2W = \frac{\|\tau\|}{2\sqrt{2}} > 0$, then $\lambda_2 = \frac{\|\tau\|}{2\sqrt{2}} - \lambda$ and $\lambda_3 = \frac{\|\tau\|}{2\sqrt{2}} + \lambda$, with $\tau \neq 0$, imply either $\lambda_2 = 0$ and $\lambda_3 > 0$ or $\lambda_2 > 0$ and $\lambda_3 = 0$.
(d) If $-\frac{\|\tau\|}{2\sqrt{2}} \neq 2W < \frac{\|\tau\|}{2\sqrt{2}}$, then $-\frac{\|\tau\|}{2\sqrt{2}} - \lambda \neq \lambda_2 < \frac{\|\tau\|}{2\sqrt{2}} - \lambda$ and $-\frac{\|\tau\|}{2\sqrt{2}} + \lambda \neq \lambda_3 < \frac{\|\tau\|}{2\sqrt{2}} + \lambda$, so we have the following possibilities: $(\lambda_2 < 0, \lambda_3 < 0)$, $(\lambda_2 < 0, \lambda_3 > 0)$, and $(\lambda_2 > 0, \lambda_3 < 0)$.

(e) If $2W = -\frac{\|\tau\|}{2\sqrt{2}} < 0$, then $\lambda_2 = -\frac{\|\tau\|}{2\sqrt{2}} - \lambda$ and $\lambda_3 = -\frac{\|\tau\|}{2\sqrt{2}} + \lambda$, with $\tau \neq 0$, imply either $\lambda_2 = 0$ and $\lambda_3 < 0$ or $\lambda_2 < 0$ and $\lambda_3 = 0$.

On the other hand, Milnor ([12], p. 307) gave a complete classification of the unimodular three-dimensional Lie groups considering the possible combinations of the signs of $\lambda_1, \lambda_2, \lambda_3$. Using this classification, the cases (a), (b), (c), (d), (e) give, respectively, the Heisenberg group H, the 3-sphere group $SU(2)$, $\tilde{E}(2)$, $\tilde{S}L(2, R)$ and $E(1, 1)$.

Non-unimodular case In this case the Lie algebra $\mathbf{g}$ is not unimodular and its unimodular kernel $\mathbf{a} = \{X \in \mathbf{g}\colon \operatorname{tr} ad_X = 0\}$ is 2-dimensional ([12], p. 320). Since

$$\begin{aligned} \operatorname{tr} ad_\xi &= g([\xi, X], X) + g([\xi, \phi X], \phi X) \\ &= -g(\nabla_X \xi, X) - g(\nabla_{\phi X} \xi, \phi X) \\ &= -\frac{1}{2}\tau(X, X) - \frac{1}{2}\tau(\phi X, \phi X) = 0, \end{aligned}$$

where X is an unit vector field orthogonal to ξ, then $\xi \in \mathbf{a}$.

Now, we consider an orthonormal basis $(e_2, e_3 = \xi)$ in $\mathbf{a}$. Then $e_1 = -\phi e_2$ does not belong to $\mathbf{a}$ and (e_1, e_2, e_3) is an orthonormal basis of $\mathbf{g}$. Moreover the map $L := ad_{e_1}$ is a linear transformation from $\mathbf{a}$ to itself ([12], p. 320). Then, we can put

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \tag{3.5}$$

$$[e_1, e_3] = \gamma e_2 + \delta e_3, \tag{3.6}$$

where $\delta = g([e_1, e_3], e_3) = -g(\nabla_\xi e_1, \xi) = g(e_1, \nabla_\xi \xi) = 0$ and hence $\alpha = \operatorname{tr} L = \operatorname{tr} ad_{e_1} \neq 0$. Since g is a metric associated to ω, we have $(d\omega)(e_1, e_2) = g(e_1, \phi e_2) = -1$. On the other hand, by (3.5), $(d\omega)(e_1, e_2) = -\frac{1}{2}\omega([e_1, e_2]) = -\frac{\beta}{2}$, and so, we get $\beta = 2$. Since $g([e_2, e_3], e_3) = g(e_2, \nabla_\xi \xi) = 0$, we can put

$$[e_2, e_3] = \mu_1 e_1 + \mu_2 e_2. \tag{3.7}$$

Jacobi's identity

$$[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = 0,$$

with (3.5), (3.6) and (3.7), implies $\alpha \mu_1 e_1 - 2\mu_2 e_3 = 0$ and hence $\mu_1 = \mu_2 = 0$. Therefore

$$[e_1, e_2] = \alpha e_2 + 2e_3, \qquad [e_1, e_3] = \gamma e_2, \qquad [e_2, e_3] = 0. \tag{3.8}$$

Using (3.8) and the first Cartan structural equations, we get

$$(\nabla_{e_i} e_j) = \begin{pmatrix} 0 & \frac{2-\gamma}{2} e_3 & \frac{\gamma-2}{2} e_2 \\ -\alpha e_2 - \frac{\gamma+2}{2} e_3 & \alpha e_1 & \frac{2+\gamma}{2} e_1 \\ \frac{-(2+\gamma)}{2} e_2 & \frac{2+\gamma}{2} e_1 & 0 \end{pmatrix}. \tag{3.9}$$

Using (3.9), by a direct calculation we find

$$\varrho = \varrho_{11} \omega \otimes \omega + \varrho_{22} \theta^2 \otimes \theta^2 + \varrho_{33} \theta^3 \otimes \theta^3 + \varrho_{23} \theta^2 \otimes \theta^3 + \varrho_{32} \theta^3 \otimes \theta^2,$$

where $(\theta^1, \theta^2, \theta^3)$ is the dual basis of (e_1, e_2, e_3), and

$$\begin{aligned}(\varrho_{11}, \varrho_{22}, \varrho_{33}, \varrho_{23}, \varrho_{32}) \\ = \left(-\alpha^2 - 2 - 2\gamma - \frac{\gamma^2}{2}, -\alpha^2 - 2 + \frac{\gamma^2}{2}, 2 - \frac{\gamma^2}{2}, -\alpha\gamma, -\alpha\gamma\right).\end{aligned} \tag{3.10}$$

Using (3.9) again we get

$$\nabla \xi = \frac{\gamma - 2}{2} \theta^1 \otimes e_2 + \frac{\gamma + 2}{2} \theta^2 \otimes e_1,$$

and hence

$$\tau = \gamma(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1). \tag{3.11}$$

Moreover, from (3.10) and (3.11), we obtain

$$r = -2\alpha^2 - 2 - 2\gamma - \frac{\gamma^2}{2}, \quad 4W = -\alpha^2 - \gamma, \quad \|\tau\|^2 = 2\gamma^2$$

from which $4\sqrt{2}W < \|\tau\|$. If $\gamma = 0$, namely the Milnor isomorphism invariant D is equal to zero, the contact metric structure (ω, g) is K-contact, the basis (e_1, e_2, ξ) diagonalizes the Ricci tensor, the scalar curvature $r = -2\alpha^2 - 2$ and the Webster curvature W is $-\frac{\alpha^2}{4}$.

Case II In this case (M, g) is a symmetric space. Since g is also a metric associated to ω, a result of [4] shows that (M, g) has constant sectional curvature $c = 0$ or $c = 1$.

Assume $c = 0$, so g is a flat metric. From (2.2), the eigenvalues of h are $+1$ and -1. Let $e.\phi e \in \operatorname{Ker}\omega$ such that $he = -e$ and $h\phi e = \phi e$. Then, by (2.1), we get

$$\nabla_e \xi = 0, \quad \nabla_{\phi e} \xi = 2e. \tag{3.12}$$

Moreover, using (3.12) and $R(X, \xi)\xi = 0 = R(X, \phi X)\xi$, we obtain

$$\nabla_\xi e = \nabla_\xi \phi e = \nabla_e \phi e = 0, \quad \nabla_{\phi e} e = -2\xi.$$

Therefore we have

$$[\phi e, \xi] = 2e, \quad [e, \xi] = 0, \quad [e, \phi e] = 2\xi. \tag{3.13}$$

Now we recall the following result of Lie group theory (see [21], p. 10).

PROPOSITION. *Let M be an n-dimensional connected and simply connected manifold and let $X_1, \dots, X_n$ be complete vector fields which are linearly independent at each point of M and satisfy $[X_i, X_j] = \sum_1^n c_{ij}^k X_k$, where the c_{ij}^k are constants. Then, for each point $p \in M$, the manifold M has a unique Lie group structure such that p is the identity and the vector fields X_i are left invariant.*

Since ξ, and also e and ϕe (because M is simply connected) are global vector fields defined on M, from (3.13) and the above proposition, we have that M has a unique Lie group structure. Moreover, in this case the Webster curvature satisfies $4\sqrt{2}W = 2\sqrt{2} = \|\tau\|$ and hence, by the study of case (I), we conclude that M is the Lie group $\tilde{E}(2)$.

Finally we assume $c = 1$. Then the Ricci curvature $\varrho(\xi, \xi) = 2$ and hence $h = 0$. So, $(M, \omega, g, \xi, \phi)$ is a complete simply connected Sasakian 3-manifold with ϕ-holomorphic sectional curvature $c = 1$. On the other hand, the 3-sphere group $SU(2)$ also has a Sasakian structure $(\omega_o, g_o, \xi_o, \phi_o)$ with ϕ-holomorphic sectional curvature $c = 1$ (which corresponds to $\lambda_1 = \lambda_2 = \lambda_3 = 2$). Then, by Proposition 4.1 of [18], there exists a diffeomorphism $f\colon SU(2) \longrightarrow M$ which maps the structure tensors of $SU(2)$ in the corresponding structure tensors of M:

$$f^\star g = g_o, \quad f^\star \omega = \omega_o, \quad f_\star \xi_o = \xi, \quad f_\star \phi_o = \phi f_\star.$$

In particular, the contact metric structure (ω, g) on M is left invariant with respect to the Lie group structure induced by $SU(2)$. In this case $W = 1$ and $\tau = 0$.

Remark 3.3. We will exhibit homogeneous contact Riemannian structure on Lie groups.

Let G be a 3-dimensional unimodular Lie group and g a left invariant metric on G. Then there exists an orthonormal basis (e_1, e_2, e_3) of the Lie algebra $\mathbf{g}$ such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3,$$

where $\lambda_1, \lambda_2 \lambda_3$ are constants. Let $(\theta^1, \theta^2, \theta^3)$ be the dual basis of (e_1, e_2, e_3). Then

$$(d\theta^1)(e_2, e_3) = -(d\theta^1)(e_3, e_2) = -\frac{\lambda_1}{2} \text{ and } (d\theta^1)(e_i, e_j) = 0$$

for $(i, j) \neq (2, 3), (3, 2)$. So θ^1 is a contact form if $\lambda_1 \neq 0$ and $\xi = e_1$ is the characteristic vector field. Assuming $\lambda_1 = 2$ and defining ϕ, with respect to the basis

(e_1, e_2, e_3), by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.14}$$

we have $d\theta^1 = g(\cdot, \phi)$. Thus g is a metric associated to θ^1 and the Riemannian connection is given by (3.2). Then it is easy to check, using Milnor's classification, that the Webster curvature and the torsion invariant distinguish the unimodular Lie groups as in Theorem 3.1.

Now let G be a 3-dimensional non-unimodular Lie group with g left invariant metric. Then there exists an orthonormal basis (e_1, e_2, e_3) of the Lie algebra $\mathbf{g}$ such that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

where $\alpha, \beta, \gamma, \delta$ are constants satisfying $\alpha + \delta \neq 0$, $\alpha\gamma + \beta\delta = 0$ ([12] p. 321). In particular, $\beta = 2$, $\delta = \gamma = 0$ and $\alpha \neq 0$ satisfy the above conditions. Moreover, if $(\theta^1, \theta^2, \theta^3)$ is the dual basis of (e_1, e_2, e_3), by a direct calculation one obtains

$$(d\theta^3)(e_1, e_2) = -(d\theta^3)(e_2, e_1) = -1 \text{ and } (d\theta^3)(e_i, e_j) = 0$$

for $(i, j) \neq (1, 2), (2, 1)$. So, θ^3 is a contact form and $\xi = e_3$ is the characteristic vector field. Defining ϕ, with respect to the basis (e_3, e_1, e_2), by the matrix (3.14), we have $d\theta^3 = g(\cdot, \phi)$. Thus g is a metric associated to θ^3 and the Riemannian connection is given by (3.9) with $\gamma = 0$. Then, by (3.10), $\varrho(\xi, \xi) = \varrho(e_3, e_3) = 2$ which gives that ξ is a Killing vector. Furthermore, the scalar curvature is given by $r = -2\alpha^2 - 2$ and the Webster scalar curvature by $W = -\alpha^2/4$.

4. Locally ϕ-symmetric spaces

T. Takahashi [20] introduced the notion of locally ϕ-symmetric spaces. These are Sasakian manifolds satisfying the curvature condition

$$\phi^2\big(\nabla_V R\big)(X, Y, Z) = 0 \tag{4.1}$$

for all $X, Y, Z, V \in \operatorname{Ker}\omega$. The geometric meaning for Takahashi's definition comes from two facts. First, a Sasakian manifold is locally ϕ-symmetric if and only if the base manifold, of the local fibering, is a Hermitian symmetric space. Second, a Sasakian manifold is locally ϕ-symmetric if and only if all ϕ-geodesic symmetries are isometric. In [6] the authors extended the Takahashi's notion, to an arbitrary contact Riemannian manifold by using the condition (4.1); it is analogous to the notion of local symmetry in the Riemanniam case. More recently, Boeckx-Vanhecke [7] give the following new definition: a contact Riemannian manifold is called locally ϕ-symmetric space if and only if all characteristic reflections are (local) isometries.

This definition seems more restrictive, but it is more geometric. We note (see [7], Theorem 9) that the standard contact Riemannian structure of the tangent sphere bundle of a Riemannian manifold of constant sectional curvature $c \neq 1$ is locally ϕ-symmetric and it is not Sasakian. In this section we show the following theorem.

THEOREM 4.1. *Let M be a homogeneous contact Riemannian 3-manifold with $\gamma = 0$ in the non-unimodular case. Then M is locally ϕ-symmetric following* [6].

Proof. We recall that in the 3-dimensional case the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= g(X,Z)QY - g(Y,Z)QX - g(QY,Z)X + g(QX,Z)Y \\ &\quad -\frac{r}{2}\{g(X,Z)Y - g(Y,Z)X\}, \end{aligned}$$

where Q is the Ricci operator. So, if $r = $ const., we have

$$\begin{aligned} (\nabla_V R)(X,Y,Z) &= g(X,Z)(\nabla_V Q)Y - g(Y,Z)(\nabla_V Q)X \\ &\quad + g((\nabla_V Q)X,Z)Y - g((\nabla_V Q)Y,Z)X. \end{aligned} \tag{4.2}$$

Now let (M, ω, g) be a homogeneous contact Riemannian manifold. From Theorem 3.1, the universal covering of M is a Lie group G equipped with a left invariant contact metric structure. In the sequel we will use the same notations as in the proof of Theorem 3.1.

Assume G unimodular. Since $\nabla_{e_i}\theta^j$ is the 1-form dual to $\nabla_{e_i}e_j$, from (3.2) and (3.3) we have

$$\begin{aligned} \nabla_\xi \varrho &= \frac{(\lambda_2-\lambda_3)(\lambda_2+\lambda_3-2)^2}{2}(\theta^2\otimes\theta^3+\theta^3\otimes\theta^2), \\ \nabla_e \varrho &= \frac{(\lambda_3-2)(\lambda_3-\lambda_2+2)^2}{2}(\theta^1\otimes\theta^3+\theta^3\otimes\theta^1), \\ \nabla_{\phi e} \varrho &= \frac{(2-\lambda_2)(\lambda_2-\lambda_3+2)^2}{2}(\theta^1\otimes\theta^2+\theta^2\otimes\theta^1). \end{aligned} \tag{4.3}$$

Assume G non-unimodular. From (3.9) and (3.10), where $\gamma = 0$, $e_1 = e$, $e_2 = \phi e$ and $e_3 = \xi$, we obtain

$$\begin{aligned} \nabla_\xi \varrho &= 0, \\ \nabla_e \varrho &= (-\alpha^2-4)(\theta^3\otimes\theta^2+\theta^2\otimes\theta^3), \\ \nabla_{\phi e} \varrho &= (\alpha^2+4)(\theta^1\otimes\theta^3+\theta^3\otimes\theta^1). \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we get $(\nabla_V \varrho)(X, Y) = 0$ for all $X, Y, V \in \operatorname{Ker}\omega$. Therefore, by (4.2), M is locally ϕ-symmetric.

Remark 4.1. It is well known that the Ricci tensor of a Sasakian 3-manifold is given by

$$\varrho = \left(\frac{r}{2} - 1\right) g + \left(-\frac{r}{2} + 3\right) \omega \otimes \omega. \tag{4.5}$$

From (4.2) and (4.5) we get the following Watanabe's result [22]: a homogeneous Sasakian 3-manifold is locally ϕ-symmetric. So our Theorem 4.1 generalizes this result.

Question. It would be interesting to find how the two definitions of locally ϕ-symmetric space, given in [6] and [7], are related.

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